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Macdonald Polynomials

Commuting Family of q-Difference Operators and Their Joint Eigenfunctions

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Preface

This book is intended to provide an introduction to the theory of Macdonald polynomials from the viewpoint of commuting q-difference operators and their joint eigenfunctions. It is an extended version of lecture notes for a series of online lectures "Introduction to Macdonald Polynomials," which I gave at KTH Royal Institute of Technology, Stockholm, during the period of February and March 2021.

Macdonald polynomials refer to a class of symmetric q-orthogonal polynomials in many variables. They include important classes of special functions such as Schur functions and Hall–Littlewood polynomials, and play important roles in various situations of mathematics and physics. After an overview of Schur functions, I will introduce Macdonald polynomials (of type A, in the GL_{n} version) as eigenfunctions of a q-difference operator, called the Macdonald–Ruijsenaars operator, in the ring of symmetric polynomials. Starting from this definition, I explain various remarkable properties of Macdonald polynomials such as orthogonality, evaluation formulas and self-duality, with emphasis on the roles of commuting q-difference operators.

The main reference for this theory is in Macdonald's book

> *Symmetric Functions and Hall Polynomials.* Second Edition. Oxford University Press, 1995, x+475 pp.
>
> Chapter VI: Symmetric functions with two parameters.

A characteristic feature of Macdonald's approach in his monograph is the use of symmetric functions in an *infinite number of variables*. In view of the introductory nature of this book, I decided to avoid the approach using infinite variables here, and to put more emphasis instead on the roles of the commuting family of q-difference operators for which Macdonald polynomials are joint eigenfunctions. I tried to make this book self-contained, and to give proofs to fundamental formulas in Macdonald theory within the framework of finite variables, as much as possible. I hope that this exposition will be helpful to a wider class of readers with various backgrounds.

In this book, I adopted the *classical* approach to Macdonald polynomials which does *not* rely on the theory of (double) affine Hecke algebras. For the Macdonald–Cherednik theory based on affine Hecke algebras, I refer the reader to Macdonald

[22], Cherednik [5] and other textbooks. In this direction, I only added a chapter on affine Hecke algebras and q-Dunkl operators, to provide an idea (without getting into the detail of proofs) about how the commuting family of q-difference operators arises in the framework of affine Hecke algebras.

I also included some materials which I could not deal with in the online lectures I gave at KTH. I really enjoyed meeting regularly online with many friends from various parts of the world, with whom I shared scientific interests and discussions. My thanks go to all the participants of the online lectures. I am grateful to the Knut and Alice Wallenberg Foundation for funding my guest professorship of the year 2020/2021 at KTH, which provided me with an invaluable opportunity of giving lectures and writing lecture notes on this subject of great concern to myself. Also, I would like to express my thanks to colleagues at KTH, especially Edwin Langmann and Jonatan Lenells, for their kind hospitality and friendship during my stay in Stockholm.

Tokyo, Japan Masatoshi Noumi

Contents

Chapter 1
Overview of Macdonald Polynomials

Abstract The Macdonald polynomials are a family of symmetric polynomials in n variables indexed by partitions. They are characterized as joint eigenfunctions of a commuting family of q-difference operators acting on the ring of symmetric polynomials. This chapter is a summary of the material which is developed in the rest of this book. Some of the notations used throughout this book is also introduced.

1.1 Macdonald Polynomials

We begin with an overview of the *Macdonald polynomials*[1]

$$P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n} \tag{1.1}$$

which we are going to discuss throughout this book. They are symmetric polynomials in n variables $x = (x_1, \ldots, x_n)$ with parameters $q, t \in \mathbb{C}$, indexed by the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\ell(\lambda) \le n$. By a *partition*, we mean a weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots); \quad \lambda_i \in \mathbb{N} = \mathbb{Z}_{\ge 0} \ \ (i = 1, 2, \ldots), \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge 0, \tag{1.2}$$

with a finite number of *parts* (nonzero components); the 0's in the tail are frequently omitted. We denote by $\ell(\lambda) \in \mathbb{N}$ the number of parts of λ, and by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ the degree (sum of all parts) of λ. We denote by $\mathcal{P}$ the set of all partitions, and by $\mathcal{P}_n$ the set of all $\lambda \in \mathcal{P}$ with $\ell(\lambda) \le n$. We identify $\mathcal{P}_n$ with the indexing set of Macdonald polynomials:

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[1] In this book, we use the term "Macdonald polynomials" in the narrow sense, meaning Macdonald polynomials of type A_{n-1} (in the GL_n version). They are called the "symmetric functions with two parameters" in Macdonald's monograph [20, Chap. VI]. They are a special case of Macdonald polynomials associated with root systems, which are Weyl group invariant Laurent polynomials with parameters q and $t = (t_\alpha)_\alpha$. The Macdonald polynomials associated with non-reduced root systems (of type $C^\vee C$ in the terminology of [22]) are called the *Koornwinder polynomials*.

