Shyam Sunder Gupta

Exploring the Beauty of Fascinating Numbers

PRAXIS

Springer

Springer Praxis Books

Popular Science

The Springer Praxis Popular Science series contains fascinating stories from around the world and across many different disciplines. The titles in this series are written with the educated lay reader in mind, approaching nitty-gritty science in an engaging, yet digestible way. Authored by active scholars, researchers, and industry professionals, the books herein offer far-ranging and unique perspectives, exploring realms as distant as Antarctica or as abstract as consciousness itself, as modern as the Information Age or as old our planet Earth. The books are illustrative in their approach and feature essential mathematics only where necessary. They are a perfect read for those with a curious mind who wish to expand their understanding of the vast world of science.

Shyam Sunder Gupta

Exploring the Beauty of Fascinating Numbers

Springer

Shyam Sunder Gupta
Indian Railways
Jaipur, Rajasthan, India

Springer Praxis Books
ISSN 2626-6113 ISSN 2626-6121 (electronic)
Popular Science
ISBN 978-981-97-2464-2 ISBN 978-981-97-2465-9 (eBook)
https://doi.org/10.1007/978-981-97-2465-9

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

This book is dedicated with great respect to the memory of my father-in-law late Shri Dwarka Prasad Gupta and mother-in-law late Smt. Angoori Devi, for their love and affection.

Preface

While I was a student, a paper titled *Pleasure Elements in Mathematics* in Junior Science Digest, India, led to my interest in recreational mathematics. While playing with numbers, some beautiful patterns were observed, which were tested for bases other than decimal. This investigation inspired me to become passionate about number recreations and the subsequent publication of my first paper, titled *Miracles of Last Digit,* in the November 1978 issue of Junior Science Digest, India. Since then, my contributions have been published in national and international journals and books.

My book, *Creative Puzzles to Ignite Your Mind*, was published by Springer Singapore in March 2023. It covers various puzzles based on applications of numbers such as square numbers, triangular numbers, Fibonacci numbers, autobiographical numbers, parasite numbers, polydivisible numbers, and so on.

This book ***Exploring the Beauty of Fascinating Numbers*** contains 23 chapters covering a large variety of topics, such as digital root wonders, the elegance of squares, triangular numbers, Smith numbers, amicable numbers, perfect, multiple perfect and sociable numbers, happy numbers, Fibonacci numbers, Lucas numbers, and golden ratio, Kaprekar numbers, Karekar Constant, self-numbers, amazing number 108, repunit numbers, equal product of reversible numbers (EPRNs), unlucky 13, rare numbers, beauty of number 153, fascinating factorials, the number of beasts, Ulam numbers, mystery of π, cab and vampire numbers, digital invariants and narcissistic numbers, special numbers like autobiographical numbers, Harshad numbers, parasite numbers, polydivisible numbers, and Ramanujan numbers, number curiosities such as lucky mistakes, Pascal's triangle, and Pythagorean triplets.

The book is a collection of several thoughts, articles, results, and patterns collected and discovered over a large period of 45 years of my passion for number recreations. When writing this book, my motivation was always to communicate the excitement and fascination of numbers to the children in schools and colleges.

The book demonstrates to the general reader that mathematics can be fun rather than dry, dull, and difficult through the journey of *Exploring the Beauty of Fascinating Numbers*.

The book can immensely benefit teachers trying to teach math, especially to students who don't like math, by supplementing their regular curriculum with the module containing material from the book, which provides an opportunity for fun and joy while encouraging students and researchers to test their mathematical, computational, and logical skills.

The late Martin Gardner ran a popular monthly column titled *Mathematical Games* in the magazine *Scientific American* for about 25 years. This monthly column about recreational mathematics was introduced to me in college by our beloved late professor, M. M. Dandekar, who has always inspired me.

A recreational mathematical idea thought to be fascinating but of no practical value can turn out to have huge practical significance. For example, in the seventeenth century, the great German mathematician G. W. Leibniz discovered and studied the idea of a binary number system, which remained a curiosity until it became a choice for the operation of electronic devices and computers in the twentieth century. Therefore, recreational mathematics can play a significant role in the advancement of human knowledge.

