Geometric Measure Theory and Real Analysis

edited by
Luigi Ambrosio

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Preface

In 2013, a school on Geometric Measure Theory and Real Analysis, organized by G. Alberti, C. De Lellis and myself, took place at the Centro De Giorgi in Pisa, with lectures by V. Bogachev, R. Monti, E. Spadaro and D. Vittone.

The lectures were so well-organized and up-to-date that we suggested publishing them as Lecture Notes. All lecturers kindly agreed to this project.

The book presents in a friendly and unitary way many recent developments which have not previously appeared in book form. Topics include: infinite-dimensional analysis, minimal surfaces and isoperimetric problems in the Heisenberg group, regularity of sub-Riemannian geodesics and the regularity theory of area-minimizing currents in any dimension and codimension.

Sobolev classes on infinite-dimensional spaces

Vladimir I. Bogachev

Contents

Introduction

Sobolev classes of functions of generalized differentiability belong to the major analytic achievements in the XX century and have found impressive applications in the most diverse areas of mathematics. So it does not come as a surprise that their infinite-dimensional analogs attract considerable attention. It was already at the end of the 60s and the beginning of the 70s of the last century that in the works of N. N. Frolov, Yu. L. Daletskiĭ, L. Gross, M. Krée, and P. Malliavin Sobolev classes with respect to Gaussian measures on infinite-dimensional spaces were introduced and studied. Their first triumph came with the development of the Malliavin calculus since the mid of the 70s. At present, such classes and their generalizations have become a standard tool of infinite-dimensional anal-

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ysis. They find applications in stochastic analysis, optimal transportation, mathematical physics, and mathematical finance.

The aim of this survey is to give a concise account of the theory of Sobolev classes on infinite-dimensional spaces with measures. We present a number of already classical cornerstone achievements, some more recent results, and open problems with relatively short formulations. There are already some books presenting elements of this rapidly developing theory (mostly in the Gaussian case), see Bogachev [13, 16] (see also [12]), Bouleau, Hirsch [25], Da Prato [34], Fang [41], Janson [58], Malliavin [66], Malliavin, Thalmaier [67], Nourdin, Peccati [73], Nualart [74], Shigekawa [83], and Üstünel, Zakai [87]. There is also another direction developing Sobolev classes on the so-called measure metric spaces, see Ambrosio, Di Marino [7], Ambrosio, Tilli [10], Cheeger [30], Hajłasz, Koskela [53], Heinonen [54], Keith [59], Reshetnyak [77–79], Vodop'janov [88], which is quite different from the topics discussed here.

The survey is based on several courses I lectured at the Scuola Normale Superiore di Pisa in the years 1995–2013.

Over the years I have had a splendid opportunity to discuss problems related to Sobolev classes in infinite dimensions with many experts in this field, including H. Airault, L. Ambrosio, G. Da Prato, D. Elworthy, S. Fang, D. Feyel, M. Fukushima, M. Hino, A. Lunardi, P. Malliavin, P.-A. Meyer, D. Nualart, M. Röckner, I. Shigekawa, S. Watanabe, N. Yoshida, and M. Zakai.

1 Measures on infinite-dimensional spaces

Given a topological space X we denote by $\mathcal{B}(X)$ its Borel σ-field. Bounded measures on $\mathcal{B}(X)$ (possibly, signed) will be called Borel measures. Such a measure μ can be uniquely written as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are mutually singular nonnegative measures called the positive and negative parts of μ, respectively. Set

$$|\mu| = \mu^+ + \mu^-, \quad \|\mu\| = |\mu|(X).$$

The class of all μ-integrable functions is denoted by $\mathcal{L}^1(\mu)$ and the corresponding Banach space of equivalence classes (where functions equal almost everywhere are identified) is denoted by $L^1(\mu)$. Similar notation $\mathcal{L}^p(\mu)$ and $L^p(\mu)$ is used for the classes of μ-measurable functions integrable to power $p \in (1, \infty)$ and the respective spaces of equivalence classes. For a Hilbert space H, the symbol $L^p(\mu, H)$ is used to denote the L^p-space of H-valued mappings.

If a measure ν on $\mathcal{B}(X)$ has the form $\nu = \varrho \cdot \mu$, where ϱ is a μ-integrable function, which means that

$$\nu(A) = \int_A \varrho(x)\, \mu(dx), \quad A \in \mathcal{B}(X),$$

then ϱ is called absolutely continuous with respect to μ, which is denoted by $\nu \ll \mu$, and ϱ is called its Radon–Nikodym density with respect to μ. A necessary and sufficient condition for that, expressed by the Radon–Nikodym theorem, is that ν vanishes on all sets of μ-measure zero. If also $\mu \ll \nu$, which is equivalent to $\varrho \neq 0$ μ-a.e., then the measures are called equivalent, which is denoted by $\nu \sim \mu$.

