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Akihiko Takahashi · Toshihiro Yamada

Asymptotic Expansion and Weak Approximation

Applications of Malliavin Calculus and Deep Learning

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To our families

Preface

Asymptotic methods have been widely used in computation or approximation of functions and quantities related to partial differential equations (PDEs), statistics and probability theory in both academics and industry. Particularly, in finance asymptotic expansions for functionals of Brownian motions provide fast and tractable approximations for intractable, but important models in financial markets driven by stochastic differential equations (SDEs). At the beginning of Chap. 7 entitled "Asymptotic Expansion and Weak convergence" in the book of Malliavin and Thalmaier (2006), the authors stated as follows: *In all these developments*, *the result of Watanabe*, *which provides the methodology of projecting as asymptotic expansion through a non-degenerated map*, *plays a key role.* Namely, S. Watanabe introduced the sophisticated theory and computational tool for analyzing Wiener functionals and heat kernels. Since then, based on the result of S. Watanabe asymptotic expansion approaches have been actively developed in computation of expectations on Wiener space within the fields of financial mathematics and statistics for the past three decades, after the earlier studies such as Kunitomo and Takahashi (1992, 2001, 2003), Takahashi (1995, 1999) and Yoshida (1992a,b).

On the other hand, weak approximation of SDEs provides time-discretized computation for expectations or integrals of the solutions of SDEs on Wiener space. Weak approximation has a long history and has been developed by G. Maruyama, G. Milstein and D. Talay and many researchers with a literature of Monte Carlo simulation. Then, at the end of 1990s and the beginning of 2000s, S. Kusuoka introduced a framework of a higher order weak approximation scheme, which works under irregular test functions with a general condition. Today, weak approximation schemes with Monte Carlo methods play important roles in computational mathematics especially in nonlinear problems.

Recently, deep learning methods have been developed as a technique of AI and widely utilized in industries. In applied mathematics, especially in the areas of PDEs and stochastic modeling, neural networks have been used as function approximations (or space-time approximations), which may be regarded as an alternative of finite difference and finite element methods. Since deep learning techniques is generally able to work in high-dimensional settings, it provides a powerful tool in scientific computing.

Our main objective is to “connect” the Watanabe expansion and high order weak approximation of SDEs, which is a continuation of the content of Chap. 7 “Asymptotic Expansion and Weak convergence” in the book of Malliavin and Thalmaier (2006). Concretely, we provide a recent development on asymptotic methods on Wiener space, and then introduce a type of higher order weak approximation of SDEs by certain Brownian polynomials based on asymptotic expansions. Furthermore, another objective of this book is to develop a high order weak approximation scheme with a deep learning method, because it provides wide applications for high-dimensional nonlinear problems. In this regard, we show how to combine our asymptotic expansion based weak approximation with a neural network approximation, which is applicable to high-dimensional nonlinear models.

Chapter 1 and 2 summarize notations and basic facts on probability theory, especially the Itô and Malliavin calculus, respectively. Then, in Chap. 3, Watanabe’s asymptotic expansion is reviewed and refined in terms of computational aspects. Chapter 4 provides a general weak approximation scheme based on our expansion method with a numerical recipe. The deep learning application in a high-dimensional nonlinear model is shown in Chap. 5.

The book is written based on a work in JST SAKIGAKE, lecture notes provided in Department of Engineering Science at Osaka University and Department of Mathematics at Kyoto University, and a talk in Bachelier Seminar Paris at H. Poincaré Institute, given by the second author. We are grateful to Professor Masaaki Fukasawa (Osaka University), Professor Emmanuel Gobet (Ecole Polytechnique), Professor Shigeo Kusuoka (University of Tokyo), Professor Takashi Sakajo (Kyoto University) and Professor Jun Sekine (Osaka University) for providing opportunities and motivations for this work. We also thank Professor Riu Naito (University of Toyama) for his continuous support and suggestions on numerical schemes and experiments. Moreover, we greatly appreciate CARF (Center for Advanced Research in Finance) and CIRJE (Center for International Research on the Japanese Economy) in University of Tokyo, CFEE (Center for Financial Engineering Education) in Hitotsubashi University, and GCI Asset Management, Inc. for their constant support of our research. Furthermore, we are grateful to Professor Naoto Kunitomo and Professor Seisho Sato (University of Tokyo) for giving us this opportunity, and we also appreciate Professor Kunitomo for his precious suggestions, which substantially improve the first version of our manuscript. Finally, we would like to thank the Springer staff, particularly, Mr. Praveen Anand Sachidanandam and Mr. Yutaka Hirachi for their support in publishing this book.

We expect that the book will help undergraduate/graduate students, researchers and practitioners who are interested in stochastic calculus, numerical analysis and machine learning to understand the theory and application of asymptotic expansion and weak approximation, as well as to find a new topic in interdisciplinary fields.

Tokyo, Japan Akihiko Takahashi
April 2025 Toshihiro Yamada

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