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$$\mathcal{P}_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}. \tag{1.3}$$

We denote by $\mathfrak{S}_n$ the *symmetric group* of degree n (the set of all bijections $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$). It acts on the ring $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ of polynomials in $x = (x_1, \dots, x_n)$ by permuting the indices of the variables x_i. We denote by $\mathbb{C}[x]^{\mathfrak{S}_n}$ the ring of symmetric ($\mathfrak{S}_n$-invariant) polynomials in x.

As a $\mathbb{C}$-vector space, $\mathbb{C}[x]^{\mathfrak{S}_n}$ has two fundamental bases,

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x) = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, s_\lambda(x), \tag{1.4}$$

both of which are indexed by $\mathcal{P}_n$. These symmetric polynomials

$$m_\lambda(x) = \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = x^\lambda + \cdots, \quad s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n} = x^\lambda + \cdots \tag{1.5}$$

are called the *monomial symmetric functions* (orbit sums) and the *Schur functions*,[2] respectively. Both $m_\lambda(x)$ and $s_\lambda(x)$ have the leading term $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ with respect to a partial order $\leq$ of partitions, called the *dominance order* (see (2.54) for the definition). The Macdonald polynomials provide a family of $\mathbb{C}$-bases of $\mathbb{C}[x]^{\mathfrak{S}_n}$ with two parameters (q, t), including $m_\lambda(x)$ and $s_\lambda(x)$ as special cases.

The Macdonald polynomials $P_\lambda(x; q, t)$ are defined (or characterized) as the eigenfunctions of the *Macdonald–Ruijsenaars q-difference operator*

$$D_x = \sum_{i=1}^n \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{t x_i - x_j}{x_i - x_j} T_{q,x_i} = \prod_{j=2}^n \frac{t x_1 - x_j}{x_1 - x_j} T_{q,x_1} + \cdots \tag{1.6}$$

acting on $\mathbb{C}[x]^{\mathfrak{S}_n}$. Here, T_{q,x_i} stands for the *q-shift operator* with respect to the variable x_i: $T_{q,x_i} f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, q x_i, \dots, x_n)$.

Theorem 1.1 (Macdonald) *For each partition $\lambda \in \mathcal{P}_n$ with $\ell(\lambda) \leq n$, there exists a unique symmetric polynomial $P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ in x, homogeneous of degree $|\lambda|$ and depending rationally on (q, t), such that*

$$(1) \quad D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad d_\lambda = q^{\lambda_1} t^{n-1} + q^{\lambda_2} t^{n-2} + \cdots + q^{\lambda_n}, \tag{1.7}$$

$$(2) \quad P_\lambda(x) = m_\lambda(x) + (\textit{lower-order terms with respect to } \leq). \tag{1.8}$$

This theorem will be proved in Sect. 4.1 (Theorem 4.1).

[2] Some people would restrict the usage of the term "symmetric functions" to the case of symmetric formal power series in an infinite number of variables $x = (x_1, x_2, \dots)$. We will not strictly follow this rule, since polynomials are functions, whereas formal power series are *not* functions in general.

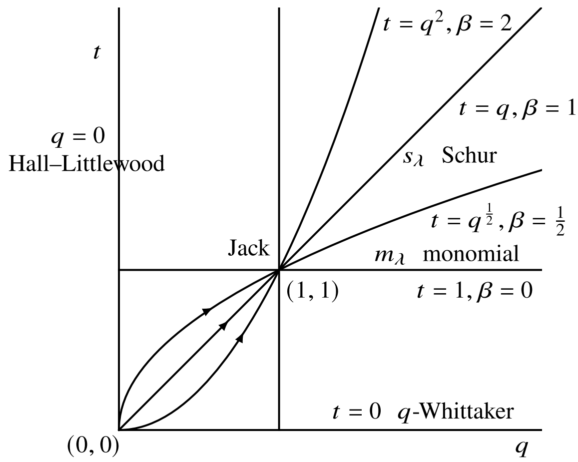

Fig. 1.1 Space of parameters (q, t)

For generic (q, t), the Macdonald polynomials $P_\lambda(x; q, t)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x; q, t). \tag{1.9}$$

They specialize to $m_\lambda(x)$ when $t = 1$, and to $s_\lambda(x)$ when $t = q$. Also, in the limit as $q \to 1$ with scaling $t = q^\beta$, they recover the *Jack polynomials* $P_\lambda^{(\beta)}(x) = \lim_{q \to 1} P_\lambda(x; q, q^\beta)$.[3] Two other important special cases are the *Hall–Littlewood polynomials* $P_\lambda(x; t) = P_\lambda(x; 0, t)$ with $q = 0$, and the *q-Whittaker functions* $P_\lambda(x; q, 0)$ with $t = 0$ (Fig. 1.1).