A nonnegative Borel measure μ on a topological space X is called Radon if, for every set $B \in \mathcal{B}(X)$ and every $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset B$ such that $\mu(B \backslash K_\varepsilon) < \varepsilon$.

Theorem 1.1. *Each Borel measure on any complete separable metric space X is Radon. Moreover, this is true for any Souslin space X,* i.e., *the image of a complete separable metric space under a continuous mapping.*

In particular, this is true for the spaces $C[0, 1]$, $\mathbb{R}^\infty$, and all separable Hilbert spaces.

For the purposes of this survey it is sufficient to have in mind the space $\mathbb{R}^\infty$, the countable power of the real line with its standard product topology (making it a complete metrizable space). The Borel σ-field in this space coincides with the smallest σ-field containing all cylinders, *i.e.*, sets of the form

$$C_B = \{x\colon\ (x_1, \dots, x_n) \in B\}, \quad B \in \mathcal{B}(\mathbb{R}^n).$$

The measure μ_n defined on $\mathbb{R}^n$ by the formula

$$\mu_n(B) = \mu(C_B)$$

is called the projection of μ on $\mathbb{R}^n$. These projections are consistent in the sense that the projection of μ_{n+1} on $\mathbb{R}^n$ equals μ_n.

By Kolmogorov's theorem, the converse is true: given a consistent sequence of probability measures μ_n on the spaces $\mathbb{R}^n$, there is a unique probability measure μ on $\mathbb{R}^\infty$ with these projections (and there is a natural extension of this result to the case of signed measures, where in the inverse implication the uniform boundedness of μ_n is required).

There is a dual concept to that of projections: conditional measures. Let us consider the one-dimensional subspace $\mathbb{R}e_1$ generated by the first coordinate vector e_1 and its natural complementing hyperplane Y_1 consisting of vectors x with $x_1 = 0$. Let ν^1 be the projection of $|\mu|$ to Y_1, *i.e.*, its image under the natural projecting to Y_1. It is known (see [15, Chapter 10]) that there are Borel measures $\mu^{1,y}$, $y \in Y_1$, on the real line (these measures are probability measures if so is μ) such that for every bounded Borel function f, writing x as $x = (x_1, y)$ with $y = (x_2, x_3, \ldots)$ and identifying y with $(0, x_2, x_3, \ldots)$, one has

$$\int f(x_1, x_2, \ldots)\, \mu(dx) = \int \int f(x_1, y)\, \mu^{1,y}(dx_1)\, \nu^1(dy),$$

where the function defined by the integral in x_1 is Borel measurable in y. Similarly, there exist conditional measures $\mu^{n,y}$, $y \in Y_n$, corresponding to the nth coordinate vector e_n and its natural complementing hyperplane Y_n consisting of vectors with zero nth coordinate. Unlike finite-dimensional distributions, conditional measures (even regarded for all n) do not uniquely determine the measure; the problem of reconstructing a measure from its conditional measures is the subject of the theory of Gibbs measures. Of course, it is not essential that we have considered basis vectors. For a general Radon measure μ (possibly, signed) on a locally vector space X that is a direct topological sum of two closed linear subspaces Z and Y, letting ν be the image of $|\mu|$ under the projection on Y, one can find Radon measures μ^y, $y \in Y$, on Z such that for each bounded Borel function f on X one has

$$\int_X f\, d\mu = \int_Y \int_Z f(z, y)\, \mu^y(dz)\, \nu(dy),$$

where we write elements of X as $x = (z, y)$, $z \in Z$, $y \in Y$, the function $y \mapsto \|\mu^y\|$ is ν-integrable, and the inner integral is also ν-integrable.

Finally, note that sometimes it is more convenient geometrically to define the conditional measures μ^y on the straight lines $\mathbb{R}h + y$ rather than on the real line. In that case the previous equality reads simply as

$$\int_X f\, d\mu = \int_Y \int_Z f(x)\, \mu^y(dx)\, \nu(dy).$$

It will be useful below to represent different measures μ and σ via conditional measures using a common measure ν on Y that may be different from their projections on Y. This is possible if take ν on Y such that both projections are absolutely continuous with respect to ν. Indeed, if $\mu = \mu^y\, \mu_Y(dy)$ and $\sigma = \sigma^y\, \sigma_Y(dy)$, where $\mu_Y = g_1 \cdot \nu$, $\sigma_Y = g_2 \cdot \nu$, then we obtain the representations

$$\mu = g_1(y)\mu^y\, \nu(dy), \quad \sigma = g_2(y)\sigma^y\, \nu(dy)$$

or

$$\mu = \mu^{y,\nu}\, \nu(dy), \quad \sigma = \sigma^{y,\nu}\, \nu(dy),$$

where ν in the symbol $\mu^{y,\nu}$ indicates that the disintegration is taken with respect to the measure ν on Y in place of μ_Y.

