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C. Y. Wang

Essential Analytic Laminar Flow

Springer

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Preface

Fluid mechanics encompasses such a wide variety of subjects that nowadays it is almost impossible for anyone to write a book and do justice to all the topics. Tomes of hundreds of pages have been written on just one major sub-topic, such as compressible flow, turbulence, computational fluid dynamics, and experimental methods. Even then, it is difficult to proclaim the presentation as complete.

Thus, it is essential to state the aim, the limitations, and the philosophy of the present work at the outset.

- Aim: To present essential fluid mechanics to students and researchers, just enough to do independent analytical work.
- Limitations: Only include the analytical (not numerical, experimental, or empirical) methods and solutions of the constant property Navier-Stokes equation and their closely related applications.
- Philosophy: To achieve the goal as directly as possible, leaving some non-essential details to the references.

There are different levels in presenting fluid mechanics. The introductory level considers statics, control volume, etc. The advanced level uses tensors, theorems about existence, etc. The present book is in the intermediate level for the early career researcher or the first-year graduate student. For best results, the reader should have had undergraduate differential equations, some fluid mechanics exposure, and know how to use simple computer software and the Science Citation Index (forwards and backwards) for additional references.

Suggested flowchart for this book is as follows: Chap. 1 "The Navier-Stokes Equation", Chap. 2 "Exact Solutions" (perhaps supplemented by Appendix A on similarity methods), Chap. 3 "Non-dimensionalization, Scaling and Approximations", Chap. 4 "Boundary Layers" (perhaps supplemented by Appendix B Perturbation Theory and Appendix C Potential Flow). Then the reader is free to choose from among the special topics in Chaps. 5–9.

In comparison to other viscous flow texts, the present work differs in the following respects:

- This book is short and concise.
- Topics and exercises are presented to the verge of original research.
- It is simple enough for self-study.
- The method is illustrated in the examples.
- The use of stream function is emphasized.

I have done teaching and research in various aspects of fluid mechanics for the past 50+ years, and this book would no doubt reflect some of my personal preferences.

Reporting any mistakes or inadequacies would be appreciated.

East Lansing, USA
2023

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About the Author

C. Y. Wang is a professor in the Department of Mathematics and an adjoint professor in the Department of Mechanical Engineering at Michigan State University, East Lansing, Michigan. He obtained his B.S. from Taiwan University and Ph.D. from Massachusetts Institute of Technology. Professor Wang published about 330 papers in fluid mechanics and thermos-fluids. He wrote a book *Essential Perturbation Methods* (Springer-Nature) and is a coauthor of *Exact Solutions for Buckling of Structural Members* (CRC Press), and *Structural Vibrations* (CRC Press). He has served as a technical editor of *Applied Mechanics Reviews.*

The Navier–Stokes Equation 1

The governing equation of viscous flow is the Navier–Stokes (N–S) equation. In this chapter we shall derive the N–S equation with the least effort possible.

The assumptions are that the fluid is a continuum (can be differentiated), isotropic (no preferred direction), Newtonian (stress proportional to strain rate) and its properties (density, viscosity) are constant.

We shall derive the N–S equation in two-dimensional Cartesian coordinates, extend to three dimensions, and finally to general orthogonal curvilinear coordinates. The choice of a proper coordinate system is important since it is necessary for analytically solving specific problems.

1.1 Deriving the N–S Equation

Consider first the continuity equation. Let (u, υ) be two-dimensional velocities in the Cartesian (x, y) directions respectively.

Figure 1.1a shows an elemental area $\Delta x \Delta y$ where $u(x, y)$ enters and when it leaves on the other side becomes $u(x + \Delta x, y)$, similarly for the υ velocity component. Since the density is constant, the net fluid loss is zero.

$$[u(x + \Delta x, y) - u(x, y)]\Delta y + [\upsilon(x, y + \Delta y) - \upsilon(x, y)]\Delta x = 0 \tag{1.1}$$

Expanding in Taylor series

$$u(x + \Delta x, y) = u(x, y) + \Delta x \frac{\partial u}{\partial x}(x, y) + \cdots \tag{1.2}$$

