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Svetlin Georgiev

Boundary Value Problems

Essential Fractional Dynamic Equations on Time Scales

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Preface

The bilateral Laplace transform on arbitrary time scales and its absolute convergence, uniform convergence, and inversion integral were investigated in the recent paper [7]. These results can be involved for investigations of initial value problems, boundary value problems, initial boundary value problems for fractional Riemann-Liouville and Caputo dynamic equations and for impulsive fractional Riemann-Liouville and Caputo dynamic equations on arbitrary time scales. The bilateral Laplace transform admits the shifting problem to be considered on the whole time scale. In the book "Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales", Springer, 2017, some initial value problems for fractional Riemann-Liouville and Caputo dynamic equations were investigated on isolated time scales, as well as the shifting problem is considered on the right-hand side of isolated time scales. The main aim of this book is to represent an investigation of boundary value problems for fractional Riemann-Liouville and Caputo dynamic equations on arbitrary time scales and to be investigated the shifting problem on the whole time scale.

"Boundary Value Problems: Essential Fractional Dynamic Equations on Time Scales" contain 4 chapters. In Chap. 1 the Laplace transform on arbitrary time scales is introduced. It investigated the decay of the exponential function and the Laplace transform is investigated for convergence and differentiability. It deducted an inverse formula for the Laplace transform and it studied the Laplace transform of power series. The bilateral Laplace transform on time scales is investigated and its absolute convergence, uniform convergence, and inversion integral are shown. Chapter 2 is devoted to the generalized convolutions of functions on arbitrary time scales and it investigated the shifting problem for existence of solutions. In Chap. 3 a short overview of the basic definitions and basic facts from fractional dynamic calculus on time scales is made. In Chap. 4 we introduced boundary value problems and initial boundary value problems for some classes Riemann-Liouville fractional dynamic equations in the cases when $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$. They deducted the main integral representations of the solutions and they are given some criteria for existence and uniqueness of the solutions.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers, and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines.

Paris, France Svetlin Georgiev
January 2023

Contents

1 The Laplace Transform on Time Scales

1.1 Definition of the Laplace Transform. Properties

Let $\mathbb{T}$ be a time scale with forward jump operator, backward jump operator, delta differentiation operator and nabla differentiation operator σ, ρ, Δ and ∇, respectively. Let also, $\sup \mathbb{T} = \infty$. Take $s \in \mathbb{T}$. If $z \in \mathbb{C}$ and $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}^\kappa$, then

$$(\ominus z)(t) = -\frac{z}{1 + \mu(t)z}$$

and

$$\begin{aligned} 1 + \mu(t)(\ominus z)(t) &= 1 - \frac{\mu(t)z}{1 + \mu(t)z} \\ &= \frac{1}{1 + \mu(t)z} \end{aligned}$$

for any $t \in \mathbb{T}^\kappa$, and hence, $e_{\ominus z}(\cdot, s)$ is well defined on $\mathbb{T}^\kappa$.

Definition 1.1 Suppose that $f : \mathbb{T} \to \mathbb{R}$ is regulated. Then the Laplace transform of f is defined by

$$\mathscr{L}(f)(z, s) = \int_s^\infty e^\sigma_{\ominus z}(t, s) f(t) \Delta t \tag{1.1}$$

for $z \in \mathbb{C}$ for which $1 + \mu(t)z \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral (1.1) exists. When $s = 0$, we will write $\mathscr{L}(f)(z)$.

Remark 1.1 Let $f : \mathbb{T} \to \mathbb{R}$ be regulated. With $\mathscr{D}\{f\}$ we will denote the set of all complex numbers z for which $1 + \mu(t)z \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral (1.1) exists.

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Theorem 1.1 (Linearity) *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are regulated. Then, for any constants α and β, we have*

$$\mathscr{L}(\alpha f + \beta g)(z, s) = \alpha \mathscr{L}(f)(z, s) + \beta \mathscr{L}(g)(z, s)$$

for $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\{g\}$, $t \in \mathbb{T}$.