We recall the definition of variation and semivariation of vector measures (see Diestel, Uhl [38] or Dunford, Schwartz [39]). Let H be a separable Hilbert space. A vector measure with values in H is an H-valued countably additive function η defined on a σ-algebra $\mathcal{A}$ of subsets of a space Ω. Such a measure automatically has bounded semivariation defined by the formula

$$V(\eta) := \sup \Bigl| \sum_{i=1}^{n} \alpha_i \eta(\Omega_i) \Bigr|_H,$$

where sup is taken over all finite partitions of Ω into disjoint parts $\Omega_i \in \mathcal{A}$ and all finite sets of real numbers α_i with $|\alpha_i| \le 1$. In other words, this is the supremum of variations of real measures $(\eta, h)_H$ over $h \in H$ with $|h|_H \le 1$. However, this does not yet mean that the vector measure η has finite variation which is defined as

$$\mathrm{Var}(\eta) := \sup \sum_{i=1}^{n} |\eta(\Omega_i)|_H,$$

where sup is taken over all finite partitions of Ω into disjoint parts $\Omega_i \in \mathcal{A}$. The variation of the measure η will be denoted by $\|\eta\|$ (but in [39] this notation is used for semivariation).

By the Pettis theorem (see Dunford, Schwartz [39, Chapter IV, §10]), an H-valued mapping Λ is a vector measure of bounded semivariation provided that $(\Lambda, h)_H$ is a bounded scalar measure for each $h \in H$.

The sets of measures of bounded variation and bounded semivariation are Banach spaces with the norms $\eta \mapsto \|\eta\|$ and $\eta \to V(\eta)$, respectively. It is easy to give an example of a measure with values in an infinite-dimensional Hilbert space having bounded semivariation, but infinite variation: consider the standard basis $\{e_n\}$ in l^2 and take Dirac's measures $\delta(e_n)$ in the points e_n and the vector measure $\eta = \sum_{n=1}^{\infty} n^{-1}\delta(e_n)e_n$. Its semivariation equals the sum of the numbers n^{-2}, but it is of infinite variation.

The space of all continuous linear functions on a locally convex space X is denoted by X^* and is called the dual (or topological dual) space.

Let $\mathcal{FC}^\infty$ denote the class of all functions f on X of the form

$$f(x) = f_0(l_1(x), \dots, l_n(x)), \quad f_0 \in C_b^\infty(\mathbb{R}^n),\ l_i \in X^*,$$

where $C_b^\infty(\mathbb{R}^n)$ is the class of all infinitely differentiable functions on $\mathbb{R}^n$ with bounded derivatives. In case of $\mathbb{R}^\infty$ we obtain just the union of all $C_b^\infty(\mathbb{R}^n)$.

2 Gaussian measures

A *Gaussian measure* on the real line is a Borel probability measure which is either concentrated at some point a (*i.e.*, is Dirac's measure δ_a at a) or has density $(2\pi\sigma)^{-1/2}\exp\bigl(-(2\sigma)^{-1}(x-a)^2\bigr)$ with respect to Lebesgue measure, where $a \in \mathbb{R}^1$ is its *mean* and $\sigma > 0$ is its *dispersion*. The measure for which $a = 0$ and $\sigma = 1$ is called *standard Gaussian*.

Similarly the standard Gaussian measure on $\mathbb{R}^d$ is defined by its density

$$(2\pi)^{-d/2}\exp(-|x|^2/2)$$

with respect to Lebesgue measure.

Although below a general concept of a Gaussian measure on a locally convex space is introduced, we define explicitly general Gaussian measures on $\mathbb{R}^d$. These are measures that are concentrated on affine subspaces in $\mathbb{R}^d$ and are standard in suitable (affine) coordinate systems. In other words, these are images of the standard Gaussian measure under affine mappings of the form $x \mapsto Ax + a$, where A is a linear operator and a is a vector. A bit more explicit representation is provided by the Fourier transform of a bounded Borel measure μ on $\mathbb{R}^d$ defined by the formula

$$\widetilde{\mu}(y) = \int \exp\bigl(i(y,x)\bigr)\,\mu(dx), \quad y \in \mathbb{R}^d.$$

In these terms, a measure μ is Gaussian if and only if its Fourier transform has the form

$$\widetilde{\mu}(y) = \exp\Bigl(i(y,a) - \frac{1}{2}Q(y,y)\Bigr),$$

where Q is nonnegative quadratic form on $\mathbb{R}^d$.

