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Elements of Mathematical Analysis

An Informal Introduction for Physics and Engineering Students

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Preface

This short textbook is by no means a complete book on mathematical analysis. It is basically a concise, informal introduction to differentiation and integration of real functions of a single variable, supplemented with an elementary discussion of first-order differential equations, an introduction to differentiation and integration in higher dimensions, and an introduction to complex analysis. Functional series (and, in particular, power series) are also discussed. The book may serve as a tutorial resource in a short-term introductory course of mathematical analysis for beginning students of physics and engineering who need to use differential and integral calculus primarily for applications.

Having taught introductory Physics at the Hellenic Naval Academy for over three decades, I have often experienced situations where my first-year undergraduates needed reinforcement of their background in advanced calculus in order to properly follow the Physics course from the outset. This need led to the idea of writing a short, practical handbook that would be especially useful for self-study “in a hurry”. The present textbook is a translated and expanded version of the author’s lecture notes written originally in Greek. Proofs of theoretical statements are limited to those considered pedagogically useful, while the theory is amply supplemented with carefully chosen examples. For a deeper study of the subject, the reader is referred to the bibliography cited at the end of the book.

Despite the essentially practical character of the book, proper attention is given to conceptual subtleties inherent in the subject. In particular, the concept of the differential of a function is carefully examined and its relation to the “differential” inside an integral is explained. For pedagogical purposes, the discussion of the indefinite integral—as an infinite collection of antiderivatives—precedes that of the definite integral; it is shown, however, that the latter concept leads in a natural way to the former by allowing variable limits of integration.

Appendix contains useful mathematical formulas and properties needed for the exercises, as well as a more detailed discussion of the concept of continuity of a function and its relationship with differentiability. Finally, answers to selected exercises are provided.

Piraeus, Greece
August 2023

Costas J. Papachristou

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Chapter 1
Functions

1.1 Real Numbers

There are various sets of numbers in mathematics, such as the set of *natural numbers*, $N = \{1, 2, 3, \ldots\}$, the set of *integers*, $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$, and the set of *rational numbers*, $Q = \{p/q$, where p, q are integers and $q \neq 0\}$. Numbers such as $\sqrt{2}$, $\sqrt{3}$, ln 3, etc., which *cannot* be expressed as quotients p/q of integers, are called *irrational*. Rational and irrational numbers together constitute the set of *real numbers*, R.

In the set R of real numbers one may define various types of *intervals*:

Open interval:	$(a, b) = \{x/x \in R, \quad a < x < b\}$
Closed interval:	$[a, b] = \{x/x \in R, \quad a \leq x \leq b\}$
Semi-closed intervals:	$[a, b) = \{x/x \in R, \quad a \leq x < b\}$ $(a, b] = \{x/x \in R, \quad a < x \leq b\}$
Infinite intervals:	$(-\infty, c), (c, +\infty), (-\infty, c], [c, +\infty), (-\infty, +\infty)$

1.2 Functions

Let $D \subseteq R$ be a subset of R. We consider a rule f: $D \to R$, such that, to every element $x \in D$ there corresponds a *unique* element $y \in R$ (two or more elements of D *may*, however, correspond to the same element of R). We write:

$$(x \in D) \overset{f}{\mapsto} (y \in R) \text{ or } y = f(x).$$

The rule f constitutes a *real function*. We say that the *dependent variable* y is a function of the *independent variable* x. The set D is called the *domain of definition* of f, while the set $\{y = f(x)/x \in D\} \equiv f(D)$ is called the *range* of f.

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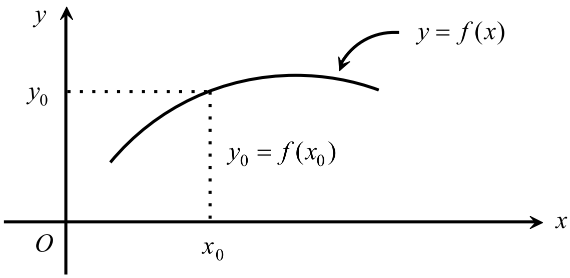

Fig. 1.1 Graph of a function

Given a function $y = f(x)$ we can draw the corresponding *graph* (Fig. 1.1). We assume that the quantities x and y are *dimensionless* and, moreover, equal lengths on the x- and y-axes correspond to *equal changes* of x and y.

A function $y = f(x)$ is said to be *continuous* at the point $x = x_0$ if its value $y_0 = f(x_0)$ at that point is defined and is equal to the limit of $f(x)$ as $x \to x_0$:

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In practical terms we may say that the graph of $f(x)$ is a continuous curve at $x = x_0$ (it does not "break" into two separate curves at this point). If we set $x–x_0 = \Delta x$ and $f(x)–f(x_0) = y–y_0 = \Delta y$, then, by the definition of a continuous function it follows that $\Delta y \to 0$ when $\Delta x \to 0$. More on continuous functions can be found in the Appendix.

Below is a list of some elementary functions:

Constant function: $y = f(x) = c \quad (c \in R)$.
Power function: $y = f(x) = x^a \quad (a \in R)$.
Exponential function: $y = f(x) = e^x$.
Logarithmic function: $y = f(x) = \ln x$.
Trigonometric functions: $y = f(x) = \sin x, \cos x, \tan x, \cot x$.
Inverse trigonometric functions: $y = f(x) = \arcsin x, \arccos x, \arctan x, \text{arc}\cot x$.

By combining elementary functions we can construct composite functions. Let us consider the functions $y = g(u)$ and $u = h(x)$. We write

$$y = g[h(x)] \equiv (g \circ h)(x).$$

We thus define the *composite function* $f = g \circ h$, so that

$$y = f(x) = g[h(x)] \equiv (g \circ h)(x).$$

To simplify our notation we may write $y = y(x)$ instead of the more explicit $y = f(x)$. Similarly, $y = y(u)$ and $u = u(x)$. Then,