We remark that the Jack polynomials $P_\lambda^{(\beta)}(x)$ are orthogonal polynomials associated with the Heckman–Opdam system (or Calogero–Sutherland system) of type A_{n-1}; we refer the reader to [15, Chap. 8] and Sect. 5.6 of this book for Heckman–Opdam and Calogero–Sutherland systems. They are the polynomial joint eigenfunctions of a commuting family of differential operators, called the *Sekiguchi–Debiard operators*. The Macdonald polynomials are also the orthogonal polynomials (polynomial joint eigenfunctions) associated with the commuting family of Macdonald–Ruijsenaars q-difference operators, which define a difference version of the differential system of Heckman–Opdam (relativistic version of the non-relativistic system of Calogero–Sutherland).

Remark 1.1 In the parameterization $t = q^\beta$, the three values $\beta = \frac{1}{2}, 1, 2$ are special in this case of type A_{n-1}. The Jack polynomials $P_\lambda^{(\beta)}(x)$ for $\beta = \frac{1}{2}, 1, 2$ arise as the zonal spherical functions associated with finite-dimensional representations of the symmetric pairs $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_n, \mathfrak{so}_n), (\mathfrak{gl}_n \times \mathfrak{gl}_n, \mathfrak{gl}_n), (\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$, respectively (see Gangolli–Varadarajan [7] or Heckman–Schlichtkrull [11]). In particu-

[3] In Macdonald's monograph [20, Sect. VI.10], the notation $P_\lambda^{(\alpha)}$ is used for Jack polynomials with the convention $\alpha = 1/\beta$.

lar, $P_\lambda^{(\beta)}(x)$ with $\beta = \frac{1}{2}$ are called the *zonal polynomials*, and play crucial roles in statistics. The Macdonald polynomials $P_\lambda(x; q, t)$ with $t = q^{\frac{1}{2}}, q, q^2$ are similarly interpreted as the zonal spherical functions of the corresponding quantum symmetric pairs $(U_q(\mathfrak{g}), U_q^{\mathrm{tw}}(\mathfrak{k}))$ (see Noumi [24], Noumi–Sugitani [28]). Here, $U_q(\mathfrak{g})$ denotes the standard quantized universal enveloping algebra of Drinfeld and Jimbo, whereas $U_q^{\mathrm{tw}}(\mathfrak{k})$ is a coideal subalgebra of $U_q(\mathfrak{g})$ corresponding to the subalgebra $U(\mathfrak{k}) \subseteq U(\mathfrak{g})$.

1.2 Fundamental Properties of Macdonald Polynomials

The Macdonald polynomials have various remarkable properties. We highlight below some of the fundamental properties of Macdonald polynomials, which are in fact intimately related with each other.

(a) Specializations: As we already mentioned above, from the Macdonald polynomials $P_\lambda(x; q, t)$, one can obtain the Schur, Jack, Hall–Littlewood and q-Whittaker functions by specializations or limiting procedures with respect to the parameters (q, t).

(b) Orthogonality: When $q, t \in \mathbb{R}$ and $|q| < 1, |t| < 1$, the Macdonald polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ are orthogonal polynomials on the torus $\mathbb{T}^n = \{|x_1| = \cdots = |x_n| = 1\}$ with respect to the scalar product defined by a certain weight function. Explicit formulas are also known for the square norms of $P_\lambda(x)$.

(c) Commuting family of q-difference operators: There exists a commuting family of higher-order q-difference operators $D_x^{(1)}, \ldots, D_x^{(n)}$ with $D_x^{(1)} = D_x$, acting on the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials. The operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ are algebraically independent, and the Macdonald polynomials $P_\lambda(x)$ are joint eigenfunctions of them. See Sect. 5.3 for the explicit formulas of these operators.

(d) Principal specialization and self-duality: The value of $P_\lambda(x)$ at the base point $x = t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$ can be evaluated explicitly as a product of simple factors. Also, the normalized Macdonald polynomials $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\delta)$ are self-dual in the sense $\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda)$ with respect to discrete sets of the position variables $x = q^\mu t^\delta$ and the spectral variables $\xi = q^\lambda t^\delta$.

(e) Pieri formula: The Macdonald polynomial $P_\mu(x)$ of degree d multiplied by the elementary symmetric function $e_r(x)$ of degree r $(r = 0, 1, 2, \ldots, n)$ can be expanded into a linear combination of Macdonald polynomials of degree $d + r$ with explicitly determined coefficients. This Pieri formula is obtained from the eigenfunction equations for the higher-order q-difference operator $D_x^{(r)}$ via the self-duality of Macdonald polynomials.

(f) Recurrence formula and tableau representation: The Macdonald polynomials of n variables $x = (x_1, \ldots, x_n)$ admit a recurrence formula regarding the number of variables with explicitly determined coefficients. A repeated application of this