The Fourier transform of the standard Gaussian measure is given by

$$\widetilde{\gamma}(y) = \exp(-|y|^2/2).$$

The change of variables formula yields the following relation between A and Q if μ is the image of γ under the affine mapping $Ax + a$:

$$\begin{aligned}\widetilde{\mu}(y) &= \int \exp\bigl(i(y, Ax+a)\bigr)\,\gamma(dx)\\ &= \exp\bigl(i(y,a)\bigr)\int \exp\bigl(i(A^*y,x)\bigr)\,\gamma(dx)\\ &= \exp\bigl(i(y,a) - |A^*y|^2/2\bigr),\end{aligned}$$

that is, $Q(y) = (AA^*y, y)$. It is readily verified that μ has a density on the whole space precisely when A is invertible.

The vector a is called the mean of μ and is expressed by the equality

$$(y, a) = \int (y, x)\, \mu(dx).$$

For the quadratic form Q we have the equality

$$Q(y, y) = \int (y, x - a)^2\, \mu(dx).$$

These equalities are verified directly (it suffices to check them in the one-dimensional case).

Let us define Gaussian measures on general locally convex spaces.

Definition 2.1. Let X be a locally convex space with the topological dual X^*. A Borel probability measure γ on X is called Gaussian if the induced measure $\gamma \circ f^{-1}$ is Gaussian for every $f \in X^*$. If all these measures are centered, then γ is called centered.

In the case of the space $\mathbb{R}^\infty$ the space $\mathbb{R}^\infty_0$ of finite sequences coincides with the dual space. Hence Gaussian measures on $\mathbb{R}^\infty$ are measures with Gaussian finite-dimensional projections.

Example 2.2. An important example of a Gaussian measure is the countable product γ of the standard Gaussian measures on the real line. This measure is defined on the space $X = \mathbb{R}^\infty$. This special example plays a very important role in the whole theory. In some sense (see Bogachev [13] for details) this is a unique up to isomorphism infinite-dimensional Gaussian measure.

Another important example of a Gaussian measure is the Wiener measure on the space $C[0, 1]$ of continuous functions or on the space $L^2[0, 1]$. This measure can be defined as the image of the standard Gaussian measure γ on $X = \mathbb{R}^\infty$ under the mapping

$$x = (x_n) \mapsto w(\cdot), \quad w(t) = \sum_{n=1}^{\infty} x_n \int_0^t e_n(s)\, ds,$$

where $\{e_n\}$ is an arbitrary orthonormal basis in $L^2[0, 1]$. One can show that this series converges in $L^2[0, 1]$ for γ-almost every x; moreover, for γ-almost every x convergence is uniform on $[0, 1]$.

We recall that a countable product $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ of probability measures μ_n on spaces $(X_n, \mathcal{B}_n)$ is defined on $X = \prod_{n=1}^{\infty} X_n$ as follows: first it is defined on sets of the form $A = A_1 \times \cdots \times A_n \times X_{n+1} \cdots$ by

$$\mu(A) = \mu_1(A_1) \times \cdots \times \mu_n(A_n),$$

then it is verified that μ is countably additive on the algebra of finite unions of such sets (called cylindrical sets), which results in a countably additive extension to the smallest σ-algebra $\mathcal{B} := \bigotimes_{n=1}^{\infty} \mathcal{B}_n$ containing such cylindrical sets.

The standard Gaussian measure γ on $\mathbb{R}^\infty$ can be restricted to many other smaller linear subspaces of full measure. For example, taking any sequence of numbers $\alpha_n > 0$ with $\sum_{n=1}^{\infty} \alpha_n < \infty$, we can restrict γ to the weighted Hilbert space of sequences

$$E := \Big\{ (x_n) \in \mathbb{R}^\infty \colon \sum_{n=1}^{\infty} \alpha_n x_n^2 < \infty \Big\},$$

making this expression the square of the norm. The fact that $\gamma(E) = 1$ follows by the monotone convergence theorem, which shows that

$$\sum_{n=1}^{\infty} \alpha_n x_n^2 < \infty$$

almost everywhere due to convergence of the integrals of the terms (the integral of x_n^2 is 1). Similarly, one can find non-Hilbert full measure Banach spaces of sequences (x_n) with $\sup_n \beta_n |x_n| < \infty$ or $\lim\limits_{n\to\infty} \beta_n |x_n| = 0$ for suitable sequences $\beta_n \to 0$; more precisely, the condition is this:

$$\sum_{n=1}^{\infty} \exp\Big(-\frac{C}{\beta_n^2}\Big) < \infty \quad \forall C > 0.$$

However, there is no minimal linear subspace of full measure. The point is that the intersection of all linear subspaces of positive (equivalently, full) measure is the subspace l^2, which has measure zero, as one can verify directly.