The theory behind the subject matter has been kept to a minimum to retain the recreational nature of the book. A section called *Further Investigations* is added as the last section of many chapters, which gives open problems and ideas to further investigate the topics. This will entertain readers and create interest in further exploring the subject.

Exploring the beauty of fascinating numbers is a delightful coverage of numerical curiosities, coincidences, and wonders, revealing many new eye-opening properties of numbers. I am sure that this book will delight readers of all levels.

Jaipur, India Shyam Sunder Gupta

Acknowledgements

The late Martin Gardner ran a popular monthly column titled *Mathematical Games* in the magazine Scientific American for about 25 years. This monthly column, filled with recreational mathematics, inspired me during my college days to become passionate about number recreation. Later, I came across wonderful books, periodicals, and other literature written by great personalities in the field of recreational mathematics, such as W. W. Rouse Ball, Maurice Kraitchik, L. E. Dickson, A. H. Beiler, Henry Dudeney, Joseph Steven Madachy, Yakov Perelman, Martin Gardner, D. R. Kaprekar, Paul Erdos, Samuel Yates, and others. I am grateful and thank them all.

I thank Tarun Kumar for the cover page design and for making all the illustrations for the book. I thank Amit Gupta for his comments after going through the manuscript.

I thank my wife, Sushil Gupta, for her encouragement and support, without which it would not have been possible to start writing and complete this book.

Since the book is a collection of several thoughts, articles, results, and patterns collected and discovered over a long period of 45 years of my passion for number recreations, it is practically not possible to make all the references available. However, I am thankful to all who were associated, directly or indirectly, including the following:

N. J. A. Sloane, Al Zimmermann, Carlos Rivera, Patrick De Geest, Paul Zimmermann, Tony Foster, Tony Sand, G. L. Honaker, Jens Kruse Andersen, Max Alekseyev, Brian Trial, John McMahon, Mauro Fiorentini, Julian Beauchamp, Okoh Ufuoma Cyrus, Alessandro Casini, Maximilian Hasler, Emmanuel Vantieghem, James Furia, Steve Homewood, Fred Schneider, and Richard Sewill.

Every effort has been made to make the book error-free; however, some errors and mistakes may always remain. Therefore, I shall be grateful for any suggestions and comments, not only for rectifying the errors and mistakes but also for improving the book.

Jaipur, India Shyam Sunder Gupta

Introduction

An equation means nothing to me unless it expresses a thought of God.
—Srinivasa Ramanujan

God created the integers; all else is the work of man.
—Leopold Kronecker

The author feels great pleasure in presenting the book, *Exploring the Beauty of Fascinating Numbers*, which is a great treasure for everybody who enjoys the beauty of the fascinating world of recreational mathematics. Apart from amateurs and math lovers, the book is considered of immense value to encourage students and researchers to test their mathematical and computational skills.

The book focuses on recreational aspects of numbers to create interest and motivate readers to learn to be creative in improving their problem-solving techniques. The book aims to show the beauty and power that are so well hidden in our numbers, with the hope that the reader will be motivated to undertake further investigations.

Srinivasa Ramanujan was one of India's greatest mathematical geniuses, and he believed that the gods gave him mathematical ideas out of his dreams. Pythagoras attributed mystical qualities to some of the numbers. Even the religious properties of numbers were extensively studied. So, four chapters are exclusively devoted to such numbers, namely, the amazing number 108, the unlucky 13, the beauty of 153, and the number of the beast, with lots of new curiosities and miraculous coincidences.

The first chapter is devoted to *digital roots*, the concept of which is over a 1000 years old and is simply the ancient process of 'casting out 9s'. Apart from digital root properties and applications, the digital roots of polygonal numbers, Fermat numbers, Mersenne primes, perfect numbers, Fibonacci and Lucas numbers, primes and twin primes, amicable numbers, Kaprekar numbers, Smith numbers, and fractions are discussed.

In the second chapter, in addition to the number of new curiosities, beautiful patterns, curious numbers, and equations, Bhaskara pairs, exclusionary, biperiod, and tridigital squares are covered. Shortcuts in computing and methods of fast detection of perfect squares are explained. To further stimulate interest in students, two puzzles based on squares are also dealt with. Automorphic numbers are also discussed in detail.

