SpringerBriefs in Mathematics

Daniel Goodair · Dan Crisan

Stochastic Calculus in Infinite Dimensions and SPDEs

Springer

SpringerBriefs in Mathematics

SpringerBriefs present concise summaries of cutting-edge research and practical applications across a wide spectrum of fields. Featuring compact volumes of 50 to 125 pages, the series covers a range of content from professional to academic. Briefs are characterized by fast, global electronic dissemination, standard publishing contracts, standardized manuscript preparation and formatting guidelines, and expedited production schedules.

Typical topics might include:

- A timely report of state-of-the art techniques
- A bridge between new research results, as published in journal articles, and a contextual literature review
- A snapshot of a hot or emerging topic
- An in-depth case study
- A presentation of core concepts that students must understand in order to make independent contributions

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians. All works are peer-reviewed to meet the highest standards of scientific literature.

Titles from this series are indexed by Scopus, Web of Science, Mathematical Reviews, and zbMATH.

Daniel Goodair • Dan Crisan

Stochastic Calculus in Infinite Dimensions and SPDEs

Springer

Daniel Goodair
Imperial College London
London, UK

Dan Crisan
Department of Mathematics
Imperial College
London, UK

ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-031-69585-8 ISBN 978-3-031-69586-5 (eBook)
https://doi.org/10.1007/978-3-031-69586-5

Mathematics Subject Classification: 60H15, 60H05, 60G44, 35R60, 35R15, 35Q35, 60H30

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The purpose of this brief is to cover the basics of infinite dimensional stochastic differential equations (defined on Hilbert spaces), in a pedagogical manner, assuming only an elementary understanding of functional analysis and probability theory. We present a robust construction of the stochastic integral in Hilbert Spaces, considering integrals driven at first by real valued martingales and later by *Cylindrical Brownian Motion*, introducing this concept and expanding into a basic set-up for Stochastic Partial Differential Equations (SPDEs). The framework that we establish facilitates a broad class of SPDEs and noise structures, notably including *unbounded* noise, in which we build upon standard Stochastic Differential Equation (SDE) theory and rigorously deduce a conversion between their Stratonovich and Itô forms. In the remainder of the brief, we explore more advanced tools to be used in the analysis of these equations.

London, UK
July 2024

Daniel Goodair
Dan Crisan

Acknowledgements

Daniel Goodair was supported by the Engineering and Physical Sciences Research Council (EPSCR) Project 2478902. Dan Crisan was partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (ERC, Grant Agreement No 856408).

Contents

1 Introduction 1
1.1 Motivation and Description of the Brief 1
1.2 Notation 4

2 Stochastic Calculus in Infinite Dimensions 7
2.1 A Classical Construction for Hilbert Space Valued Processes 7
2.2 Martingale and Local Martingale Integrators 15
2.3 Cylindrical Brownian Motion 23
2.4 Martingale Theory in Hilbert Spaces 27
2.5 Integration with Respect to Cylindrical Brownian Motion 45

3 Stochastic Differential Equations in Infinite Dimensions 57
3.1 The Stratonovich Integral 57
3.2 Strong Solutions in the Abstract Framework 59
3.3 Uniqueness and Maximality 63
3.4 Stratonovich SPDEs in the Abstract Framework 71
3.5 Weak Solutions in the Abstract Framework 77
3.6 Time-Dependent Operators 80

4 A Toolbox for Nonlinear SPDEs 83
4.1 Existence and Uniqueness in Finite Dimensions 83
4.2 Tightness Criteria 92
4.3 Cauchy Criteria 97
4.4 Enhanced Regularity and an Energy Equality 104
4.5 SPDEs with Constant Multiplicative Noise 123

Appendix A 127
A.1 Classical Results from the Real Valued Theory 127
A.2 Classical Tightness Criteria 129
A.3 Stochastic Grönwall Lemma 130

References 131

Index 135

Chapter 1
Introduction

This chapter introduces the brief, outlining its structure and motivating the core themes. We also establish and collect notation used throughout the book.