It is known that any Radon Gaussian measure γ has mean $m \in X$, *i.e.*, m is a vector in X such that

$$f(m) = \int_X f(x)\,\gamma(dx) \quad \forall f \in X^*.$$

If $m = 0$, *i.e.*, the measures $\gamma \circ f^{-1}$ for $f \in X^*$ have zero mean, then γ is called *centered*. Any Radon Gaussian measure γ is a shift of a centered Gaussian measure γ_m defined by the formula $\gamma_m(B) := \gamma(B + m)$. Hence for many purposes it suffices to consider only centered Gaussian measures.

For a centered Radon Gaussian measure γ we denote by X_γ^* the closure of X^* in $L^2(\gamma)$. The elements of X_γ^* are called *γ-measurable linear* *functionals*. There is an operator $R_\gamma\colon X_\gamma^* \to X$, called the *covariance operator* of the measure γ, such that

$$f(R_\gamma g) = \int_X f(x)g(x)\,\gamma(dx) \quad \forall f \in X^*,\ g \in X_\gamma^*.$$

Set

$$g := \widehat{h} \quad \text{if } h = R_\gamma g.$$

Then $\widehat{h}$ is called the γ-measurable linear functional generated by h. The following vector equality holds (if X is a Banach space, then it holds in Bochner's sense):

$$R_\gamma g = \int_X g(x)x\,\gamma(dx) \quad \forall g \in X_\gamma^*.$$

For example, if γ is a centered Gaussian measure on a separable Hilbert space X, then there exists a nonnegative nuclear operator K on X for which $Ky = R_\gamma y$ for all $y \in X$, where we identify X^* with X. Then we obtain

$$(Ky, z) = (y, z)_{L^2(\gamma)} \quad \text{and} \quad \widetilde{\gamma}(y) = \exp\bigl(-(Ky, y)/2\bigr).$$

Let us take an orthonormal eigenbasis $\{e_n\}$ of the operator K with eigenvalues $\{k_n\}$. Then γ coincides with the image of the countable power γ_0 of the standard Gaussian measure on $\mathbb{R}^1$ under the mapping

$$\mathbb{R}^\infty \to X, \quad (x_n) \mapsto \sum_{n=1}^{\infty} \sqrt{k_n}x_n e_n.$$

This series converges γ_0-a.e. in X by convergence of the series $\sum_{n=1}^\infty k_n x_n^2$, which follows by convergence of the series of k_n and the fact that the integral of x_n^2 against the measure γ_0 equals 1. Here X_γ^* can be identified with the completion of X with respect to the norm $x \mapsto \|\sqrt{K}x\|_X$, *i.e.*, the embedding $X = X^* \to X_\gamma^*$ is a Hilbert–Schmidt operator.

The space

$$H(\gamma) = R_\gamma(X_\gamma^*)$$

is called the *Cameron–Martin space* of the measure γ. It is a Hilbert space with respect to the inner product

$$(h, k)_H := \int_X \widehat{h}(x)\widehat{k}(x)\,\gamma(dx).$$

The corresponding norm is given by the formula

$$|h|_H := \|\widehat{h}\|_{L^2(\gamma)}.$$

Moreover, it is known that $H(\gamma)$ with the indicated norm is separable and its closed unit ball is compact in the space X. Note that the same norm is given by the formula

$$|h|_H = \sup\bigl\{f(h)\colon\ f \in X^*,\ \|f\|_{L^2(\gamma)} \le 1\bigr\}.$$

It should be noted that if $\dim H(\gamma) = \infty$, then $\gamma\bigl(H(\gamma)\bigr) = 0$.

In terms of the inner product in H the vector $R_\gamma(l)$ is determined by the identity

$$\bigl(j_H(f), R_\gamma g\bigr)_H = f(R_\gamma g) = \int_X fg\,d\gamma, \quad f \in X^*, g \in X^*_\gamma. \tag{2.1}$$

In the above example of a Gaussian measure γ on a Hilbert space we have

$$H(\gamma) = \sqrt{K}(X).$$

Let us observe that $H(\gamma)$ coincides also with the set of all vectors of the form

$$h = \int_X f(x)x\,\gamma(dx), \quad f \in L^2(\gamma).$$

Indeed, letting f_0 be the orthogonal projection of f onto X^*_γ in $L^2(\gamma)$, we see that the integral of the difference $[f(x) - f_0(x)]x$ over X vanishes since the integral of $[f(x) - f_0(x)]l(x)$ vanishes for each $l \in X^*$.

Theorem 2.3. *The mapping $h \mapsto \widehat{h}$ establishes a linear isomorphism between $H(\gamma)$ and X^*_γ preserving the inner product. In addition, $R_\gamma\widehat{h} = h$.*

If $\{e_n\}$ is an orthonormal basis in $H(\gamma)$, then $\{\widehat{e_n}\}$ is an orthonormal basis in X^*_γ and $\widehat{e_n}$ are independent random variables.

