Gödel Forever

Through 90 Years of Foundational Claims

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To the memory of Mrs. B.T. Lallas, the 10th grade Ensley High School Algebra-2 teacher who one day set aside her lesson-plan to clarify an arithmetical homework locution that no one was able to sort, neither at home nor during the entire class period that followed.

“Is,” she announced, raising a finger when the bell rang signaling the end of the class, “means equals.”

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Introduction

Nothing in the known history of mathematics is quite like the Incompleteness of Arithmetic (GI), either in substance, in the circumstances around its discovery, or in the effect it has had on mathematical thinking. Unlike many others, the Incompleteness revolution was not the culmination of a group or industry effort that began a new School (like Newton's Calculus or Frege's Logicism), but came very much out of the blue from the single-mindedness of Kurt Gödel. Its effect has been negative, too, decimating two of the leading mathematical schools of the day, with its academic research a professional cul-de-sac (Girard 2011).

It is remarkable that on this 90th anniversary, and after countless learned articles and entire texts on the subject, a leading authority can still write in good conscience that, "As regards Gödel's First Incompleteness Theorem and the matter of its proof, Gödel's own paper has yet to be improved upon" (Smorynski 2009, 122). Indeed, the simple irrefutability of Gödel's original presentation played a large part of its initial appeal. Take the Chinese Remainder theorem; while unknown to most new to Gödel's derivations, it is a tidbit, among others, that the unfamiliar upper-level undergraduate student will easily pick up on along the way.

This is a story of a proofing and its influence over ninety years that will unfold in four acts. We begin with the GI derivation itself, expanded over details that for lack of necessity at the time, or space, Gödel did not include in the original production. By the derivation we refer solely to Gödel's original. This is also a story of the peculiar regularity with which GI citations turn up in the literature of 20th century philosophy. Such literature subjects include the limits to machine intelligence, the redundancy of truth, and number ontology. For the first, there is the indirect self-referencing of the Gödel sentence, g, that presumably evades machine proof; for the other two, the universal generalization in its construction. It is only the well-put and narrowly detailed former story, we think, that properly informs our understanding of the latter, broader story, bringing us up to where we find ourselves in it today. If there is one thing we have learned in the course of this study, it is that the facts regarding GI's proper place in these matters lies in the mathematical details of its derivation.

There are statements *in* arithmetic that cannot be shown either true or false *by* arithmetic. If the arithmetic statement "3+5 = 8" can be shown, i.e., proven true or false, it would seem then that "3+5 = 8 and/or 7+1 = 3" could also be proven true or false by the same schoolhouse arithmetic. But for the statement referred to in the Incompleteness claim (presumably, some conjoined and disjoined combination of like simple arithmetic statements) apparently the reasoning does not hold. How could this be?

There may be no answer nor need for one. Besides the Incompleteness claim, there are other interpretations of Kurt Gödel's formal deduction, the Gödel result (GI),[1] that range from the banal acknowledgement that consistent axiomatic formulations of number theory include undecidable propositions, to those that have it that the human mind is unable to comprehend itself, and by which computers can never outsmart humans (Jones and Wilson 2009).[2]

Here are a few others:

- All mathematics can be formalized: however, mathematics can *never* be exhausted in *any one* system but requires an infinite sequence of discourses which get progressively more comprehensive (Carnap 1934).
- Every arithmetic is incomplete (Waismann 2003).
- Truth transcends proof (Vidal-Rosset 2006).
- Every system of arithmetic contains arithmetical propositions which can neither be proved nor disproved within the system (Gödel 1962).
- An axiomatic approach to number theory cannot fully characterize the nature of number-theoretical truth; what we know and understand about mathematics transcends what can be expressed through our mathematical systems (Lipscomb 2010).

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1 Bertrand Russell, philosopher and author of *Principia Mathematica*, which Gödel calls out by name (Whitehead and Russell 1910), seems to have interpreted Incompleteness as a verdict on mathematical consistency and openly worried whether 3+5 indeed *is* 8. There is no record he ever advocates we stop teaching it in schools.

2 Both seemingly at odds with a reasoning from GI that a mind may *not* claim superiority over a machine (Makey 1995). For others still, GI settles the question whether a machine has a soul (A. Turing 1950).

- Relying on words to lead you to the truth is like relying on an incomplete formal system to lead you to the truth. A formal system will give you some truths, but a formal system, no matter how powerful, cannot lead to all truths (Hofstadter 2000).
- In any language there exist true but unprovable statements (Uspenski 1989).

The list may be continued.[3]

Once the *pons asinorum* of *Mathematical Logic* (Lucas 2002), a new trend describes GI as “so simple, and so sneaky, that it is almost embarrassing to relate” (Rucker 2008, 162), with the details unnecessary to contemplate.[4] Here we maintain that even as valid interpretations may differ, the sound interpretation of Incompleteness is based on knowledge of precisely what it is Gödel has done. Common misconceptions of Gödel Incompleteness (GI) discussed in the coming “Indirect Self-Reference” section make this point clear. While covering the Incompleteness derivation in good detail, we do not run the full circuit. At one point in the original derivation, Gödel launches a sequence of 46 functional definitions essential to the derivation. It is tedious and takes up a lot of space. Of the 46, we analyze an essential few and describe others, leaving the rest for review in cited sources. Among these we recommend a modern translation of the original paper, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems* (Gödel 1962), a popular review (Smith 2013), and a recent online rewrite (Gödel 2000) from which we freely borrow notation.

