

SpringerBriefs on PDEs and Data Science

Emmanuel Trélat



# Control in Finite and Infinite Dimension

# **SpringerBriefs on PDEs and Data Science**

## **Editor-in-Chief**

Enrique Zuazua, Department of Mathematics, University of Erlangen-Nuremberg, Erlangen, Bayern, Germany

## **Series Editors**

Irene Fonseca, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

Franca Hoffmann, Hausdorff Center for Mathematics, University of Bonn, Bonn, Germany

Shi Jin, Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai, Shanghai, China

Juan J. Manfredi, Department of Mathematics, University Pittsburgh, Pittsburgh, PA, USA

Emmanuel Trélat, CNRS, Laboratoire Jacques-Louis Lions, Sorbonne University, PARIS CEDEX 05, Paris, France

Xu Zhang, School of Mathematics, Sichuan University, Chengdu, Sichuan, China

SpringerBriefs on PDEs and Data Science targets contributions that will impact the understanding of partial differential equations (PDEs), and the emerging research of the mathematical treatment of data science.

The series will accept high-quality original research and survey manuscripts covering a broad range of topics including analytical methods for PDEs, numerical and algorithmic developments, control, optimization, calculus of variations, optimal design, data driven modelling, and machine learning. Submissions addressing relevant contemporary applications such as industrial processes, signal and image processing, mathematical biology, materials science, and computer vision will also be considered.

The series is the continuation of a former editorial cooperation with BCAM, which resulted in the publication of 28 titles as listed here: <https://www.springer.com/gp/mathematics/bcam-springerbriefs>

Emmanuel Trélat

# Control in Finite and Infinite Dimension

Emmanuel Trélat  
Laboratoire Jacques-Louis Lions  
Sorbonne Université  
Paris, France

ISSN 2731-7595                      ISSN 2731-7609 (electronic)  
SpringerBriefs on PDEs and Data Science  
ISBN 978-981-97-5947-7              ISBN 978-981-97-5948-4 (eBook)  
<https://doi.org/10.1007/978-981-97-5948-4>

Mathematics Subject Classification: 93C15, 93C20, 93C25

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Singapore Pte Ltd. 2024

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd. The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

If disposing of this product, please recycle the paper.

# Foreword

This short book is the result of various master and summer school courses I have taught. The objective is to introduce the readers to mathematical control theory, both in finite and infinite dimension. In the finite-dimensional context, we consider controlled ordinary differential equations (ODEs); in this context, existence and uniqueness issues are easily resolved thanks to the Picard-Lindelöf (Cauchy-Lipschitz) theorem. In infinite dimension, in view of dealing with controlled partial differential equations (PDEs), the concept of well-posed system is much more difficult and requires to develop a bunch of functional analysis tools, in particular semigroup theory—and this, just for the setting in which the control system is written and makes sense. This is why I have split the book into two parts, the first being devoted to finite-dimensional control systems, and the second to infinite-dimensional ones.

In spite of this splitting, it may be nice to learn basics of control theory for finite-dimensional linear autonomous control systems (e.g., the Kalman condition) and then to see in the second part how some results are extended to infinite dimension, where matrices are replaced by operators, and exponentials of matrices are replaced by semigroups. For instance, the reader will see how the Gramian controllability condition is expressed in infinite dimension, and leads to the celebrated Hilbert Uniqueness Method (HUM).

Except the very last section, in the second part I have only considered linear autonomous control systems (the theory is already quite complicated), providing anyway several references to other textbooks for the several techniques existing to treat some particular classes of nonlinear PDEs. In contrast, in the first part on finite-dimensional control theory, there are much less difficulties to treat general nonlinear control systems, and I give here some general results on controllability, optimal control, and stabilization.

Of course, whether in finite or infinite dimension, there exist much deeper results and methods in the literature, established however for specific classes of control

systems. Here, my objective is to provide the reader with an introduction to control theory and to the main tools allowing to treat general control systems. I hope this will serve as motivation to go deeper into the theory or numerical aspects that are not covered here.