One can take an orthonormal basis in X^*_γ consisting of elements $\xi_n \in X^*$. The general form of an element $l \in X^*_\gamma$ is this:

$$l = \sum_{n=1}^{\infty} c_n\xi_n,$$

where the series converges in $L^2(\gamma)$. Since ξ_n are independent Gaussian random variables, this series converges also γ-a.e. The domain of its convergence is a Borel linear subspace L of full measure. One can take a version of l which is linear on all of X in the usual sense; it is called a *proper linear version*. It is easy to show that such a version is automatically continuous on $H(\gamma)$ with the norm $|\cdot|_H$; more precisely,

$$f_0(h) = (R_\gamma f, h)_H = \int_X f\widehat{h}\,d\gamma, \quad h \in H.$$

Conversely, any continuous linear functional l on the Hilbert space $H(\gamma)$ admits a unique extension to a γ-measurable proper linear functional $\widehat{l}$ such that $\widehat{l}$ coincides with l on $H(\gamma)$. For every $h \in H(\gamma)$, such an extension of the functional $x \mapsto (x, h)_H$ is exactly $\widehat{h}$. If $h = \sum_{n=1}^{\infty} c_n e_n$, then $\widehat{h} = \sum_{n=1}^{\infty} c_n \widehat{e_n}$. Two γ-measurable linear functionals are equal almost everywhere precisely when their proper linear versions coincide on $H(\gamma)$.

If a measure γ on $X = \mathbb{R}^\infty$ is the countable power of the standard Gaussian measure on the real line, then X^* can be identified with the space of all sequences of the form $f = (f_1, \dots, f_n, 0, 0, \dots)$. Here we have

$$(f, g)_{L^2(\gamma)} = \sum_{i=1}^{\infty} f_i g_i.$$

Hence X_γ^* can be identified with l^2; any element $l = (c_n) \in l^2$ defines an element of $L^2(\gamma)$ by the formula $l(x) := \sum_{n=1}^{\infty} c_n x_n$, where the series converges in $L^2(\gamma)$. Therefore, the Cameron–Martin space $H(\gamma)$ coincides with the space l^2 with its natural inner product. An element l represents a continuous linear functional precisely when only finitely many numbers c_n are nonzero. For the Wiener measure on $C[0, 1]$ the Cameron–Martin space coincides with the class $W_0^{2,1}[0, 1]$ of all absolutely continuous functions h on $[0, 1]$ such that $h(0) = 0$ and $h' \in L^2[0, 1]$; the inner product is given by the formula

$$(h_1, h_2)_H := \int_0^1 h_1'(t) h_2'(t)\, dt.$$

The next classical result, called the *Cameron–Martin formula*, relates measurable linear functionals and vectors in the Cameron–Martin space to the Radon–Nikodym density for shifts of the Gaussian measure.

Theorem 2.4. *The space $H(\gamma)$ is the set of all $h \in X$ such that $\gamma_h \sim \gamma$, where $\gamma_h(B) := \gamma(B + h)$, and the Radon–Nikodym density of the measure γ_h with respect to γ is given by the following Cameron–Martin formula*:

$$d\gamma_h / d\gamma = \exp\bigl(-\widehat{h} - |h|_H^2/2\bigr).$$

For every $h \notin H(\gamma)$ we have $\gamma \perp \gamma_h$.

It follows from this formula that for every bounded Borel function f on X we have

$$\int_X f(x + h)\, \gamma(dx) = \int_X f(x) \exp\bigl(\widehat{h}(x) - |h|_H^2/2\bigr)\, \gamma(dx).$$

In the case of the standard Gaussian measure on $\mathbb{R}^\infty$ this formula is a straightforward extension of the obvious finite-dimensional expression, one just needs to define $\widehat{h}(x)$ as the sum of a series.

A centered Radon Gaussian measure is uniquely determined by its Cameron–Martin space (with the indicated norm!): if μ and ν are centered Radon Gaussian measures such that $H(\mu) = H(\nu)$ and $|h|_{H(\mu)} = |h|_{H(\nu)}$ for all $h \in H(\mu) = H(\nu)$, then $\mu = \nu$. The Cameron-Martin space is also called the reproducing Hilbert space.

Definition 2.5. A Radon Gaussian measure γ on a locally convex space X is called nondegenerate if for every nonzero functional $f \in X^*$ the measure $\gamma \circ f^{-1}$ is not concentrated at a point.

The nondegeneracy of γ is equivalent to that $\gamma(U) > 0$ for all nonempty open sets $U \subset X$. This is also equivalent to that the Cameron-Martin space $H(\gamma)$ is dense in X. For every degenerate Radon Gaussian measure γ there exists the smallest closed linear subspace $L \subset X$ for which $\gamma(L + m) = 1$, where m is the mean of the measure γ. Moreover, $L + m$ coincides with the topological support of γ. If $m = 0$, then on L the measure γ is nondegenerate.