The third chapter deals with *triangular numbers,* first studied by ancient Greek mathematicians. In this chapter, several new curiosities and observations, harmonic triples of triangular numbers, special triangular numbers such as palindromic, reversible, Smith, Harshad, happy, Kaprekar, abundant, deficient, and exclusionary triangular numbers are covered. The latest results about magic squares containing triangular numbers and the existence of infinite families of triangular numbers containing only odd digits are discussed.

Chapter 4 deals with the construction and distribution of *Smith numbers***,** highly decomposable Smith numbers, consecutive Smith numbers, and special Smith numbers such as Repdigit, Fibonacci, and sphenic. New applications of digital roots for speeding up Smith number computations have been discussed. Hoax numbers and Ruth-Aaron numbers are also covered.

Chapters 5 and 6 are devoted to *amicable and perfect numbers*, which were extensively studied by the Greeks, especially Euclid, who devised a method for obtaining even perfect numbers. Divisibility and the digital roots of known amicable numbers are discussed in detail in Chap. 5. Based on the updated list of the known perfect numbers, curious properties, digital roots, and endings of perfect numbers, along with multiple perfect and sociable numbers, are discussed in Chap. 6.

Though happy numbers are infinite, observing the proportion of *happy numbers* is interesting. Special happy numbers such as happy Pythagorean triplets, repdigit happy numbers, palindromic happy numbers, happy amicable pairs, and happy triangular numbers are discussed in Chap. 7. In addition, consecutive happy numbers, happy primes, and happy cubes are also covered.

The Fibonacci numbers covered in Chap. 8 were first described in India; however, they are named after the Italian mathematician Leonardo of Pisa, also known as Fibonacci, who introduced the sequence in his 1202 book Liber Abaci. Fibonacci numbers are related to the golden ratio, which has a unique characteristic in that it differs from its reciprocal by 1. This characteristic leads to several fascinating properties, which are discussed. In addition, Fibonacci identities and Fibonacci factorials are covered.

Chapter 9 includes some of the marvellous numbers like *Kaprekar numbers*, self-numbers, and Kaprekar constant 6174, discovered by Indian recreational mathematician D. R. Kaprekar. Readers may enjoy and explore further some of the peculiarities of intra-differences and self-gaps discussed in this chapter.

The number 108 discussed in Chap. 10 is considered sacred and auspicious in Hinduism, Sikhism, Buddhism, and Jainism. In addition to many other amazements of 108, geometrical properties and astronomical coincidences are worth noting. Mathematically, 108 is the smallest number whose divisors (i.e., 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, and 108) contain every digit at least once and is also the smallest number that can be partitioned into six distinct primes such that the sum of any five is prime.

Repunits comprising only the digit 1, such as 11 or 111, are closely related to repetends and period lengths of primes. Some beautiful number patterns and curiosities involving repunits, like cubes of certain n-digit numbers ending in R_n, like $88471^3 = 692472942511111$, are covered in Chap. 11.

The numbers that can be expressed as the product of a number and its reversal in two different ways, like $144648 = 861 \times 168 = 492 \times 294$, are termed EPRNs (Equal Product of Reversible Numbers). Based on new properties, methods of computing EPRNs and the distribution of EPRNs are discussed in Chap. 12.

The abnormal fear of the number 13, often known as triskaidekaphobia, is one of the most prominent myths in science and was fuelled by the failed mission of Apollo 13, which launched at 13.13 h and exploded on April 13, 1970. Some interesting coincidences, like Microsoft skipping Office version 13, are discussed in Chap. 13. The number of observations, like 13 major joints in our body and 13 essential vitamins for our body, along with many interesting mathematical curiosities like 'the sum of all prime numbers up to 13 is equal to the 13th prime number', and '13 composite numbers between 13 and its reversal' are discussed.

The numbers that give a perfect square on adding as well as subtracting their reverse are rare, like 65, 621770, etc., and hence have been termed as *rare numbers* by the author. Chapter 14 is extensively devoted to computing and investigating rare numbers up to 10^{24}. Some properties are listed, and conjectures are made to enable the readers to further explore the subject.