Paris, France  
March 2024

Emmanuel Trélat

# Contents

## Part I Control in Finite Dimension

<b>1</b>	<b>Controllability</b>	3
1.1	Controllability of Linear Systems	4
1.1.1	Controllability of Autonomous Linear Systems	4
1.1.1.1	Without Control Constraints: Kalman Condition	5
1.1.1.2	With Control Constraints	8
1.1.1.3	Similar Systems	9
1.1.2	Controllability of Time-Varying Linear Systems	11
1.1.2.1	Case Without Control Constraints	11
1.1.2.2	Case with Control Constraints	14
1.1.3	Geometry of Accessible Sets	15
1.2	Controllability of Nonlinear Systems	17
1.2.1	Local Controllability Results	17
1.2.2	Geometry of the Accessible Set	23
1.2.3	Global Controllability Results	25
	References	27
<b>2</b>	<b>Optimal Control</b>	29
2.1	Existence of an Optimal Control	30
2.2	Pontryagin Maximum Principle (PMP)	33
2.2.1	General Statement	33
2.2.2	Proof in a Simplified Context	36
2.2.3	Generalizations and Additional Comments	39
2.3	Particular Cases and Examples	41
2.3.1	Minimal Time Problem for Linear Control Systems	41
2.3.2	Linear Quadratic Theory	44
2.3.2.1	The Basic LQ Problem	44
2.3.2.2	Tracking Problem	46
2.3.3	Examples of Nonlinear Optimal Control Problems	48
	References	58



<b>3</b>	<b>Stabilization</b>	61
3.1	Stabilization of Autonomous Linear Systems	61
3.1.1	Reminders on Stability Notions	61
3.1.2	Pole-Shifting Theorem	64
3.2	Stabilization of Instationary Linear Systems	66
3.3	Stabilization of Nonlinear Systems	67
3.3.1	Local Stabilization by Linearization	67
3.3.2	Global Stabilization by Lyapunov Theory	68
	References	74
 <b>Part II Control in Infinite Dimension</b>		
<b>4</b>	<b>Semigroup Theory</b>	77
4.1	Homogeneous Cauchy Problems	79
4.1.1	Semigroups of Linear Operators	79
4.1.2	The Cauchy Problem	85
4.1.2.1	Classical Solutions	85
4.1.2.2	Weak Solutions	87
4.1.3	Scale of Banach Spaces	90
4.2	Nonhomogeneous Cauchy Problems	94
	References	96
<b>5</b>	<b>Linear Control Systems in Banach Spaces</b>	97
5.1	Admissible Control Operators	98
5.1.1	Definition	98
5.1.2	Dual Characterization of the Admissibility	100
5.1.3	Degree of Unboundedness of the Control Operator	102
5.1.4	Examples	104
5.1.4.1	Dirichlet Heat Equation with Internal Control	104
5.1.4.2	Heat Equation with Dirichlet Boundary Control	105
5.1.4.3	Heat Equation with Neumann Boundary Control	107
5.1.4.4	Second-Order Equations	107
5.2	Controllability	109
5.2.1	Definitions	109
5.2.2	Duality Controllability / Observability	111
5.2.3	Hilbert Uniqueness Method (HUM)	113
5.2.4	Example: The Wave Equation	114
5.2.4.1	1D Wave Equation with Dirichlet Boundary Control	115
5.2.4.2	1D Dirichlet Wave Equation with Internal Control	117
5.2.4.3	Multi-D Dirichlet Wave Equation with Internal Control	119
5.2.5	Example: The Heat Equation	120
5.2.5.1	Dirichlet Heat Equation with Internal Control	120
5.2.5.2	Heat Equation with Dirichlet Boundary Control	121

5.2.6	Pontryagin Maximum Principle .....	122
5.3	Further Results .....	124
5.3.1	Kalman Condition in Infinite Dimension .....	124
5.3.2	Necessary Conditions for Exact Controllability .....	125
5.3.3	Moment Method .....	126
5.3.4	Equivalence Between Observability and Exponential Stability .....	129
5.3.5	1D Semilinear Heat Equation .....	133
	References .....	137

**Part I**  
**Control in Finite Dimension**

# Chapter 1

## Controllability



Let  $n$  and  $m$  be two positive integers. In this chapter we consider a control system in  $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(t), u(t)) \quad (1.1)$$

where  $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is of class  $C^1$  with respect to  $(x, u)$  and locally integrable with respect to  $t$ , and the controls are measurable essentially bounded functions of time taking their values in some measurable subset  $\Omega$  of  $\mathbb{R}^m$  (set of control constraints).