Let γ be a centered Radon Gaussian measure on a locally convex space X; as usual, one can assume that this is the standard Gaussian measure on $\mathbb{R}^\infty$. *The Ornstein–Uhlenbeck semigroup* is defined by the formula

$$T_t f(x) = \int_X f\left(e^{-t}x - \sqrt{1 - e^{-2t}}\, y\right) \gamma(dy), \quad f \in \mathcal{L}^p(\gamma). \tag{2.2}$$

A simple verification of the fact that $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on all $L^p(\gamma)$, $1 \leq p < \infty$, can be found in [13]; the semigroup property means that

$$T_{t+s} f = T_s T_s f, \quad t, s \geq 0.$$

An important feature of this semigroup is that the measure γ is invariant for it, that is,

$$\int_X T_t f(x)\, \gamma(dx) = \int_X f(x)\, \gamma(dx).$$

Theorem 2.6. *For every $p \in [1, +\infty)$ and $f \in L^p(\gamma)$ one has*

$$\lim_{t\to 0} \|T_t f - f\|_{L^p(\gamma)} = 0, \quad \lim_{t\to+\infty} \left\| T_t f - \int f\, d\gamma \right\|_{L^p(\gamma)} = 0$$

and if $1 < p < \infty$, then also $\lim_{t\to 0} T_t f(x) = f(x)$ a.e.

It is also known that in the finite-dimensional case $\lim_{t\to 0} T_t f(x) = f(x)$ a.e. for all $f \in L^1(\gamma)$. It remains an open problem whether this is true in infinite dimensions.

The generator L of the Ornstein–Uhlenbeck semigroup is called the *Ornstein–Uhlenbeck operator* (more precisely, for every $p \in [1, +\infty)$, there is such a generator on the corresponding domain in $L^p(\gamma)$; if p is not explicitly indicated, then usually $p = 2$ is meant). By definition, $Lf = \lim_{t\to 0}(T_t f - f)/t$ if this limit exists in the norm of $L^p(\gamma)$. This operator will be important Section 4. In the case of $\mathbb{R}^\infty$, on smooth functions $f(x) = f(x_1, \dots, x_n)$ in finitely many variables one can explicitly calculate that

$$Lf(x) = \Delta f(x) - \big(x, \nabla f(x)\big) = \sum_{i=1}^{n} \big[\partial_{x_i}^2 f(x) - x_i \partial_{x_i} f(x)\big].$$

This representation can be also extended to some functions in infinitely many variables. In the general case Lf is the sum of a similar series, but its two parts need converge separately.

In the theory of Gaussian measures an important role is played by the Hermite (or Chebyshev–Hermite) polynomials H_n defined by the equalities

$$H_0 = 1, \quad H_n(t) = \frac{(-1)^n}{\sqrt{n!}} e^{t^2/2} \frac{d^n}{dt^n}\big(e^{-t^2/2}\big), \quad n > 1.$$

They have the following properties:

$$H_n'(t) = \sqrt{n} H_{n-1}(t) = t H_n(t) - \sqrt{n+1} H_{n+1}(t).$$

In addition, the system of functions $\{H_n\}$ is an orthonormal basis in $L^2(\gamma)$, where γ is the standard Gaussian measure on the real line.

For the *standard Gaussian* measure γ_n on $\mathbb{R}^n$ (the product of n copies of the standard Gaussian measure on $\mathbb{R}^1$) an orthonormal basis in $L^2(\gamma_n)$ is formed by the polynomials of the form

$$H_{k_1,\dots,k_n}(x_1, \dots, x_n) = H_{k_1}(x_1) \cdots H_{k_n}(x_n), \quad k_i \ge 0.$$

If γ is a centered Radon Gaussian measure on a locally convex space X and $\{l_n\}$ is an orthonormal basis in X_γ^*, then a basis in $L^2(\gamma)$ is formed by the polynomials

$$H_{k_1,\dots,k_n}(x) = H_{k_1}\big(l_1(x)\big) \cdots H_{k_n}\big(l_n(x)\big), \quad k_i \ge 0, n \in \mathbb{N}.$$

For example, for the countable power of the standard Gaussian measure on the real line such polynomials are $H_{k_1,\dots,k_n}(x_1, \dots, x_n)$. It is convenient to arrange polynomials $H_{k_1,\dots,k_n}$ according to their degrees $k_1+\cdots+k_n$. For $k=0,1,\dots$ we denote by $\mathcal{X}_k$ the closed linear subspace of $L^2(\gamma)$ generated by the functions $H_{k_1,\dots,k_n}$ with $k_1+\cdots+k_n=k$. The functions $H_{k_1,\dots,k_n}$ are mutually orthogonal and, for the fixed value $k=k_1+\cdots+k_n$, form an orthonormal basis in $\mathcal{X}_k$.