1.1 Motivation and Description of the Brief

This brief comprises three chapters, increasing in complexity, described below:

- In Chap. 2 we present a "classical" construction of the Itô stochastic integral, for processes evolving in a Hilbert space. This is introduced first for a one dimensional driving Brownian motion, before generalizations to other one dimensional martingales and, further, to *cylindrical Brownian motion*. Our construction is direct and designed to be familiar to a reader who has undertaken the real valued study as covered, for example, in [44, 55]. In defining the infinite dimensional Brownian motion, that is the cylindrical Brownian motion, we cover the fundamentals of martingale theory in Hilbert spaces broadly by finite dimensional projections along with the real valued theory. The hope is again that this approach is entirely accessible to a reader with background in the real valued integration theory. Precise attention must be paid to the martingale theory in order to properly consider Stratonovich equations; in our opinion the most thorough presentation of this material is in [60], yet more details and results are needed, such as the cross-variation between a Hilbert space valued and real valued martingale. The cylindrical Brownian motion is the only infinite dimensional driving process that we integrate with respect to; while we present a background on general Q-cylindrical processes which could be viable integrators, limiting ourselves to cylindrical Brownian motion enables the integral to be established as a straightforward limit of the integrals against finite dimensional Brownian motions. In particular we avoid the operator theoretic technicalities necessary in the general case, present in the all of the constructions of the classical works

D. Goodair, D. Crisan, *Stochastic Calculus in Infinite Dimensions and SPDEs*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-69586-5_1

[20, 38, 51, 52, 57, 60]. We hope that removing some generality makes our approach more accessible for newcomers to the field, and we note that our construction is sufficient for the framework and applications that follow.

- Chapter 3 details a framework for the study of stochastic partial differential equations (SPDEs), which are evolution equations involving integrals of the form introduced in the previous chapter. Through this framework we define notions of solutions for an abstract SPDE, motivated in particular by the recent attention given to *transport type noise* (where the stochastic integral is dependent on the gradient of the solution) and *Stratonovich* equations. Motivation for such study is given below this list of contents. A rigorous mathematical understanding of these equations presents difficulty for two key reasons. The first is the gradient dependency in the noise, taking us beyond the most general "variational frameworks" seen in the literature as these are posed for a noise operator which is bounded on some Hilbert space. The second is the Stratonovich integration, which we are likely to only understand as a corrected Itô integral, yet this conversion is highly nontrivial for a noise which is not bounded on a Hilbert space. Furthermore we wish to consider *nonlinear* SPDEs, such as the evolution equations of fluid dynamics, rendering the well-established linear theory insufficient.

  To be precise, we present a framework which shares its spirit with the variational approach to SPDEs pioneered by Pardoux in the 1970s and now best represented in the more recent books [51, 56, 57]. This classical framework considers an evolution equation with respect to a Gelfand Triple, say $V \hookrightarrow H \hookrightarrow V^*$, where solutions have paths which are square integrable in V, continuous in H, and satisfying an identity in V^*. Recalling our motivation of fluid equations, the prototypical example in this framework is the Navier–Stokes equation. While analytically weak solutions fit this framework seamlessly, analytically strong solutions fit to the spaces $W^{2,2} \hookrightarrow W^{1,2} \hookrightarrow L^2$, which prompts our choice of a triplet of embedded Hilbert spaces without any necessary duality structure. Furthermore, the aforementioned works allow only for a noise operator bounded in H (thus not of first order in applications) and do not consider Stratonovich equations. To include a Stratonovich transport type noise, we introduce a fourth Hilbert space, necessary as the Itô–Stratonovich correction requires an additional derivative to cover the transport type noise. We can then properly define weak, strong, and local solutions of nonlinear PDEs with Stratonovich transport noise, alongside other more classical additive and multiplicative stochastic perturbations. We believe that presenting the technical details, in such generality, of these notions here facilitates the rigorous and free analysis of the equations in future works.
- Chapter 4 contains advanced novel techniques in the existence theory for non-linear SPDEs. The beauty of the classical variational approach comes from the existence results, which certainly cannot be matched as elegantly in a framework built for 3D Navier–Stokes equations and related stochastic fluid models. Instead we focus on techniques that can be used in this direction, centered around the *Galerkin Method* in which finite dimensional approximations of the SPDE are

considered and some properties are used to deduce their limit. Immediately then an existence result for the finite dimensional equations is required, more precisely for where the Hilbert space in which the equation evolves is finite dimensional, but the driving Brownian motion is still infinite dimensional. We assume standard Lipschitz and linear growth conditions, and to the best of our knowledge this result is not present in the literature. There are two predominant ways to deduce the existence of a limit of the finite dimensional approximations, which we detail now.