First of all, given an arbitrary initial point  $x_0 \in \mathbb{R}^n$ , and an arbitrary control  $u$ , we claim that there exists a unique solution  $x(\cdot)$  of (1.1) such that  $x(0) = x_0$ , maximally defined on some open interval of  $\mathbb{R}$  containing 0. We use here a generalization of the usual Picard-Lindelöf theorem (sometimes called Carathéodory theorem), where the dynamics here can be discontinuous (because of the control). For a general version of this existence and uniqueness theorem, we refer to [6, Theorem 5.3] and [12, Appendix C3]. We stress that the differential equation (1.1) holds for almost every  $t$  in the maximal interval. Given a time  $T > 0$  and an initial point  $x_0$ , we say that a control  $u \in L^\infty([0, T], \mathbb{R}^m)$  is admissible if the corresponding trajectory  $x(\cdot)$ , such that  $x(0) = x_0$ , is well defined on  $[0, T]$ .

We say that the control system is *linear* if  $f(t, x, u) = A(t)x + B(t)u + r(t)$ , with  $A(t)$  a  $n \times n$  matrix,  $B(t)$  a  $n \times m$  matrix (with real coefficients),  $r(t) \in \mathbb{R}^n$ , and in that case we will assume that  $t \mapsto A(t)$ ,  $t \mapsto B(t)$  and  $t \mapsto r(t)$  are of class  $L^\infty$  on every compact interval (actually,  $L^1$  would be enough). The linear control system is said to be *autonomous* if  $A(t) \equiv A$  and  $B(t) \equiv B$ , otherwise it is said to be *instationary* or *time-varying*. Note that, for linear control systems, there is no blow-up in finite time (i.e., admissibility holds true on any interval).

**Definition 1.1** Let  $x_0 \in \mathbb{R}^n$  and let  $T > 0$  arbitrary. A control  $u \in L^\infty([0, T], \Omega)$  is said to be *admissible* on  $[0, T]$  if the trajectory  $x_u(\cdot)$ , solution of (1.1), corresponding to the control  $u$ , and such that  $x_u(0) = x_0$ , is well defined on  $[0, T]$ . The *end-point mapping*  $E_{x_0, T}$  is then defined by  $E_{x_0, T}(u) = x_u(T)$ .

The set of admissible controls on  $[0, T]$  is denoted by  $\mathcal{U}_{x_0, T, \Omega}$ . It is the domain of definition of  $E_{x_0, T}$  (indeed one has to be careful with blow-up phenomena), when considering controls taking their values in  $\Omega$ .

**Definition 1.2** The control system (1.1) is said to be (globally) *controllable from*  $x_0$  *in time*  $T$  if  $E_{x_0, T}(\mathcal{U}_{x_0, T, \Omega}) = \mathbb{R}^n$ , i.e., if  $E_{x_0, T}$  is surjective.

Accordingly, defining the *accessible set* from  $x_0$  in time  $T$  by  $\text{Acc}_\Omega(x_0, T) = E_{x_0, T}(\mathcal{U}_{x_0, T, \Omega})$ , the control system (1.1) is (globally) controllable from  $x_0$  in time  $T$  if  $\text{Acc}_\Omega(x_0, T) = \mathbb{R}^n$ .

Since such a global surjectivity property is certainly a very strong property which may not hold in general, it is relevant to define *local controllability*.

**Definition 1.3** Let  $x_1 = E_{x_0, T}(\bar{u})$  for some  $\bar{u} \in \mathcal{U}_{x_0, T, \Omega}$ . The control system (1.1) is said to be *locally controllable from*  $x_0$  *in time*  $T$  *around*  $x_1$  if  $x_1$  belongs to the interior of  $\text{Acc}_\Omega(x_0, T)$ , i.e., if  $E_{x_0, T}$  is locally surjective around  $x_1$ .

Other variants of controllability can be defined. A clear picture will come from the geometric representation of the accessible set.

In this chapter we will provide several tools in order to analyze the controllability properties of a control system, first for linear (autonomous, and then instationary) systems, and then for nonlinear systems.

