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Apostolos Syropoulos

Fuzzy Mathematics

A Fundamental Introduction

Springer

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Dedicated
to my son Demetrios-Georgios
and Linda

Preface

In 2022, I received an invitation to contribute a chapter for a book about the use of neutrosophic sets (an extension of fuzzy sets) in education. I asked some of my colleagues whether they would like to work on a small-scale project on the introduction of fuzzy sets to secondary education students. They liked the idea and we started working on the project. Two of my colleagues taught the material that I prepared to two different classes consisting of 25 students. The fourth member, who is the principal of the school where we work, provided all the necessary help and assistance required. The result of our endeavor was documented in a chapter that was published as a chapter of a book.[1]

When the project was concluded, I thought it would be a nice idea to *transform* the material I had prepared for this mini course into a little book. At that time, I also noticed that there was no simple but complete introduction to fuzzy mathematics. Thus, the idea seemed reasonable and I am really glad that Springer agreed with me. The book that you hold in your hands (or view on a computer screen, a tablet, or a smartphone) is the result of this endeavor. Our mini project revealed certain things (e.g., what is the minimum mathematical background to follow such a course or read a book based on it) and I took under serious consideration all of them. Consequently, the book has been designed for anyone who has a very basic background in mathematics and wants to learn what are these weird mathematical objects that are called *fuzzy sets*. Anyone who wants to learn about fuzzy sets needs to know what is a set and what are the properties and operations between sets. Thus, this book introduces all these notions and ideas. In different words, it is a self-contained introduction to the basics of fuzzy mathematics. However, fuzzy sets are a mathematical model of *vagueness* and Chap. 1 of the book is an introduction to vagueness and related ideas. This chapter presents the various concepts and ideas from a philosophical and a practical point of view. Since fuzzy sets are sets, it is necessary to

[1] Syropoulos, A., Giakati, I., Prountzos, I., & Tatsiou, E. (2023). *Introducing Vagueness in the Mathematical Curriculum of Secondary Education: Experience in Greece*. In S. Broumi (Ed.), *Handbook of Research on the Applications of Neutrosophic Sets Theory and Their Extensions in Education* (pp. 205–214). IGI Global. https://doi.org/10.4018/978-1-6684-7836-3.ch010.

have a basic understanding of sets and the operations between them. This is the subject of Chap. 2. Chapter 3 introduces fuzzy sets, their operations, their representations, and some of their most common extensions. Chapter 4 is about fuzzy numbers and their operations. Fuzzy numbers have found many applications and, naturally, most people working on practical problems using fuzzy sets, use fuzzy numbers. Certainly, this book could have more about fuzzy sets and their mathematics, but this was not my initial intention. The reader who will read this book and get an understanding of fuzzy sets and their operations can continue studying fuzzy sets and their mathematics—there are many good books on the subject.

Xanthi, Greece Apostolos Syropoulos
July 2024

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