Proof Let $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\{g\}$, $t \in \mathbb{T}$. Then, we have

$$\begin{aligned}
\mathscr{L}(\alpha f + \beta g)(z, s) &= \int_s^\infty e^\sigma_{\ominus z}(t, s)(\alpha f + \beta g)(t)\Delta t \\
&= \int_s^\infty e^\sigma_{\ominus z}(t, s)(\alpha f(t) + \beta g(t))\Delta t \\
&= \int_s^\infty e^\sigma_{\ominus z}(t, s)\alpha f(t)\Delta t + \int_s^\infty e^\sigma_{\ominus z}(t, s)\beta g(t)\Delta t \\
&= \alpha \int_s^\infty e^\sigma_{\ominus z}(t, s) f(t)\Delta t + \beta \int_s^\infty e^\sigma_{\ominus z}(t, s) g(t)\Delta t \\
&= \alpha \mathscr{L}(f)(z, s) + \beta \mathscr{L}(g)(z, s).
\end{aligned}$$

This completes the proof.

Lemma 1.1 *Let $z \in \mathbb{C}$. Then*

$$\begin{aligned}
e^\sigma_{\ominus z}(t, s) &= \frac{e_{\ominus z}(t, s)}{1 + \mu(t) z} \\
&= -\frac{(\ominus z)(t)}{z} e_{\ominus z}(t, s), \quad t \in \mathbb{T}.
\end{aligned}$$

Proof For $z \in \mathbb{C}$ and $t \in \mathbb{T}$, we have

$$\begin{aligned}
e^\sigma_{\ominus z}(t, s) &= e_{\ominus z}(t, s) + \mu(t) e^\Delta_{\ominus z}(t, s) \\
&= e_{\ominus z}(t, s) + (\ominus z)(t)\mu(t) e_{\ominus z}(t, s) \\
&= (1 + (\ominus z)(t)\mu(t)) e_{\ominus z}(t, s) \\
&= \left(1 - \frac{\mu(t) z}{1 + \mu(t) z}\right) e_{\ominus z}(t, s) \\
&= \frac{1}{1 + \mu(t) z} e_{\ominus z}(t, s) \\
&= -\frac{1}{z}\left(-\frac{z}{1 + \mu(t) z}\right) e_{\ominus z}(t, s) \\
&= -\frac{(\ominus z)(t)}{z} e_{\ominus z}(t, s).
\end{aligned}$$

This completes the proof.

Theorem 1.2 *Assume $f : \mathbb{T} \to \mathbb{C}$ is such that f^{Δ^l}, $l \in \{0, \ldots, k\}$, are regulated. Then*

$$\mathscr{L}\left(f^{\Delta^k}\right)(z, s) = z^k \mathscr{L}(f)(z, s) - \sum_{l=0}^{k-1} z^l f^{\Delta^{k-1-l}}(s) \tag{1.2}$$

for those $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\{f^{\Delta}\} \bigcap \ldots \bigcap \mathscr{D}\left\{f^{\Delta^k}\right\}$ satisfying

$$\lim_{t\to\infty}\left(f^{\Delta^l}(t) e_{\ominus z}(t, s)\right) = 0, \quad l \in \{0, \ldots, k-1\}, \tag{1.3}$$

for any $t \in \mathbb{T}$.

Proof Let $t \in \mathbb{T}$. We will use induction.

1. Let $k = 1$. Take $z \in D\{f\} \bigcap D\left\{f^{\Delta}\right\}$ so that

$$\lim_{t\to\infty}(f(t) e_{\ominus z}(t, s)) = 0.$$

   Integrating by parts and using Lemma 1.1 and the condition (1.3), we get

$$\begin{aligned}
\mathscr{L}\left(f^{\Delta}\right)(z, s) &= \int_s^{\infty} f^{\Delta}(t) e^{\sigma}_{\ominus z}(t, s)\, \Delta t \\
&= f(t) e_{\ominus z}(t, s)\Big|_{t=s}^{t\to\infty} - \int_s^{\infty} f(t)(\ominus z)(t)\, e_{\ominus z}(t, s)\, \Delta t \\
&= -f(s) + z \int_s^{\infty} f(t) e^{\sigma}_{\ominus z}(t, s)\, \Delta t \\
&= -f(s) + z \mathscr{L}(f)(z, s).
\end{aligned}$$

