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Emmanuel Trélat

Control in Finite and Infinite Dimension

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Foreword

This short book is the result of various master and summer school courses I have taught. The objective is to introduce the readers to mathematical control theory, both in finite and infinite dimension. In the finite-dimensional context, we consider controlled ordinary differential equations (ODEs); in this context, existence and uniqueness issues are easily resolved thanks to the Picard-Lindelöf (Cauchy-Lipschitz) theorem. In infinite dimension, in view of dealing with controlled partial differential equations (PDEs), the concept of well-posed system is much more difficult and requires to develop a bunch of functional analysis tools, in particular semigroup theory—and this, just for the setting in which the control system is written and makes sense. This is why I have split the book into two parts, the first being devoted to finite-dimensional control systems, and the second to infinite-dimensional ones.

In spite of this splitting, it may be nice to learn basics of control theory for finite-dimensional linear autonomous control systems (e.g., the Kalman condition) and then to see in the second part how some results are extended to infinite dimension, where matrices are replaced by operators, and exponentials of matrices are replaced by semigroups. For instance, the reader will see how the Gramian controllability condition is expressed in infinite dimension, and leads to the celebrated Hilbert Uniqueness Method (HUM).

Except the very last section, in the second part I have only considered linear autonomous control systems (the theory is already quite complicated), providing anyway several references to other textbooks for the several techniques existing to treat some particular classes of nonlinear PDEs. In contrast, in the first part on finite-dimensional control theory, there are much less difficulties to treat general nonlinear control systems, and I give here some general results on controllability, optimal control, and stabilization.

Of course, whether in finite or infinite dimension, there exist much deeper results and methods in the literature, established however for specific classes of control systems. Here, my objective is to provide the reader with an introduction to control theory and to the main tools allowing to treat general control systems. I hope this will serve as motivation to go deeper into the theory or numerical aspects that are not covered here.

Paris, France
March 2024

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Part I
Control in Finite Dimension

Chapter 1
Controllability

Let n and m be two positive integers. In this chapter we consider a control system in $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 with respect to (x, u) and locally integrable with respect to t, and the controls are measurable essentially bounded functions of time taking their values in some measurable subset Ω of $\mathbb{R}^m$ (set of control constraints).

First of all, given an arbitrary initial point $x_0 \in \mathbb{R}^n$, and an arbitrary control u, we claim that there exists a unique solution $x(\cdot)$ of (1.1) such that $x(0) = x_0$, maximally defined on some open interval of $\mathbb{R}$ containing 0. We use here a generalization of the usual Picard-Lindelöf theorem (sometimes called Carathéodory theorem), where the dynamics here can be discontinuous (because of the control). For a general version of this existence and uniqueness theorem, we refer to [6, Theorem 5.3] and [12, Appendix C3]. We stress that the differential equation (1.1) holds for almost every t in the maximal interval. Given a time $T > 0$ and an initial point x_0, we say that a control $u \in L^\infty([0, T], \mathbb{R}^m)$ is admissible if the corresponding trajectory $x(\cdot)$, such that $x(0) = x_0$, is well defined on $[0, T]$.

We say that the control system is *linear* if $f(t, x, u) = A(t)x + B(t)u + r(t)$, with $A(t)$ a $n \times n$ matrix, $B(t)$ a $n \times m$ matrix (with real coefficients), $r(t) \in \mathbb{R}^n$, and in that case we will assume that $t \mapsto A(t)$, $t \mapsto B(t)$ and $t \mapsto r(t)$ are of class L^∞ on every compact interval (actually, L^1 would be enough). The linear control system is said to be *autonomous* if $A(t) \equiv A$ and $B(t) \equiv B$, otherwise it is said to be *instationary* or *time-varying*. Note that, for linear control systems, there is no blow-up in finite time (i.e., admissibility holds true on any interval).

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Definition 1.1 Let $x_0 \in \mathbb{R}^n$ and let $T > 0$ arbitrary. A control $u \in L^\infty([0, T], \Omega)$ is said to be *admissible* on $[0, T]$ if the trajectory $x_u(\cdot)$, solution of (1.1), corresponding to the control u, and such that $x_u(0) = x_0$, is well defined on $[0, T]$. The *end-point mapping* $E_{x_0,T}$ is then defined by $E_{x_0,T}(u) = x_u(T)$.

The set of admissible controls on $[0, T]$ is denoted by $\mathcal{U}_{x_0,T,\Omega}$. It is the domain of definition of $E_{x_0,T}$ (indeed one has to be careful with blow-up phenomena), when considering controls taking their values in Ω.

Definition 1.2 The control system (1.1) is said to be (globally) *controllable from* x_0 *in time* T if $E_{x_0,T}(\mathcal{U}_{x_0,T,\Omega}) = \mathbb{R}^n$, i.e., if $E_{x_0,T}$ is surjective.