The first is through *tightness*, which is the stochastic route to relative compactness arguments used in PDE theory. The idea is that from tightness we can deduce relative compactness of the laws of the processes over some suitable function space, at which point Skorohod's Representation Theorem enables the deduction of a limiting process almost surely on a new probability space. Criteria to deduce tightness in relevant function spaces are thus of great significance, and our criteria come largely from the works of [3, 43, 59]. The second is through a Cauchy type argument in the relevant spaces, difficult to execute in the case of local solutions, but recently this has been overcome to great effect due to Glatt-Holtz and Ziane in [33] and extended by the authors here. We defer a greater discussion of this highly technical result to Chap. 4 and emphasize that this is a new result in the cutting edge theory of SPDEs.

An energy equality in this setting is also presented. This is well understood in the typical variational framework, for which we again refer to [51, 57], but we take care in addressing some subtle differences. The first is the loss of the duality structure, though for this result we do assume a bilinear form relation which behaves similarly. The second is that we conduct the proof for *local* solutions, necessary for our motivating class of equations, so it is important for us to explicitly address how the localization affects the proof. Indeed, the consideration of local solutions, as well as the related localization in the construction of the integral, martingale theory, and analytical techniques, is an important extension of the framework of [51, 56, 57]. Similarly, the final key change is that we do not assume any integrability over the probability space of our processes, demanding again another source of localization which we find worthy of detailing in this brief. The chapter rounds out with a demonstration that the infinite dimensional noise can be reduced to one dimensional objects if it is constant multiplicative in each direction.

Before setting up notation and beginning with our exposition, we give some more explicit motivation for this brief. First of all, why work in infinite dimensions? Finite dimensional stochastic differential equations have rich applications in physics and finance, for example, in Langevin equations modeling the movement of a particle in space [13] or the Black–Scholes options pricing model for the dynamics of the price of a stock [8]. These are applications of classical Itô calculus, where the integral of a process takes values in Euclidean spaces. While this theory is adequate in such applications, mathematical models for physical phenomena far exceed those for the position of a particle or that of a tradable stock price. The extension to infinite dimensions is necessary for stochastic differential equations modeling functions of both space and time, such as the velocity or temperature of a fluid. It is therefore necessary to define the stochastic integral

$$\int_0^t \boldsymbol{\Psi}_s dW_s, \tag{1.1}$$

for a class of stochastic processes $\boldsymbol{\Psi} : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^d$. We regard $\boldsymbol{\Psi}$ not as a pointwise defined function but rather an element of a function space, which is our motivating context for stochastic integration of Hilbert space valued processes. The recent attention toward Stratonovich and transport noise SPDEs is inspired from the seminal work [42], in which Holm establishes a new class of stochastic equations which serve as fluid dynamics models by adding uncertainty in the transport of fluid parcels to reflect the unresolved scales. For recent literature on the analysis of equations under this stochastic scheme, please see [4, 11, 15–19, 22, 34, 35, 37, 40, 41, 47, 50, 61] to list only a few, all of which rely on an Itô–Stratonovich conversion and a framework such as we present here, which is yet to see any rigorous justification. In fact the pertinence of Stratonovich transport noise in fluid dynamics equations was demonstrated as early as 1992 in the paper [9], and the analysis of SPDEs with general Stratonovich transport noise can be seen in the papers [1, 2, 5, 6, 12, 23–25, 27–31, 39, 48, 49, 53, 54] as well as the recent book [26]. We believe that our framework and results facilitate the rigorous and free analysis of such equations in future works.

1.2 Notation

Throughout the brief, we work with a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, which is complete with respect to $\mathcal{F}_0$. We always consider Banach spaces as measure spaces equipped with the corresponding Borel σ-algebra and shall use λ to denote the Lebesgue Measure. All of our Hilbert Spaces are assumed to be separable.