The one-dimensional space $\mathcal{X}_0$ consists of constants and $\mathcal{X}_1 = X_\gamma^*$. One can show that every element $f\in\mathcal{X}_2$ can be written in the form

$$f=\sum_{n=1}^{\infty}\alpha_n(l_n^2-1),$$

where $\{l_n\}$ is an orthonormal basis in X_γ^* and $\sum_{n=1}^{\infty}\alpha_n^2<\infty$ (*i.e.*, the series for f converges in $L^2(\gamma)$).

The spaces $\mathcal{X}_k$ are mutually orthogonal and their orthogonal sum is the whole $L^2(\gamma)$:

$$L^2(\gamma)=\bigoplus_{k=0}^{\infty}\mathcal{X}_k,$$

which means that, denoting by I_k the operator of orthogonal projection onto $\mathcal{X}_k$, we have an orthogonal decomposition

$$F=\sum_{k=0}^{\infty}I_k(F),\qquad F\in L^2(\gamma).$$

One can check that $T_tH_{k_1,\dots,k_n}=e^{-k_1-\cdots-k_n}H_{k_1,\dots,k_n}$, which yields that

$$T_tF=\sum_{k=0}^{\infty}e^{-kt}I_k(F),\qquad F\in L^2(\gamma).$$

Given a separable Hilbert space E, one defines similarly the space $\mathcal{X}_k(E)$ of polynomials with values in E as the closure in $L^2(\gamma,E)$ of the liner span of the mappings $f\cdot v$, where $f\in\mathcal{X}_k$, $v\in E$.

3 Integration by parts and differentiable measures

Suppose that f is a bounded Borel function on a locally convex space X with a centered Radon Gaussian measure γ such that the partial derivative

$$\partial_h f(x)=\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}$$

exists for some vector h in the Cameron-Martin space of γ and is bounded. Applying the Cameron-Martin formula and Lebesgue's dominated convergence theorem, we arrive at the equality

$$\int_X\partial_h f(x)\,\gamma(dx)=\int_X f(x)\widehat{h}(x)\,\gamma(dx),$$

where we also use that the derivative of $t \mapsto e^{t\widehat{h}-t^2|h|_H^2/2}$ at zero is $\widehat{h}$. This simple formula, called the integration by parts formula for the Gaussian measure, plays a very important role in stochastic analysis and is a starting point for far-reaching generalizations connected with differentiabilities of measures in the sense of Fomin [45, 46] and in the sense of Skorohod [85].

A measure μ on X is called Skorohod differentiable along a vector h if there exists a measure $d_h\mu$, called the Skorohod derivative of the measure μ along the vector h, such that

$$\lim_{t\to 0}\int_X \frac{f(x-th)-f(x)}{t}\,\mu(dx) = \int_X f(x)d_h\mu(dx) \tag{1}$$

for every bounded continuous function f on X. If the measure $d_h\mu$ is absolutely continuous with respect to the measure μ, then the measure μ is called Fomin differentiable along the vector h, the Radon–Nikodym density of the measure $d_h\mu$ with respect to μ is denoted by β_h^μ and called the logarithmic derivative of μ along h. The Skorohod differentiability of μ along h is equivalent to the identity

$$\int_X \partial_h f(x)\,\mu(dx) = -\int_X f(x)\,d_h\mu(dx), \quad f \in \mathcal{FC}^\infty.$$

The Fomin differentiability is the equality

$$\int_X \partial_h f(x)\,\mu(dx) = -\int_X f(x)\,\beta_h^\mu(x)\,\mu(dx), \quad f \in \mathcal{FC}^\infty.$$

On the real line the Fomin differentiability is equivalent to the membership of the density in the Sobolev class $W^{1,1}$, and the Skorohod differentiability is the boundedness of variation of the density; the picture is similar also in $\mathbb{R}^n$. A detailed discussion of these types of differentiability of measures can be found in Bogachev [16].

It follows from our previous discussion that for the centered Gaussian measure μ we have

$$\beta_h^\mu = -\widehat{h}, \quad h \in H(\mu).$$

In the case of a probability measure on $\mathbb{R}^\infty$ efficient conditions for both types of differentiability can be expressed in terms of finite-dimensional distributions. The Skorohod differentiability along a vector $h = (h_n)$ is equivalent to the following condition: for every n, the generalized derivative of the projection μ_n on $\mathbb{R}^n$ along the vector $(h_1, \dots, h_n)$ is a bounded measure and such measures are uniformly bounded. For Fomin's differentiability more is needed: the corresponding logarithmic derivatives