Equation (1.1) simplifies to

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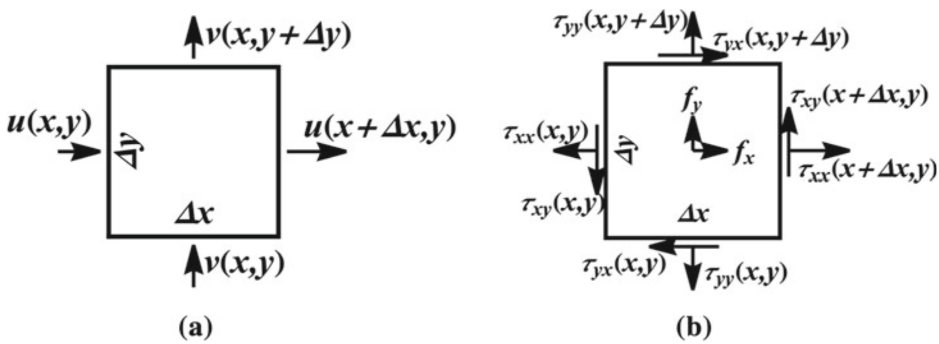

Fig. 1.1 An elemental area **a** velocities through the boundary **b** stresses and forces

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{1.3}$$

Extending to three dimensions and in vector form the continuity equation is

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.4}$$

where $\boldsymbol{u}$ is the velocity vector. Equation (1.4) is also valid when the flow is unsteady (time dependent).

Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the velocity vector, t be the time and $\boldsymbol{x}$ be the spatial (Eulerian) position. $\boldsymbol{x}$ is function of some original position $\boldsymbol{\xi}$ and t. Thus, the acceleration following the mass (Lagrangian) is

$$\boldsymbol{a} = \frac{d}{dt}\boldsymbol{u}\big[\boldsymbol{x}(\boldsymbol{\xi}, t), t\big] = \frac{\partial \boldsymbol{u}}{\partial t} + \sum \frac{\partial \boldsymbol{u}}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \tag{1.5}$$

The forces on an elemental volume of fluid consist of body forces which act on the whole mass and surface forces which act on the bonding surface. Figure 1.1b shows a two-dimensional elemental area subjected to body forces per area (f_x, f_y) and stresses $(\tau_{xx}, \tau_{xy}, \tau_{yx}, \tau_{yy})$. Similar to Eq. (1.1) the net force in the x direction is

$$F_x = \frac{\partial \tau_{xx}}{\partial x}\Delta x \Delta y + \frac{\partial \tau_{yx}}{\partial y}\Delta y \Delta x + f_x \Delta x \Delta y \tag{1.6}$$

Since Newton's momentum law holds, the mass times acceleration is equal to the net force. Using the x component of Newton's law gives

$$\rho \Delta x \Delta y \left[\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)u\right] = F_x \tag{1.7}$$

or

$$\rho\left[\frac{\partial u}{\partial t} + (\boldsymbol{u}\cdot\nabla)u\right] = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x \tag{1.8}$$

Similarly in the y direction

$$\rho\left[\frac{\partial \upsilon}{\partial t} + (\boldsymbol{u}\cdot\nabla)\upsilon\right] = \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + f_y \tag{1.9}$$

A moment balance of the elemental area shows the stress (tensor) is symmetric.

$$\tau_{xy} = \tau_{yx} \tag{1.10}$$

Define a strain rate tensor

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.11}$$

For an isotropic Newtonian fluid (most fluids) the stress is proportional to strain rate

$$\tau_{ij} = -p\delta_{ij} + 2\mu d_{ij} \tag{1.12}$$

Here δ_{ij} is the Kronecker delta, p is the pressure and μ is the viscosity. In two dimensions

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right), \quad \tau_{yy} = -p + 2\mu\frac{\partial \upsilon}{\partial y} \tag{1.13}$$

Upon substituting into Eq. (1.8) and using Eq. (1.3), we obtain

$$\rho\left[\frac{\partial u}{\partial t} + (\boldsymbol{u}\cdot\nabla)u\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f_x \tag{1.14}$$

Extending to three dimensions yields the N–S equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f} \tag{1.15}$$

where $\boldsymbol{f}$ is the body force per mass and $\nu = \mu/\rho$ is the kinematic viscosity. Equation (1.15) is supplemented by continuity Eq. (1.4), initial condition and boundary conditions.

The three-dimensional form of the N–S equation is seldom used in analytic work. Here we present the two-dimensional form in Cartesian coordinates.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f_x \tag{1.16}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2}\right) + f_y \tag{1.17}$$

From Eq. (1.3) a stream function ψ can be defined

$$u = \frac{\partial \psi}{\partial y}, \quad \upsilon = -\frac{\partial \psi}{\partial x} \tag{1.18}$$

Then eliminating pressure, the N–S equation becomes

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \nu\nabla^4\psi + \left(\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}\right) \tag{1.19}$$

In two dimensions, the vorticity is

$$\zeta = \nabla^2\psi \tag{1.20}$$

Notice, if the body forces f_x, f_y are conservative (such as gravity), then they can be absorbed into the pressure term and the last parenthesis in Eq. (1.19) can be set to zero.

