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The Art of Gluing Space-Time Manifolds

Methods and Applications

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Preface

In 1996, one of us (RM), while on sabbatical at Potsdam University, Germany, gave a series of lectures with the same title as this monograph. Thereafter came the idea of writing it up as a review and the first draft of the beginning chapters was even written down. We are very glad to see now, exactly 100 years after the first paper on the problem of gluing manifolds published by Lanczos in Berlin (1922), not far from Potsdam, and 27 years after those lectures, we were successful in finishing our longstanding effort of reviewing this very special topic with widespread application in vast areas of gravitation and cosmology.

In the centennial of conceiving this "art", it is a pleasure to dedicate this monograph posthumously to Lanczos and his student Sen, who did an excellent job in a flourishing academic atmosphere in Berlin.

We would like to acknowledge the Cosmology Group at the Department of Physics, Sharif University of Technology, especially past students and new colleagues Mohammad Khorrami, Sima Ghassemi, Shahram Khosravi, and Kourosh Nozari, for all the lively discussions and contributions to the topic of this work. RM would like also to thank IPM for providing a motivating atmosphere and financial support for this work. SKh would like to thank Nuclear Science and Technology Research Institute for the support over the years.

Tehran, Iran — Samad Khakshournia
April 2023 — Reza Mansouri

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Chapter 1
Introduction

Modeling natural phenomena using mathematical language is the basis for understanding nature. Now, many natural features and phenomena within spacetime are not by any means as smooth as could be defined mathematically in a way suitable to be used in a physical model. Cosmology as a dynamic and flourishing science could not exist without smearing out all non-smooth structures like planets, stars, galaxies and so on. This, however, is the beginning of a long journey called the science of cosmology, its dynamics, and astrophysics, including all different structures, using a wide spectrum of mathematics, details of which are the aim of this study.

Manifolds and the simplified geometries on them used in cosmology give us a vast possibility to grasp notions like Branes, walls, domain walls, solitonic objects, D-branes, p-branes, phase transitions in the early universe, and the formation of topological defects, just to name some of our needs in physics. Early attempts to model such phenomena go back to the dawn of relativity [59, 135, 136, 187]. Since then, understanding localized matter distributions in astrophysics and cosmology has always been of interest. Realizing the difficulty of handling thick shells mathematically, it was too natural to consider the idealization of a singular hypersurface as a thin shell and try to formulate its dynamics within general relativity, though Einstein and Straus made the first attempt to implicitly use the concept of a thick shell [75] to embed a spherical star within a FRW universe (see also [119]). It was then too natural to continue studying singular hypersurfaces, started first by Sen [187], Lanczos [135, 136], and Darmois [59], a development which then was summed up by Israel [112]. The next era of intense interest in thin shells began with ideas related to phase transitions in the early universe and the formation of topological defects [126]. Strings and domain walls were assumed to be infinitesimally thin [55] (see [206] for a review), mainly due to technical difficulties. In most of the above examples, the way to make a reasonable model is to glue two separate spacetime manifolds, usually submanifolds of existing solutions to Einstein equations, at a common boundary. We may differentiate two distinct cases of boundaries: hypersurfaces without or with supporting energy-momentum. The first case is simply called a *boundary surface*, and the second one being a singular hypersurface is coined a *thin shell* or a *surface layer*.

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Thin shell formalism has been used to play an important role in various dynamical contexts ranging from microscopic to astrophysical scales [91]. For instance, by applying a charged shell as an electron model, one may avoid the appearance of negative gravitational mass caused by the concentration of charge at the center [141, 202]. Macroscopically stable quark-gluon matter can also be studied with a toy model in which relativistic thin shells and the MIT bag model are combined [211]. Accordingly, the quantization of systems comprising thin shells is shown to be tractable [29, 53, 69]. On the other side, this formalism is also suitable to describe cosmic bubble dynamics and interior structures of black holes (see Chap. 10 and references therein), gravitationally-induced decoherence [98], exotic objects such as gravastars [150] and wormholes (see Chap. 10 and references therein). Moreover, gluing spacetime domains have also been considered to analyze cosmological phase transitions in the early universe, to describe cosmological voids, to construct semi-classical creation models avoiding the initial singularity of the Big Bang scenario by quantum tunneling (see Chap. 10 and references therein), etcetera.

Now, gluing spacetime manifolds in a mathematically consistent way needs geometrical assumptions and leads to dynamical conditions on the gravitational and matter fields related to the gluing hypersurface. These assumptions and conditions are usually summed up under terms such as junction- or matching conditions, originally studied by Darmois [59] and Lichnerowiz [140], and much later for non-null thin shells derived by Israel [112], and later extended to null thin shells by Barrabes and Israel [23]. Although the geometrical part of these so-called conditions is just requirements inherent to the definition of a manifold, the dynamical gluing conditions depend on the gravitational field theory and in each case have to be derived independently.