## 1.1 Controllability of Linear Systems

Throughout this section, we consider the linear control system  $\dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t)$ , with  $u \in L^\infty([0, +\infty), \Omega)$ . Since there is no finite-time blow-up for linear systems, we have  $\mathcal{U}_{x_0, T, \Omega} = L^\infty([0, T], \Omega)$  for every  $T > 0$ .

### 1.1.1 Controllability of Autonomous Linear Systems

In this section, we assume that  $A(t) \equiv A$  and  $B(t) \equiv B$ , where  $A$  is a  $n \times n$  matrix and  $B$  is a  $n \times m$  matrix.

### 1.1.1.1 Without Control Constraints: Kalman Condition

In this section, we assume that  $\Omega = \mathbb{R}^m$  (no control constraint). The celebrated Kalman theorem provides a necessary and sufficient condition for autonomous linear control systems without control constraint.

**Theorem 1.1** *We assume that  $\Omega = \mathbb{R}^m$  (no control constraint). The control system  $\dot{x}(t) = Ax(t) + Bu(t) + r(t)$  is controllable (from any initial point, in arbitrary time  $T > 0$ ) if and only if the Kalman matrix*

$$K(A, B) = (B, AB, \dots, A^{n-1}B)$$

(which is of size  $n \times nm$ ) is of maximal rank  $n$ .

**Proof of Theorem 1.1** Given any  $x_0 \in \mathbb{R}^n$ ,  $T > 0$  and  $u \in L^\infty([0, T], \mathbb{R}^m)$ , the Duhamel formula gives

$$E_{x_0, T}(u) = x_u(T) = e^{TA}x_0 + \int_0^T e^{(T-t)A}r(t) dt + L_T u \quad (1.2)$$

where  $L_T : L^\infty([0, T], \mathbb{R}^m) \rightarrow \mathbb{R}^n$  is the linear continuous operator defined by  $L_T u = \int_0^T e^{(T-t)A}Bu(t) dt$ . Clearly, the system is controllable in time  $T$  if and only if  $L_T$  is surjective. Then to prove the theorem it suffices to prove the following lemma.

**Lemma 1.1** *The Kalman matrix  $K(A, B)$  is of rank  $n$  if and only if  $L_T$  is surjective.*

**Proof of Lemma 1.1** We argue by contraposition. If  $L_T$  is not surjective, then there exists  $\psi \in \mathbb{R}^n \setminus \{0\}$  which is orthogonal to the range of  $L_T$ , that is,

$$\psi^\top \int_0^T e^{(T-t)A}Bu(t) dt = 0 \quad \forall u \in L^\infty([0, T], \mathbb{R}^m).$$

This implies that  $\psi^\top e^{(T-t)A}B = 0$ , for every  $t \in [0, T]$ . Taking  $t = T$  yields  $\psi^\top B = 0$ . Then, derivating first with respect to  $t$ , and taking  $t = T$  then yields  $\psi^\top AB = 0$ . By immediate iteration we get that  $\psi^\top A^k B = 0$ , for every  $k \in \mathbb{N}$ . In particular  $\psi^\top K(A, B) = 0$  and thus the rank of  $K(A, B)$  is less than  $n$ .

Conversely, if the rank of  $K(A, B)$  is less than  $n$ , then there exists  $\psi \in \mathbb{R}^n \setminus \{0\}$  such that  $\psi^\top K(A, B) = 0$ , and therefore  $\psi^\top A^k B = 0$ , for every  $k \in \{0, 1, \dots, n-1\}$ . From the Hamilton-Cayley theorem, there exist real numbers  $a_0, a_1, \dots, a_{n-1}$  such that  $A^n = \sum_{k=0}^{n-1} a_k A^k$ . Therefore we get easily that  $\psi^\top A^n B = 0$ . Then, using the fact that  $A^{n+1} = \sum_{k=1}^n a_k A^k$ , we get as well that  $\psi^\top A^{n+1} B = 0$ . By immediate recurrence, we infer that  $\psi^\top A^k B = 0$ , for every  $k \in \mathbb{N}$ , and therefore, using the series expansion of the exponential, we get that  $\psi^\top e^{(T-t)A}B = 0$ , for