2. Assume that (1.2) holds for some $k \in \mathbb{N}$ and for those $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\left\{f^{\Delta}\right\} \bigcap \ldots \bigcap \mathscr{D}\left\{f^{\Delta^k}\right\}$ for which (1.3) hold.
3. We will prove

$$\mathscr{L}\left(f^{\Delta^{k+1}}\right)(z, s) = z^{k+1} \mathscr{L}(f)(z, s) - \sum_{l=0}^{k} z^l f^{\Delta^{k-l}}(s)$$

   for those $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\left\{f^{\Delta}\right\} \bigcap \ldots \bigcap \mathscr{D}\left\{f^{\Delta^{k+1}}\right\}$ for which

$$\lim_{t\to\infty}\left(f^{\Delta^l}(t) e_{\ominus z}(t, s)\right) = 0, \quad l \in \{0, \ldots, k\}. \tag{1.4}$$

Really, let $z \in \mathscr{D}\{f\} \bigcap \mathscr{D}\left\{f^{\Delta}\right\} \bigcap \ldots \bigcap \mathscr{D}\left\{f^{\Delta^{k+1}}\right\}$ be such that (1.4) hold. Integrating by parts and using Lemma 1.1 and (1.2), we get

$$
\begin{aligned}
\mathscr{L}\left(f^{\Delta^{k+1}}\right)(z, s) &= \int_s^{\infty} f^{\Delta^{k+1}}(t) e_{\ominus z}^{\sigma}(t, s)\,\Delta t \\
&= f^{\Delta^k}(t) e_{\ominus z}(t, s)\Big|_{t=s}^{t \to \infty} - \int_s^{\infty} f^{\Delta^k}(t)(\ominus z)(t) e_{\ominus z}(t, s)\,\Delta t \\
&= z \int_s^{\infty} f^{\Delta^k}(t) e_{\ominus z}^{\sigma}(t, s)\,\Delta t - f^{\Delta^k}(s) \\
&= z \mathscr{L}\left(f^{\Delta^k}\right)(z, s) - f^{\Delta^k}(s) \\
&= z\left(z^k \mathscr{L}(f)(z, s) - \sum_{l=0}^{k-1} z^l f^{\Delta^{k-1-l}}(s)\right) - f^{\Delta^k}(s) \\
&= z^{k+1} \mathscr{L}(f)(z, s) - \sum_{l=0}^{k-1} z^{l+1} f^{\Delta^{k-1-l}}(s) - f^{\Delta^k}(s) \\
&= z^{k+1} \mathscr{L}(f)(z, s) - \sum_{l=1}^{k} z^l f^{\Delta^{k-l}}(s) - f^{\Delta^k}(s) \\
&= z^{k+1} \mathscr{L}(f)(z, s) - \sum_{l=0}^{k} z^l f^{\Delta^{k-l}}(s).
\end{aligned}
$$

This completes the proof.

Example 1.1 Let $z \in \mathbb{C}$ be such that

$$1 + \mu(t) z \neq 0 \quad \textit{and} \quad \lim_{t \to \infty} e_{\ominus z}(t, s) = 0. \tag{1.5}$$

Then

$$
\begin{aligned}
\mathscr{L}(1)(z) &= \int_s^{\infty} e_{\ominus z}^{\sigma}(t, s)\Delta t \\
&= \int_s^{\infty} (1 + \mu(t)(\ominus z)(t))\, e_{\ominus z}(t, s)\Delta t \\
&= \int_s^{\infty} \left(1 - \frac{z\mu(t)}{1 + z\mu(t)}\right) e_{\ominus z}(t, s)\Delta t \\
&= -\frac{1}{z} \int_s^{\infty} \left(-\frac{z}{1 + z\mu(t)}\right) e_{\ominus z}(t, s)\Delta t \\
&= -\frac{1}{z} \int_s^{\infty} (\ominus z)(t) e_{\ominus z}(t, s)\,\Delta t
\end{aligned}
$$