Today, the technology of gluing manifolds in order to model localized phenomena, being space-, time-, or light-like, shells, has grown into a standard research tool to be used widely. Having this in mind, we elaborate on this technology with the aim of having a sound foundation to be used for further research on any area of physics in which N-dimensional manifolds and their geometry, being a configuration- or momentum-space, play a role. This monograph is organized as follows:

Chapter 2 is devoted to relevant concepts and definitions needed to set up the necessary ingredients for gluing spacetimes, including a brief historical survey of different junction conditions differentiating geometrical requirements from dynamical conditions.

Depending on the way we choose the coordinates on manifolds at each side of the timelike/spacelike hypersurface to be glued, different approaches may be used to obtain basic equations governing the dynamics of thin shells. In the case of arbitrary coordinates, the pill-box integration of Gauss–Codazzi equations over the singular hypersurface leads us to the desired junction conditions [112]. This approach is elaborated on in Chap. 3. It is, however, possible and sometimes more feasible to use a unique coordinate system and metric on the manifolds supposed to be glued. This gives us the opportunity to use distribution-valued tensors on the glued manifold including the boundary surface, leading to a simple and aesthetic form of the gluing conditions similar to the Einstein equations [145]. This approach is introduced in

Chap. 4. Both chapters are limited to timelike/spacelike boundary surfaces due to the intricate behavior of lightlike hypersurfaces.

The case of null hypersurfaces is studied in Chap. 5. We will see there that in this case there is no other way than to start with continuous coordinates and use a distributional approach. As the induced metric on the null hypersurface is degenerate, its normal vector is at the same time tangent to it. Therefore, the extrinsic curvature is not defined uniquely. Addressing this subtle point, the necessary prescription similar to Chap. 4 is given, while highlighting some peculiar features of null shells, such as a related impulsive gravitational wave absent in timelike/spacelike shells, and the gluing freedom for a null shell placed at the horizon of static black holes.

To have a deeper insight into the dynamics of boundary hypersurfaces, we use the variational principle to derive the junction conditions for gluing spacetimes through a timelike or spacelike hypersurface in Chap. 6.

Gravity theories other than general relativity (GR) have always been of interest for different reasons. In Chaps. 7–9, we study the gluing technology in widely discussed alternatives to GR. Chapter 7 is devoted to the gluing conditions in Einstein–Cartan theory of gravity, taking into account the spacetime torsion. Our review covers both non-null and null hypersurfaces.

In Chap. 8, we study higher-order gravity theories being motivated due to difficulties of general relativity in providing well-accepted explanations to problems such as singularities, the nature of dark matter, understanding the accelerated expansion of the universe, or quantum gravity. We constrain ourselves to the simplest proposal, the so-called $f(R)$ theories, where higher powers of the Ricci scalar are added to the standard Einstein–Hilbert action. The study requires special attention due to the unavoidable products of singular distributions to be handled for a consistent mathematical framework needed to derive the gluing conditions. Among several contributions, the work of Senovilla is the first to show the presence of a dipole-type term in the energy-momentum content supported on the shell.

Chapter 9 is then confined to the most general quadratic theory, which is defined by the addition of terms quadratic in curvature to the standard general relativity action.

Finally, in Chap. 10, we give various examples covering timelike/spacelike and null shells to illustrate the methods presented, highlighting the significance of gluing spacetimes.

This monograph is not the beginning of an end! Many topics have been omitted, such as intersecting thin shells [107, 138], signature change on thin shells and its controversy [70, 80, 81, 105, 146], general hypersurfaces whose timelike, spacelike or null character can change from point to point [149, 190], thick shells and their thin shell limit [72, 90, 121], and deriving gluing conditions in more complicated higher-order gravitational theories such as $f(T)$ [204] $f(R, T)$ [25, 180], Palatini $f(R)$ theory [161], Palatini $f(R, T)$ theory [181], and last but not least, the Penrose junction conditions corresponding to metrics with a delta distribution representing an impulsive gravitational wave [170].

Conventions and Definitions:
We use the signature $(-+++)$, and adopt the curvature conventions of Misner, Thorn, and Wheeler (MTW) [151], with a Riemann tensor defined by $R^{\sigma}_{\mu\rho\nu} = \Gamma^{\sigma}_{\mu\nu,\rho} + \cdots$, and a Ricci tensor defined by $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$. Greek indices run from 0 to 3 and Latin indices from 1 to 3. A semicolon indicates the covariant derivative with respect to either the four-metric of whole spacetime or to the three-metric of shell. There will, however, be no confusion because the kind of indices used makes the difference transparent. Symbol $\nabla^{\pm}$ denotes the covariant derivative with respect to either of the metrics of partial manifolds $\mathcal{M}^{\pm}$ which are to be glued together.

The square brackets $[F]$ are used to indicate the jump of any quantity F at the layer, and bars $\overline{F}$ is the arithmetic mean of it. As we are going to work with distribution-valued tensors, there may be terms in a tensor quantity F proportional to some δ-function. These terms are indicated by $\breve{F}$.