In terms of a Jacobian, the N–S equation without body forces is

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nu\nabla^4\psi \tag{1.21}$$

Equation (1.21) is a partial differential equation to be solved with an initial condition, four boundary conditions in x, and four boundary conditions in y. On solid boundaries we assume the no-slip condition, unless partial slip occurs (see Chap. 7).

1.2 N–S Equation in Other Coordinates

A coordinate system that fits (or closely fits), the boundary is necessary for obtaining an analytic solution.

For non-orthogonal coordinate systems, tensors and Christoffel symbols are necessary. An example is the flow in a helical tube with non-zero pitch.

For orthogonal curvilinear coordinate systems, let $\boldsymbol{x}(\boldsymbol{\xi})$ denote the relationship between the new system $\boldsymbol{\xi}$ and the Cartesian system $\boldsymbol{x}$. Construct the elemental distance squared

$$ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = \sum_{i=1}^{3} h_i^2 d\xi_i^2 \tag{1.22}$$

where h_i are scale factors, and let $\boldsymbol{a}_i$ be the unit vectors in the directions ξ_i of the curvilinear system. Equation (1.22) also ascertains the curvilinear system is orthogonal. Vector calculus shows

$$\nabla\phi = \sum_{1}^{3} \frac{\boldsymbol{a}_i}{h_i}\frac{\partial \phi}{\partial \xi_i} \tag{1.23}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\sum\nolimits_1^3 \frac{\partial}{\partial\xi_i}\left(\frac{h_{i+1}h_{i+2}}{h_i}\frac{\partial\phi}{\partial\xi_i}\right) \tag{1.24}$$

where the index i is mod[3], i.e. $h_4 = h_1, h_5 = h_2$. Any vector can be decomposed

$$\boldsymbol{F} = \sum\nolimits_1^3 F_i\boldsymbol{a}_i \tag{1.25}$$

then

$$\nabla\cdot\boldsymbol{F} = \frac{1}{h_1h_2h_3}\sum\nolimits_1^3 \frac{\partial}{\partial\xi_i}(h_{i+1}h_{i+2}F_i) \tag{1.26}$$

$$\nabla\times\boldsymbol{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{a}_1 & h_2\boldsymbol{a}_2 & h_3\boldsymbol{a}_3 \\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \tag{1.27}$$

$$(\boldsymbol{G}\cdot\nabla)\boldsymbol{F} = \sum\nolimits_1^3 \boldsymbol{a}_i\left[\begin{aligned} (\boldsymbol{G}\cdot\nabla)F_i + \frac{F_{i+1}}{h_ih_{i+1}}\left(G_i\frac{\partial h_i}{\partial\xi_{i+1}} - G_{i+1}\frac{\partial h_{i+1}}{\partial\xi_i}\right) \\ + \frac{F_{i+2}}{h_ih_{i+2}}\left(G_i\frac{\partial h_i}{\partial\xi_{i+2}} - G_{i+2}\frac{\partial h_{i+2}}{\partial\xi_i}\right)\end{aligned}\right] \tag{1.28}$$

Notice for Eq. (1.15) the cross product should be used

$$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) = -\nabla\times(\nabla\times\boldsymbol{u}) \tag{1.29}$$

where Eq. (1.4) has been applied. Typical strain rate tensors are

$$d_{11} = \frac{1}{h_1}\frac{\partial u_1}{\partial\xi_1} + \frac{u_2}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} + \frac{u_3}{h_1h_3}\frac{\partial h_1}{\partial\xi_3} \tag{1.30}$$

$$d_{12} = \frac{h_2}{2h_1}\frac{\partial}{\partial\xi_1}\left(\frac{u_2}{h_2}\right) + \frac{h_1}{2h_2}\frac{\partial}{\partial\xi_2}\left(\frac{u_1}{h_1}\right) \tag{1.31}$$

1.2.1 Cylindrical Coordinates

Let (u, υ, w) be velocities in the directions of the cylindrical coordinates (r, θ, z). Since $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, Eq. (1.22) gives $h_1 = 1, h_2 = r, h_3 = 1$. For two dimensional flow independent of the z direction, the N–S equation (without the body force terms) is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{\upsilon}{r}\frac{\partial u}{\partial\theta} - \frac{\upsilon^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial\upsilon}{\partial\theta}\right) \tag{1.32}$$