Notation 1.1 *Let $(\mathcal{X}, \mu)$ denote a general measure space, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ and $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ be Banach Spaces, and $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be general Hilbert spaces:*

- *$L^p(\mathcal{X}; \mathcal{Y})$ is the usual class of measurable p-integrable functions from $\mathcal{X}$ into $\mathcal{Y}$, $1 \leq p < \infty$, which is a Banach space with norm*

$$\|\phi\|^p_{L^p(\mathcal{X};\mathcal{Y})} := \int_{\mathcal{X}} \|\phi(x)\|^p_{\mathcal{Y}} \, \mu(dx).$$

  The space $L^2(\mathcal{X}; \mathcal{Y})$ is a Hilbert Space when $\mathcal{Y}$ itself is Hilbert, with the standard inner product

$$\langle \phi, \psi \rangle_{L^2(\mathcal{X};\mathcal{Y})} = \int_{\mathcal{X}} \langle \phi(x), \psi(x) \rangle_{\mathcal{Y}} \, \mu(dx).$$

- *$L^\infty(\mathcal{X}; \mathcal{Y})$ is the usual class of measurable functions from $\mathcal{X}$ into $\mathcal{Y}$ which are essentially bounded, which is a Banach Space when equipped with the norm*

$$\|\phi\|_{L^\infty(\mathcal{X};\mathcal{Y})} := \inf\{C \geq 0 : \|\phi(x)\|_Y \leq C \text{ for } \mu\text{-}a.e.\ x \in \mathcal{X}\}.$$

- *$C(\mathcal{X}; \mathcal{Y})$ is the space of continuous functions from $\mathcal{X}$ into $\mathcal{Y}$.*
- *$C_w(\mathcal{X}; \mathcal{Y})$ is the space of "weakly continuous" functions from $\mathcal{X}$ into $\mathcal{Y}$, by which we mean continuous with respect to the given topology on $\mathcal{X}$ and the weak topology on $\mathcal{Y}$.*
- *$\mathscr{L}(\mathcal{Y}; \mathcal{Z})$ is the space of bounded linear operators from $\mathcal{Y}$ to $\mathcal{Z}$. This is a Banach Space when equipped with the norm*

$$\|F\|_{\mathscr{L}(\mathcal{Y};\mathcal{Z})} = \sup_{\|y\|_{\mathcal{Y}}=1} \|Fy\|_{\mathcal{Z}}\,.$$

  $\mathscr{L}(\mathcal{Y}; \mathcal{Z})$ is the dual space $\mathcal{Y}^*$ when $\mathcal{Z} = \mathbb{R}$, with operator norm $\|\cdot\|_{\mathcal{Y}^*}$.
- *$\mathscr{L}^1(\mathcal{U}; \mathcal{H})$ is the space of trace-class operators from $\mathcal{U}$ to $\mathcal{H}$, defined as the elements $F \in \mathscr{L}(\mathcal{U}; \mathcal{H})$ such that for some basis (e_i) of $\mathcal{U}$,*

$$\sum_{i=1}^{\infty} \|Fe_i\|_{\mathcal{H}} < \infty.$$

  This is independent of the choice of basis (see, e.g., [14, pp. 267 Ex 20]).
- *$\mathscr{L}^2(\mathcal{U}; \mathcal{H})$ is the space of Hilbert–Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$, defined as the elements $F \in \mathscr{L}(\mathcal{U}; \mathcal{H})$ such that for some basis (e_i) of $\mathcal{U}$,*

$$\sum_{i=1}^{\infty} \|Fe_i\|_{\mathcal{H}}^2 < \infty.$$

  This is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathscr{L}^2(\mathcal{U};\mathcal{H})} = \sum_{i=1}^{\infty} \langle Fe_i, Ge_i \rangle_{\mathcal{H}}\,,$$

  which is independent of the choice of basis (see, e.g., [14, pp. 267 Ex 20]).
- *For any $T > 0$, $\mathscr{S}_T$ is the subspace of $C([0,T]; [0,T])$ of strictly increasing functions.*
- *For any $T > 0$, $\mathcal{D}([0,T]; \mathcal{Y})$ is the space of cádlág functions from $[0,T]$ into $\mathcal{Y}$. It is a complete separable metric space when equipped with the metric*