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Optimal Transport and Applications to Geometric Optics

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Preface

This book focuses on the theory of Optimal Transport and its applications in solving problems in geometric optics. It provides a comprehensive presentation that includes a thorough analysis of key problems, namely the Monge problem, the Monge-Kantorovich problem, the transshipment problem, and the network flow problem. It also establishes the interconnections between these problems. Additionally, the book dedicates a chapter to Monge-Ampère measures, offering exercises for further understanding.

Furthermore, the book conducts a detailed analysis of the disintegration of measures and its application to the Wasserstein metric, showcasing its realization using the continuity equation. A chapter on the Sinkhorn algorithm is also included.

In terms of optics applications, the book covers the essential background knowledge on light refraction, addressing both the far-field and near-field refraction problems. It also sheds light on current research directions in this area.

The presentation of the book is self-contained, providing detailed explanations and complete proofs of the theorems and results. It is ideal for researchers, practitioners, and students interested in utilizing optimal transport principles for the design of non-rotationally symmetric lenses.

To fully grasp the content of this book, readers are expected to have a solid understanding of measure theory and integration, as well as a basic knowledge of functional analysis.

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Contents

Chapter 1
Introduction

Abstract Three problems motivating the theory of optimal transportation are introduced: the distribution problem, the Monge problem and the Kantorovich problem. Also, the network flow problem is analyzed, solved in detail and it is described how to convert it into an optimal transport problem.

We begin presenting three problems that are the motivation of the theory of optimal transportation.

1.1 The Transportation or Distribution Problem

If $X_1, \cdots, X_m$ are sources (for example warehouses) and $Y_1, \cdots, Y_n$ are destinations (for example shops), *the transportation problem* consists of transporting commodities or items from the sources to the destinations assuming the cost of transporting one item from X_i to Y_j is c_{ij}. We are assuming also that u_i is the supply at X_i and v_j is the demand at Y_j. In summary,

$$\begin{aligned} m &= \#\text{ of sources of goods} \\ n &= \#\text{ of destinations} \\ u_i &= \text{ capacity of source } i \\ v_j &= \text{ need or demand of destination } j \\ c_{ij} &= \text{ unit transportation cost from source } i \text{ to destination } j \\ x_{ij} &= \text{ quantity shipped from source } i \text{ to destination } j. \end{aligned}$$

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A transportation plan is a matrix $X = (x_{ij})$ with $1 \le i \le m, 1 \le j \le n$. Each transportation plan gives rise to a cost

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\, c_{ij} = \langle X, C\rangle.$$

The objective is then to find a transportation plan X so that the cost is minimum.

Restrictions/constraints:

1. the shipments are non negative, i.e., $x_{ij} \ge 0$;
2. capacity constraints: $\sum_{j=1}^{n} x_{ij} = u_i$ for $1 \le i \le m$;
3. needs constraints: $\sum_{i=1}^{m} x_{ij} = v_j$ for $1 \le j \le n$.

The problem is feasible if the total of sources is at least the total of needs, i.e., $\sum_{i=1}^{m} u_i \ge \sum_{i=1}^{n} v_i$. One may assume that $\sum_{i=1}^{m} u_i = \sum_{i=1}^{n} v_i$ because if $\sum_{i=1}^{m} u_i > \sum_{i=1}^{n} v_i$ we may introduce an imaginary destination Y_{n+1} with $v_{n+1} = \sum_{i=1}^{m} u_i - \sum_{i=1}^{n} v_i$ and cost $c_{i(n+1)} = 0$ for $1 \le i \le n$. That is, the excess is placed at an imaginary destination with cost zero. See [12, pp. 61–62] and [19, pp. 3–8] containing illuminating examples of application. [12, pp. 61–62] contains also a historical description and evolution of linear programming.

Let H be the Hilbert space of real matrices with m rows and n columns, $H = \mathbb{R}^{m\times n}$, with the inner product

$$\langle A, B\rangle = \text{trace}\,\left(A\, B^t\right).$$

Let $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ be vectors with non negative components. Consider the set of matrices $A \in \mathbb{R}^{m\times n}$ with non negative entries such that the vector sum of their rows is $\mathbf{u}$ and the vector sum of their columns is $\mathbf{v}$, that is,

$$\sum_{j=1}^{n} a_{ij} = u_i, \quad 1 \le i \le m, \text{ and } \sum_{i=1}^{m} a_{ij} = v_j, \quad 1 \le j \le n. \tag{1.1}$$

Let us denote this set of matrices by $\mathcal{N}(\mathbf{u}, \mathbf{v})$ which is referred as the transportation polytope, see [20] for many examples and the simplex method, also [35, 43]; and [8] for properties of this set of matrices; $\mathcal{N}(\mathbf{u}, \mathbf{v})$ is a compact convex set in $H = \mathbb{R}^{m\times n}$.

