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V. Parameswaran Nair

Geometric Quantization and Applications to Fields and Fluids

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V. Parameswaran Nair
Department of Physics
City College of New York
New York, NY, USA

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To my sons Nityan and Haris

Preface

The fundamental dynamical variables of any physical system take values in what is referred to as the phase space. The geometry and topology of this space play a guiding role in the dynamics of the physical system. While this was well-appreciated and well-understood in classical dynamics, early formulations of quantum mechanics did not have an easy flexibility to accommodate features of geometry and topology. Over the years, this problem was addressed with increasing levels of sophistication. Geometric quantization gives an elegant framework for accommodating geometrical and topological features of the phase space. By now there are many books and mathematically sophisticated reviews of this topic. Most of these focus on the formalism and some of the subtleties involved. While this is of great value, I think that highlighting a variety of diverse applications, especially those which are physically motivated and interesting, can be a very useful complementary approach. This book is an attempt in this direction. In keeping with this motivation, most of the material here is presented from a physicist's point of view.

Some of the topics were covered in lectures at different summer schools in theoretical physics. More recently, a skeletal draft of most of the topics was prepared for lectures at the Second Autumn School on High Energy Physics & Quantum Field Theory in Yerevan, Armenia, in October 2014. This book is an augmented and updated version of the lecture notes.

I thank all the organizers of the summer school in Armenia for the invitation to speak there. I have discussed some of these topics with my colleagues and express my thanks for their insights and comments. I also thank my wife Dimitra for collaborations, discussions and for a span of time free from mundane worries while working on this. This work was supported in part by the U.S. National Science Foundation Grant PHY-2112729.

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Chapter 1
Introduction

A physical theory, as a logical explanation of physical phenomena, is to be constructed taking account of general principles and incorporating data and information from experiments. Any effects we attribute to the quantum nature of phenomena should be included from the outset. A classical description may then be obtained, in a suitable regime of parameters, as a useful and simpler working approximation. The flow of logic should thus be:

$$\left.\begin{array}{c}\text{General principles} + \\ \text{experimental input}\end{array}\right\} \Longrightarrow \text{Quantum theory} \Longrightarrow \text{Classical approximation.}$$

But the build-up of a theory along these lines is almost never done in practice. Primarily, this is because, at the human level of direct experience, most phenomena are well described by classical dynamics, and hence our intuition about physical systems is mostly classical. So we tend to start there and try to "quantize" the classical theory. This is a process with many ambiguities, but over the course of many years, we have learned to understand the structure of this procedure of quantization. In this book, we will attempt to describe some aspects of geometric quantization and consider a few examples or applications.

We will begin with some general observations on why we need such a procedure as geometric quantization. This is best illustrated by an example. Consider the elementary quantum mechanics of a single particle in three spatial dimensions. The operators of position $\hat{x}_i$, $i = 1, 2, 3$ and momentum $\hat{p}_i$ obey the Heisenberg algebra

$$\begin{aligned} \hat{x}^i\, \hat{x}^j - \hat{x}^j\, \hat{x}^i &= 0 \\ \hat{x}^i\, \hat{p}_j - \hat{p}_j\, \hat{x}^i &= \mathrm{i}\, \delta^i{}_j \\ \hat{p}_i\, \hat{p}_j - \hat{p}_j\, \hat{p}_i &= 0 \end{aligned} \tag{1.1}$$

As is well known these have the standard Schrödinger representation on the x-diagonal wave functions $\psi(x)$,

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$$\hat{x}^i \, \psi = x^i \, \psi, \qquad \hat{p}_i \, \psi = -\mathrm{i} \frac{\partial}{\partial x^i} \, \psi \tag{1.2}$$

Notice that the commutation rules (1.1) and the specific representation (1.2) are expressed in terms of Cartesian coordinates. While we know that we should have the freedom of choosing any set of coordinates for the classical description, constructing the commutation rules and the operators in coordinate systems other than the Cartesian one is not straightforward. What is usually done in textbook solutions, say, of the Hydrogen atom in spherical polar coordinates is to set up the quantum theory and the Schrödinger equation first in the Cartesian basis (with $\hat{p}^2 = -\nabla^2$) and then make a change of coordinates. While this is an adequate working procedure for many situations, it is clearly unsatisfactory; one would like a procedure that works directly without the crutch of the Cartesian system. Also, in situations where we may have a curved space or a curved phase space, a quantization procedure which takes account of the geometry of the manifold is not just a desirable choice, but is actually needed.

There are also situations, such as in field theory, where the dynamical variables are components of fields and have no obvious Cartesian-like structure. In assigning commutation rules to the components of fields, a more general procedure is then called for. Geometric quantization is a partial answer to these concerns. It highlights the geometry and topology of the phase space and gives insights into many physical situations. But as it stands, it is still not a complete answer to the issues mentioned above. We will comment on some of these inadequacies later in the text.

There are many other approaches to quantization as well. Quantum theory may be viewed as a unitary irreducible representation (UIR) of the algebra of observables, the latter being selected by physical criteria [1]. The algebra itself must satisfy certain conditions so as to have the correct physical requirements. Generally it ends up as a C^*-algebra with further additional conditions equivalent to symmetries or other desirable properties (such as Lorentz invariance, relativistic causality) and so on. In relativistic field theory, this would lead to a von Neumann algebra. Here we are not going to pursue such an algebraic approach to quantization. Instead, we will consider the essential geometry (which has to do with the symplectic structure) of the classical theory and work out how a quantum theory can be constructed. This will be done in the language of Hamiltonians and Hilbert space. The key principle of quantization, as always, is that canonical transformations of the classical theory should be represented as unitary transformations on the Hilbert space of states in the quantum theory.

There is yet another approach to the quantum theory, the functional integral approach, which is formulated directly in terms of the action and can be made manifestly covariant if the theory of interest has relativistic invariance [2, 3]. Here we will not discuss this formulation either, but some points of overlap will be pointed out as the occasion arises.

Chapter 2
Symplectic Form and Poisson Brackets

We start with the formulation of theories in the symplectic language [4, 5]. Later, we will briefly discuss how this is connected to the action which may be used to specify the physical theory.

2.1 Symplectic Structure

In the analytical formulation of classical physics, the key concept is the phase space, which is a smooth even dimensional orientable manifold M (say, of dimension $2n$) endowed with a symplectic structure Ω. By this we mean that there is a differential 2-form Ω defined on M which is closed and nondegenerate. Closure means that $\mathrm{d}\Omega = 0$, where d denotes exterior differentiation. The qualification "nondegenerate" refers to the fact that for any vector field ξ on M, if $i_\xi \Omega = 0$, then ξ must be zero. Here i_ξ indicates interior contraction with the vector field ξ. We will use q^μ to denote local coordinates on M. In terms of these, we can write

$$\Omega = \frac{1}{2}\, \Omega_{\mu\nu}\, \mathrm{d}q^\mu \wedge \mathrm{d}q^\nu \tag{2.1}$$

The closure condition $\mathrm{d}\Omega = 0$ can be written out as

$$\begin{aligned} d\Omega &\equiv \frac{1}{2} \frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^\mu \wedge \mathrm{d}q^\nu \\ &= \frac{1}{3!} \left[\frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} + \frac{\partial \Omega_{\alpha\mu}}{\partial q^\nu} + \frac{\partial \Omega_{\nu\alpha}}{\partial q^\mu} \right] \mathrm{d}q^\alpha \wedge \mathrm{d}q^\mu \wedge \mathrm{d}q^\nu \\ &= 0 \end{aligned} \tag{2.2}$$

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