Chapter 15 deals with the beauty of the *number 153*, references to which can be found in the New Testament, where, in the net, Simon Peter drew from the Sea of Tiberias 153 fish. The number 153 has puzzled and intrigued Christian thinkers for centuries. Interesting coincidences include the mention of 153 in 'The Ascent of Rum Doodle', a novel by W. E. Bowman. Mathematically, when the cubes of the digits of any number, i.e., a multiple of 3, are added, and then this process is repeated, the final result is always 153, where the process ends because $153 = 1^3 + 5^3 + 3^3$.

Factorion, i.e., positive integers that are equal to the sum of the factorials of their digits like 1, 2, 145, and 40585, along with amicable factorion and magic factorion, are discussed in Chap. 16. Using factorial digits, the possibility of drawing amazing shapes like triangles, rhombuses, hexagons, octagons, etc. is discussed. Various kinds of factorials like half, double, multi, hyper, super factorials, and subfactorials are covered, along with some curiosities like 372973, the only number that is one less than the sum of the factorials of their digits.

The biblical *number of the beast*, 666, is covered in Chap. 17. US Route 666, known as 'The Highway of the Beast', was renamed Route 491 in 2003 after controversy in connection with the beast number. Intel introduced the 666 MHz Pentium III CPU in 1999, but it was marketed as the Pentium III 667 instead of the Pentium III 666. Mathematically, 666 contains delightful curiosities like 'the sum of cubes of ascending and descending numbers is 666, i.e., $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3 = 666$'.

The distribution of *Ulam numbers*, unulam numbers, and superulam numbers, along with observations on the distribution of gaps between Ulam numbers, is dealt with in Chap. 18. Ulam numbers in arithmetic progressions are also explored.

π has many real-world uses; however, the digits of π beyond the first few decimal places are of value mainly in testing computer systems. Apart from digit curiosities, the new concepts of π factorial and piorial primes are discussed in Chap. 19. Using π digits, the drawing of various shapes like triangles, rhombuses, hexagons, octagons, etc. is also demonstrated.

Chapter 20 covers the *cab numbers* mentioned by Henry E. Dudeney in his book, *Amusements in Mathematics*, like $8745231 \times 96 = 839542176$, and the vampire numbers, like $12417993 = 1317 \times 9429 = 1347 \times 9219$.

Apart from perfect and recurring digital invariants, *narcissistic numbers* like $548834 = 5^6 + 4^6 + 8^6 + 8^6 + 3^6 + 4^6$, Münchhausen numbers, i.e., $3435 = 3^3 + 4^4 + 3^3 + 5^5$, *Friedman numbers*, which can be written in some non-trivial way using their digits, like $736 = 7 + 3^6$, and printer error numbers like $3^4 \times 7^2 \times 875 = 3472875$ are discussed in detail in Chap. 21.

Chapter 22 discusses some special numbers, like autobiographical numbers, Harshad numbers, parasite numbers, polydivisible numbers, and Ramanujan numbers.

The last chapter, 23, contains details about some number curiosities like lucky mistakes, including anomalous cancellations, Pascal's triangle, and Pythagorean triples.

Hundreds of books and journals have been trawled through libraries in search of curious and interesting numbers. So, it would be impossible to credit all the sources referred to. However, references for further exploring the subject have been given at the end of the chapters.

So let us not waste time and start enjoying the treasure full of wonders of numbers, as many unexplored areas are waiting for you to unlock the doors of your imagination.

Contents

About the Author

Shyam Sunder Gupta is the former Principal Chief Engineer, Indian Railways, Government of India. He is an Indian Railway Service of Engineers (IRSE) officer of batch 1981. He has more than 35 years of experience in various managerial, administrative, and technical positions, such as Principal Chief Engineer, Executive Director, Divisional Railway Manager, and Director/RDSO on Indian Railways. He is an Indian recreational mathematician who is actively involved in popularising mathematics at the national and international levels. His website on number recreations, https://www.shyamsundergupta.com, is very popular with visitors from more than 150 countries. His major discoveries are Equal Product of Reversible Numbers (EPRNs), rare numbers, unique numbers, palindromic pseudoprimes, fifth-order prime polynomial, 17350-digit memorable prime, etc.

His interest in number recreations dates back to 1978, when his first paper, *Miracles of Last Digit* was published. Since then, his contributions have been published in *Science Reporter*, *Science Today*, *Math Education*, *At Right Angles*, *The American Mathematical Monthly*, *The Mathematical Gazette*, *Mathematical*