Therefore each matrix $A \in \mathcal{N}(\mathbf{u}, \mathbf{v})$ represents a transportation plan which yields a cost $\langle A, C\rangle$, where $C = c_{ij}$ is the cost matrix. The question is then to find a transportation plan that minimizes the total cost, that is, to find

$$\min_{A\in\mathcal{N}(\mathbf{u},\mathbf{v})} \langle A, C\rangle,$$

and a matrix A attaining this minimum. Since $\mathcal{N}(\mathbf{u}, \mathbf{v})$ is compact there must be a transportation plan that attains the minimum. However to find the optimal plan can be extremely long and computationally costly. Linear programming was invented to solve these type of problems efficiently, in particular, a method developed with this purpose is the simplex method, see [12, Chapter 5] and [19, Chapter 1].

1.2 Monge Problem

From [24]: “When we have to transport land from one place to another, we usually give the name of excavation to the volume of land that we must transport, and the name of embankment to the space they must occupy after transport. The cost of transporting a molecule being, all other things being equal, proportional to its weight and the space it is made to travel, and therefore the product of total transport must be proportional to the sum of the products of molecules multiplied by the space covered, it follows that the cut and fill being given in figure and position, it is not unimportant that such molecule of the cut is transported in such or such other place of the fill, but that 'There is a certain distribution of molecules from the first to the second, according to which the sum of these products will be the smallest possible, and the price of total transport will be a minimum.”

This problem can be formally described as follows. Suppose (X, μ) and (Y, ν) are given measure spaces with $\mu(X) = \nu(Y)$, and let $c : X \times Y \to [0, +\infty)$ be a function, the cost. A function $T : X \to Y$ preserves the measures μ and ν if $\mu\left(T^{-1}(E)\right) = \nu(E)$ for each set $E \subset Y$; T *is called a transport map.* Let $\mathcal{S}(\mu, \nu)$ be the class of maps preserving μ and ν. Monge question can then be phrased as follows: Find $T \in \mathcal{S}(\mu, \nu)$ such that the integral

$$\int_X c\,(x, Tx)\; d\mu$$

is minimum among all $T \in \mathcal{S}(\mu, \nu)$. In Monge problem, the cost is the Euclidean distance $c(x, y) = |x - y|$.

At this point, all this is formal and measurability properties are needed for the precise formulation. We introduce the *push forward of the measure* μ *through* T by $T_{\#}\mu(E) = \mu\left(T^{-1}(E)\right)$ for $E \subset Y$. It will be proved later that $T_{\#}\mu$ is a measure and the problem above can be precisely formulated and solved under conditions on the cost c.

Notice that for certain measures μ, ν we might have $\mathcal{S}(\mu, \nu) = \emptyset$, i.e., there might not exist any measure preserving map. In fact, this is the case if for example, $X = Y = \mathbb{R}$, $\mu = \delta_0$, and $\nu = \frac{1}{2}\left(\delta_{-1} + \delta_1\right)$.

Remark 1.1 Suppose X, Y are two domains in $\mathbb{R}^n$, the measures μ and ν have continuous densities ρ and σ respectively with respect to Lebesgue measure, and $T : X \to Y$ is a measure preserving map that is a C^1 diffeomorphism. Let $\phi \in C(Y)$. From the formula of change of variables

$$\int_Y \phi(y)\,\sigma(y)\,dy = \int_X \phi(Tx)\,\sigma(Tx)\,|\det DT(x)|\,dx.$$

Since T is measure preserving $\mu\left(T^{-1}E\right) = \nu(E)$ for each Borel set $E \subset Y$ which can be rewritten as

$$\int_Y \chi_E(y)\,\sigma(y)\,dy = \int_X \chi_{T^{-1}E}(x)\,\rho(x)\,dx = \int_X \chi_E(Tx)\,\rho(x)\,dx.$$

If ϕ is a simple function, $\phi(y) = \sum_{j=1}^k \alpha_j\,\chi_{E_j}(y)$, then

$$\int_Y \phi(y)\,\sigma(y)\,dy = \int_X \phi(Tx)\,\rho(x)\,dx, \tag{1.2}$$

and since simple functions are dense in $C(Y)$ we obtain that (1.2) holds for each $\phi \in C(Y)$ (see Lemma 5.4 for a more general result). Therefore we obtain the formula

$$\int_X \phi(Tx)\,\rho(x)\,dx = \int_X \phi(Tx)\,\sigma(Tx)\,|\det DT(x)|\,dx$$

for each $\phi \in C(Y)$ which implies that T satisfies the differential equation