$$= -\frac{1}{z}\int_s^\infty e^{\Delta}_{\ominus z}(t,s)\Delta t$$

$$= -\frac{1}{z} e_{\ominus z}(t,s)\Big|_{t=s}^{t\to\infty}$$

$$= \frac{1}{z}.$$

Example 1.2 Let $\mathbb{T} = \mathbb{Z}$, $s = 0$ and

$$f(t) = t^2 + 1, \quad t \in \mathbb{T}.$$

We will find $\mathscr{L}(f)(z)$ for $z \in \mathscr{D}\{f\}$. We have

$$\sigma(t) = t + 1, \quad \mu(t) = 1, \quad t \in \mathbb{T}.$$

We set

$$f_1(t) = t, \quad t \in \mathbb{T}.$$

Let $z \in \mathbb{C}$ be such that $z \in D\{f_1\} \bigcap D\{f_1^{\Delta}\} \bigcap D\{f\} \bigcap D\{f^{\Delta}\}$ and (1.5) hold. By (1.2), we have

$$\mathscr{L}\left(f_1^{\Delta}\right)(z) = z\mathscr{L}(f_1)(z) - f_1(0),$$

or

$$\mathscr{L}(1)(z) = z\mathscr{L}(f_1)(z),$$

or

$$\frac{1}{z} = z\mathscr{L}(f_1)(z),$$

or

$$\mathscr{L}(f_1)(z) = \frac{1}{z^2}.$$

Note that

$$\begin{aligned} f^{\Delta}(t) &= \sigma(t) + t \\ &= t + 1 + t \\ &= 2t + 1, \quad t \in \mathbb{T}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathscr{L}\left(f^{\Delta}\right)(z) &= 2\mathscr{L}(f_1)(z) + \mathscr{L}(1)(z) \\ &= \frac{2}{z^2} + \frac{1}{z} \\ &= \frac{z+2}{z^2}. \end{aligned}$$

Now, we use (1.2) to get

$$\mathscr{L}\left(f^{\Delta}\right)(z) = z\mathscr{L}(f)(z) - f(0),$$

or

$$\frac{z+2}{z^2} = z\mathscr{L}(f)(z) - 1,$$

or

$$\frac{z+2}{z^2} + 1 = z\mathscr{L}(f)(z),$$

or

$$\frac{z^2+z+2}{z^2} = z\mathscr{L}(f)(z),$$

or

$$\mathscr{L}(f)(z) = \frac{z^2+z+2}{z^3}.$$

Example 1.3 Let $\mathbb{T} = \mathbb{N}_0^2$, $s = 0$ and

$$f(t) = \frac{2t + 2\sqrt{t} + 3}{(t+1)^2\left(t + 2\sqrt{t} + 2\right)^2}, \quad t \in \mathbb{T}.$$

We will find $\mathscr{L}(f)(z)$ for $z \in \mathscr{D}\{f\}$. Let

$$h(t) = \frac{1}{(t+1)^2}, \quad t \in \mathbb{T}.$$

Take $z \in D\{f\} \bigcap D\{h\}$. We have

$$\begin{aligned} \sigma(t) &= t + 2\sqrt{t} + 1, \\ \mu(t) &= 2\sqrt{t} + 1, \quad t \in \mathbb{T}. \end{aligned}$$

Note that

$$\begin{aligned} e^{\sigma}_{\ominus z}(t,0) &= e_{-\frac{z}{1+z\mu(t)}}(\sigma(t), 0) \\ &= e^{-\int_0^{\sigma(t)} \frac{1}{\mu(\tau)} \mathrm{Log}(1+z\mu(\tau))\Delta\tau} \\ &= e^{-\sum_{\tau\in[0,t]} \mathrm{Log}(1+z\mu(\tau))} \\ &= \prod_{\tau\in[0,t]} \frac{1}{(1+z\mu(\tau))} \\ &= \prod_{\tau\in[0,t]} \frac{1}{\left(1 + z\left(1+2\sqrt{\tau}\right)\right)}, \quad t \in \mathbb{T}. \end{aligned}$$

Then

$$\begin{aligned}\mathscr{L}(h)(z) &= \int_0^\infty \frac{1}{(1+t)^2} e^\sigma_{\ominus z}(t,0)\,\Delta t \\ &= \sum_{t\in[0,\infty)} \frac{2\sqrt{t}+1}{(t+1)^2} \prod_{\tau\in[0,t]} \frac{1}{\left(1+z\left(1+2\sqrt{\tau}\right)\right)}, \quad t\in\mathbb{T}.\end{aligned}$$