Accordingly, defining the *accessible set* from x_0 in time T by $\mathrm{Acc}_\Omega(x_0, T) = E_{x_0,T}(\mathcal{U}_{x_0,T,\Omega})$, the control system (1.1) is (globally) controllable from x_0 in time T if $\mathrm{Acc}_\Omega(x_0, T) = \mathbb{R}^n$.

Since such a global surjectivity property is certainly a very strong property which may not hold in general, it is relevant to define *local controllability*.

Definition 1.3 Let $x_1 = E_{x_0,T}(\bar{u})$ for some $\bar{u} \in \mathcal{U}_{x_0,T,\Omega}$. The control system (1.1) is said to be *locally controllable from* x_0 *in time* T around x_1 if x_1 belongs to the interior of $\mathrm{Acc}_\Omega(x_0, T)$, i.e., if $E_{x_0,T}$ is locally surjective around x_1.

Other variants of controllability can be defined. A clear picture will come from the geometric representation of the accessible set.

In this chapter we will provide several tools in order to analyze the controllability properties of a control system, first for linear (autonomous, and then instationary) systems, and then for nonlinear systems.

1.1 Controllability of Linear Systems

Throughout this section, we consider the linear control system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)$, with $u \in L^\infty([0, +\infty), \Omega)$. Since there is no finite-time blow-up for linear systems, we have $\mathcal{U}_{x_0,T,\Omega} = L^\infty([0, T], \Omega)$ for every $T > 0$.

1.1.1 Controllability of Autonomous Linear Systems

In this section, we assume that $A(t) \equiv A$ and $B(t) \equiv B$, where A is a $n \times n$ matrix and B is a $n \times m$ matrix.

1.1.1.1 Without Control Constraints: Kalman Condition

In this section, we assume that $\Omega = \mathbb{R}^m$ (no control constraint). The celebrated Kalman theorem provides a necessary and sufficient condition for autonomous linear control systems without control constraint.

Theorem 1.1 *We assume that $\Omega = \mathbb{R}^m$ (no control constraint). The control system $\dot{x}(t) = Ax(t) + Bu(t) + r(t)$ is controllable (from any initial point, in arbitrary time $T > 0$) if and only if the Kalman matrix*

$$K(A, B) = (B, AB, \ldots, A^{n-1}B)$$

(which is of size $n \times nm$) is of maximal rank n.

Proof of Theorem 1.1 Given any $x_0 \in \mathbb{R}^n$, $T > 0$ and $u \in L^\infty([0, T], \mathbb{R}^m)$, the Duhamel formula gives

$$E_{x_0,T}(u) = x_u(T) = e^{TA}x_0 + \int_0^T e^{(T-t)A}r(t)\,dt + L_T u \tag{1.2}$$

where $L_T : L^\infty([0, T], \mathbb{R}^m) \to \mathbb{R}^n$ is the linear continuous operator defined by $L_T u = \int_0^T e^{(T-t)A}Bu(t)\,dt$. Clearly, the system is controllable in time T if and only if L_T is surjective. Then to prove the theorem it suffices to prove the following lemma.

Lemma 1.1 *The Kalman matrix $K(A, B)$ is of rank n if and only if L_T is surjective.*

Proof of Lemma 1.1 We argue by contraposition. If L_T is not surjective, then there exists $\psi \in \mathbb{R}^n \setminus \{0\}$ which is orthogonal to the range of L_T, that is,

$$\psi^\top \int_0^T e^{(T-t)A}Bu(t)\,dt = 0 \qquad \forall u \in L^\infty([0, T], \mathbb{R}^m).$$

This implies that $\psi^\top e^{(T-t)A}B = 0$, for every $t \in [0, T]$. Taking $t = T$ yields $\psi^\top B = 0$. Then, derivating first with respect to t, and taking $t = T$ then yields $\psi^\top AB = 0$. By immediate iteration we get that $\psi^\top A^k B = 0$, for every $k \in \mathbb{N}$. In particular $\psi^\top K(A, B) = 0$ and thus the rank of $K(A, B)$ is less than n.

Conversely, if the rank of $K(A, B)$ is less than n, then there exists $\psi \in \mathbb{R}^n \setminus \{0\}$ such that $\psi^\top K(A, B) = 0$, and therefore $\psi^\top A^k B = 0$, for every $k \in \{0, 1, \ldots, n-1\}$. From the Hamilton-Cayley theorem, there exist real numbers $a_0, a_1, \ldots, a_{n-1}$ such that $A^n = \sum_{k=0}^{n-1} a_k A^k$. Therefore we get easily that $\psi^\top A^n B = 0$. Then, using the fact that $A^{n+1} = \sum_{k=1}^{n} a_k A^k$, we get as well that $\psi^\top A^{n+1}B = 0$. By immediate recurrence, we infer that $\psi^\top A^k B = 0$, for every $k \in \mathbb{N}$, and therefore, using the series expansion of the exponential, we get that $\psi^\top e^{(T-t)A}B = 0$, for