$$\rho(x) = \sigma(Tx)\,|\det DT(x)|.$$

1.3 Kantorovitch Problem

Let X and Y be metric spaces, (X, μ) and (Y, ν) Borel measure spaces with $\mu(X) = \nu(Y) = 1$,[1] and let $c : X \times Y \to \mathbb{R}_{\geq 0}$ be a measurable function in the product space $(X \times Y, \mu \otimes \nu)$. Consider the class $\Pi(\mu, \nu)$ of all measures γ in $X \times Y$ satisfying $\gamma(A \times Y) = \mu(A)$ for all μ-measurable subsets $A \subset X$ and $\gamma(X \times B) = \nu(B)$ for all ν-measurable subsets $B \subset Y$ (that is, the marginals of γ are μ and ν). Notice that this implies that $\gamma(X \times Y) = 1$. *The measure γ is called a transport plan.* Notice the measure $\mu \otimes \nu \in \Pi(\mu, \nu)$ so the class of admissible measures $\Pi(\mu, \nu)$ is always a non empty convex set. The Kantorovitch problem consists in minimizing

$$\int_{X\times Y} c(x, y)\,d\gamma$$

over all $\gamma \in \Pi(\mu, \nu)$.

Remark 1.2 We show that when the measures μ and ν are discrete Kantorovitch's problem is the transportation problem explained in Sect. 1.1. Indeed, let $\mu =$

[1] If $\mu(X) = \nu(Y)$ not necessarily equal one, we normalize the measures taking $\tilde{\mu} = \mu/\mu(X)$ and $\tilde{\nu} = \nu/\nu(Y)$.

$\sum_{i=1}^{m} u_i\, \delta_{X_i}$ and $\nu = \sum_{j=1}^{n} v_j\, \delta_{Y_j}$ with $\sum_{i=1}^{m} u_i = \sum_{j=1}^{n} v_j = 1$. Take $\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} u_i\, v_j\, \delta_{(X_i, Y_j)}$. Then

$$\gamma(A \times Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} u_i\, v_j\, \delta_{(X_i,Y_j)}\,(A \times Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} u_i\, v_j\, \delta_{X_i}\,(A)$$
$$= \sum_{j=1}^{n} v_j \sum_{i=1}^{m} u_i\, \delta_{X_i}\,(A) = \nu(Y)\,\mu(A) = \mu(A).$$

Similarly, $\gamma(X \times B) = \nu(B)$ for $B \subset Y$. So $\gamma \in \Pi(\mu, \nu)$. On the other hand, if $\pi \in \Pi(\mu, \nu)$, we shall prove that

$$\pi = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, \delta_{(X_i,Y_j)}$$

with $A = (a_{ij}) \in \mathcal{N}(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (u_1, \cdots, u_m)$ and $\mathbf{v} = (v_1, \cdots, v_n)$. Indeed, first notice that

$$\operatorname{supp}(\pi) = \{(x, y) : \text{there is a neighborhood } N_{(x,y)} \text{ such that } \pi(N_{(x,y)}) = 0\}^c$$
$$= \{(X_i, Y_j) : 1 \le i \le m, 1 \le j \le n\}$$

because if $(x, y) \neq (X_i, Y_j)$ for all i, j, then $x \neq X_i$ or $y \neq Y_j$ so there is a neighborhood N_x such that $X_i \notin N_x$ and so $\pi(N_x \times Y) = \mu(N_x) = 0$, or there is a neighborhood N_y such that $Y_j \notin N_y$ and so $\pi(X \times N_y) = \nu(N_y) = 0$. Hence $\pi = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, \delta_{(X_i,Y_j)}$ with $a_{ij} = \pi\left((X_i, Y_j)\right)$.

Therefore $\Pi(\mu, \nu)$ can be identified with $\mathcal{N}(\mathbf{u}, \mathbf{v})$ so

$$\int_{X \times Y} c(x, y)\, d\pi = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, c(X_i, Y_j)$$

and Kantorovitch's problem is the transportation problem.

Remark 1.3 Suppose μ and ν are probability Borel measures in X and Y respectively and let $T \in \mathcal{S}(\mu, \nu)$ be a measure preserving map, i.e., $T_{\#}\mu = \nu$. We show that T gives rise to a measure $\gamma \in \Pi(\mu, \nu)$ as follows. Let $I : X \to X$ be the identity map, and let $S : X \to X \times Y$ be defined by $Sx = (x, Tx)$. Define $\gamma_T = S_{\#}\mu$, that is, for $E \subset X \times Y$, $\gamma_T(E) = \mu\left(S^{-1}(E)\right)$. If $A \subset X$ and $B \subset Y$, then

$$\gamma_T(A \times Y) = \mu\left(S^{-1}\,(A \times Y)\right) = \mu\,(A)$$
$$\gamma_T(X \times B) = \mu\left(S^{-1}\,(X \times B)\right) = \mu\left(T^{-1}(B)\right) = \nu(B),$$