Next,

$$\begin{aligned}h^\Delta(t) &= -\frac{\sigma(t)+t+2}{(t+1)^2(\sigma(t)+1)^2} \\ &= -\frac{2t+2\sqrt{t}+3}{(t+1)^2\left(t+2\sqrt{t}+2\right)^2} \\ &= -f(t), \quad t\in\mathbb{T}.\end{aligned}$$

Hence and (1.2), we obtain

$$\mathscr{L}\left(h^\Delta\right)(z) = z\mathscr{L}(h)(z) - h^\Delta(0),$$

or

$$-\mathscr{L}(f)(z) = z\mathscr{L}(h)(z) - \frac{3}{4},$$

or

$$\mathscr{L}(f)(z) = -z\sum_{t\in[0,\infty)} \frac{2\sqrt{t}+1}{(t+1)^2} \prod_{\tau\in[0,t]} \frac{1}{\left(1+z\left(1+2\sqrt{\tau}\right)\right)} + \frac{3}{4}.$$

Example 1.4 Let $\mathbb{T} = 3^{\mathbb{N}_0}\bigcup\{0\}$, $s=0$ and

$$f(t) = \begin{cases} \frac{4}{(t+1)(3t+1)(9t+1)}, & t\neq 0, \quad t\in\mathbb{T}, \\ \frac{3}{8}, & t=0. \end{cases}$$

We will find $\mathscr{L}(f)(z)$ for $z\in\mathscr{D}\{f\}$. Let

$$h(t) = \frac{1}{1+t}, \quad t\in\mathbb{T}.$$

Take $z\in D\{f\}\bigcap D\{h\}\bigcap D\{h^\Delta\}\bigcap D\{h^{\Delta^2}\}$. We have

$$\begin{aligned}\sigma(t) &= 3t, \\ \mu(t) &= 2t, \quad t\in 3^{\mathbb{N}_0}, \\ \sigma(0) &= 1, \\ \mu(0) &= 1.\end{aligned}$$

Next,

$$
\begin{aligned}
e_{\ominus z}^{\sigma}(t,0) &= e^{\int_0^{\sigma(t)} \frac{1}{\mu(\tau)} \mathrm{Log}(1+(\ominus z)(\tau)\mu(\tau))\Delta\tau} \\
&= e^{\int_0^{\sigma(t)} \frac{1}{\mu(\tau)} \mathrm{Log}\left(1-\frac{z\mu(\tau)}{1+z\mu(\tau)}\right)\Delta\tau} \\
&= e^{\int_0^{\sigma(t)} \frac{1}{\mu(\tau)} \mathrm{Log}\frac{1}{1+z\mu(\tau)}\Delta\tau} \\
&= e^{\sum_{\tau\in[0,t]} \mathrm{Log}\frac{1}{1+z\mu(\tau)}} \\
&= \prod_{\tau\in[0,t]} \frac{1}{(1+z\mu(\tau))} \\
&= \begin{cases} \frac{1}{1+z} \prod\limits_{\tau\in[1,t]} \frac{1}{(1+2z\tau)}, & t\in 3^{\mathbb{N}_0}, \\ \frac{1}{z+1}, & t=0. \end{cases}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathscr{L}(h)(z) &= \int_0^{\infty} \frac{1}{t+1} e_{\ominus z}^{\sigma}(t,0)\,\Delta t \\
&= \sum_{t\in[0,\infty)} \frac{\mu(t)}{t+1} e_{\ominus z}^{\sigma}(t,0) \\
&= \frac{1}{1+z}\left(1+\sum_{t\in[1,\infty)} \frac{2t}{t+1} \prod_{\tau\in[1,t]} \frac{1}{(1+2z\tau)}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
h^{\Delta}(t) &= -\frac{1}{(1+t)(1+\sigma(t))} \\
&= \begin{cases} -\frac{1}{2}, & t=0, \\ -\frac{1}{(1+t)(1+3t)}, & t\neq 0, \end{cases}
\end{aligned}
$$

for $t=0$ we get

$$
\begin{aligned}
h^{\Delta^2}(0) &= \frac{h^{\Delta}(\sigma(0)) - h^{\Delta}(0)}{\sigma(0)-0} \\
&= h^{\Delta}(1) + \frac{1}{2} \\
&= -\frac{1}{8} + \frac{1}{2} \\
&= \frac{3}{8} \\
&= f(0),
\end{aligned}
$$

for $t\neq 0$ we obtain