Of the differences here with these and other reviews there is our emphasis on the question of expressibility of primitive recursion in the formal language of arithmetic. That the Gödel sentence, g, appears there, as required for its Incompleteness, is a straightforward but lengthy demonstration usually taken as given. Another is that we make more explicit Gödel’s original “arithmetization” of the meta-language, as consequent on an isomorphism between the original language, P, and another, P′ that has a meta-language all its own. This helps draw out more clearly the underlying premises involved in the Incompleteness mechanics.

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3 See e.g. chapter 11 of Sokal’s *Fashionable Nonsense* (Sokal and Bricmont 1998).

4 “Understanding Gödel isn't about following his formal proof, which would make a mockery of everything Gödel was up to” (Jones 2008).

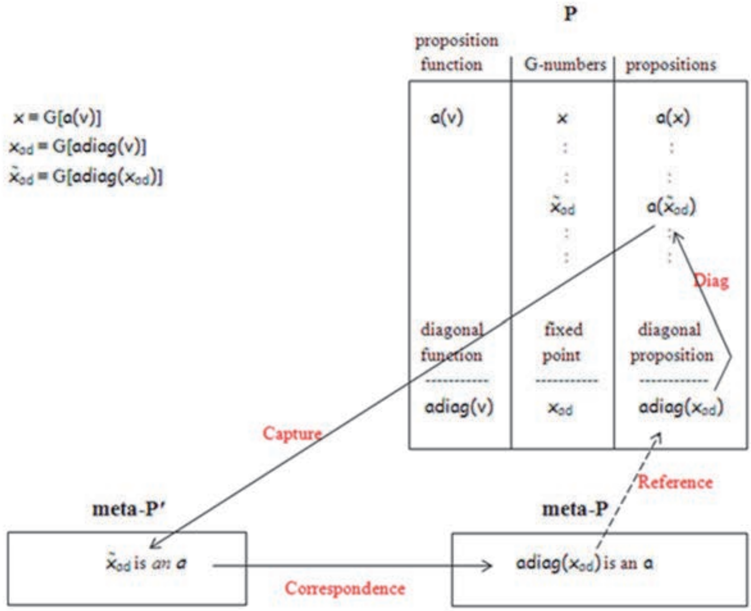
As will become apparent by the "Correspondence"–sub and "Indirect Self-Reference" sections below, it also helps clarify an important limitation on inferences from Incompleteness that might be made in other fields of study, such as the Philosophy of Mind that typically follow from the misconception of Gödel's arithmetization as an isomorphism between mathematical P and its meta-language, meta-P. If there is an isomorphism at play, and only two languages, then the isomorphism can only be between those two, inviting such speculations as exceed that limit; if there is but one reason for fleshing out this four-language, indirect self-referencing mechanism that infers "truth" without "meaning" and upon which GI proceeds, that is it. While there are two Incompleteness theorems derived in (Gödel 1962), the first and second, there are also two distinct proofs of the first—a semantic and a syntactic proof. In our first two sections we consider only the semantic derivation of the first, said to be the weaker proof.

Our first three sections are: "I. Arithmetization", "II. Indirect Self-Reference", and "III. An Odious Turn". Given the four flavors of Gödelian Incompleteness:

- Semantic Incompleteness, semantically derived (applying the notion of truth)
- Semantic Incompleteness, syntactically derived (absent the notion of truth)
- Syntactic Incompleteness (Undecidability), semantically derived
- Syntactic Incompleteness (Undecidability), syntactically derived

When the subject in question does not concern formal soundness itself, one or other of the simpler semantical derivations is usually cited; otherwise not. Accordingly, our beginning "I. Arithmetization" section is an elementary examination of the simplest semantic GI derivational details. While it is well known that the means by which g is said to indirectly refer to itself is established via the "arithmetization of a metalanguage" whose object is isomorphic to the language in which g appears, the crucial question becomes *where* precisely in this mechanism is the much-talked-about linguistic isomorphism applied,[5] and where not. The section closes out with the following Indirect Self-Reference inference diagram

5 Alternate interpretation, translation, what have you.



by means of which toward the end of our short middle section, "II. Indirect Self-Reference", we closely examine how GI is cited in the context of the machine-intelligence limit discussion and consider whether, as usually assumed, *g* is ambiguous between object- and meta-linguistic readings.

Any presentation of Gödel's Incompleteness that aims at clarity must balance between deductive detail and economy of exposition, particularly with regard to details of its more novel features such as the formal "Definability" of primitive recursion relations and their "Diagonal Compositions" that drive the indirect self-referencing mechanism and is common to all four flavors. On the other hand, not only *g*, but all of the captured diagonally-composed metalinguistic predicates are locked into this mechanism, so that even the most elementary of these puts its inner workings on full display, as in the above inference diagram for arbitrary predicate "ɑ". For these reasons, the 'Definability' and "Diagonal Composition" of primitive recursion in our formal system P, as well as P's Capture of its most elementary linguistic syntax "ɑ is a variable", we derive in great detail, with examples and visual aids that together leave little room