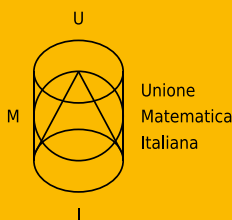


Lecture Notes of the Unione Matematica Italiana

Andrea Loi · Michela Zedda

Kähler Immersions of Kähler Manifolds into Complex Space Forms



 Springer

Editorial Board

Ciro Ciliberto

(Editor in Chief)
Dipartimento di Matematica
Università di Roma Tor Vergata
Via della Ricerca Scientifica
00133 Roma, Italy
e-mail: cilibert@axp.mat.uniroma2.it

Susanna Terracini

(Co-editor in Chief)
Università degli Studi di Torino
Dipartimento di Matematica "Giuseppe Peano"
Via Carlo Alberto 10
10123 Torino, Italy
e-mail: susanna.teraccini@unito.it

Adolfo Ballester-Bollinches

Department d'Àlgebra
Facultat de Matemàtiques
Universitat de València
Dr. Moliner, 50
46100 Burjassot (València), Spain
e-mail: Adolfo.Ballester@uv.es

Annalisa Buffa

IMATI – C.N.R. Pavia
Via Ferrata 1
27100 Pavia, Italy
e-mail: annalisa@imati.cnr.it

Lucia Caporaso

Dipartimento di Matematica
Università Roma Tre
Largo San Leonardo Murialdo
I-00146 Roma, Italy
e-mail: caporaso@mat.uniroma3.it

Fabrizio Catanese

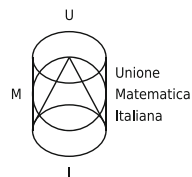
Mathematisches Institut
Universitätstraße 30
95447 Bayreuth, Germany
e-mail: fabrizio.catanese@uni-bayreuth.de

Corrado De Concini

Dipartimento di Matematica
Università di Roma "La Sapienza"
Piazzale Aldo Moro 5
00185 Roma, Italy
e-mail: deconcini@mat.uniroma1.it

Camillo De Lellis

School of Mathematics
Institute for Advanced Study
Einstein Drive
Simonyi Hall
Princeton, NJ 08540, USA
e-mail: camillo.delellis@math.ias.edu



Franco Flandoli

Dipartimento di
Matematica Applicata
Università di Pisa
Via Buonarroti 1c
56127 Pisa, Italy
e-mail: flandoli@dma.unipi.it

Angus MacIntyre

Queen Mary University of London
School of Mathematical Sciences
Mile End Road
London E1 4NS, United Kingdom
e-mail: a.macintyre@qmul.ac.uk

Giuseppe Mingione

Dipartimento di Matematica e Informatica
Università degli Studi di Parma
Parco Area delle Scienze, 53/a (Campus)
43124 Parma, Italy
e-mail: giuseppe.mingione@math.unipr.it

Mario Pulvirenti

Dipartimento di Matematica
Università di Roma "La Sapienza"
P.le A. Moro 2
00185 Roma, Italy
e-mail: pulvirenti@mat.uniroma1.it

Fulvio Ricci

Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa, Italy
e-mail: fricci@sns.it

Valentino Tosatti

Northwestern University
Department of Mathematics
2033 Sheridan Road
Evanston, IL 60208, USA
e-mail: tosatti@math.northwestern.edu

Corinna Ulcigrai

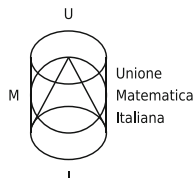
Forschungsinstitut für Mathematik
HG G 44.1
Rämistrasse 101
8092 Zürich, Switzerland
e-mail: corinna.ulcigrai@bristol.ac.uk

The Editorial Policy can be found
at the back of the volume.

Andrea Loi • Michela Zedda

Kähler Immersions of Kähler Manifolds into Complex Space Forms

 Springer



Andrea Loi
Department of Mathematics & Computer
Science
University of Cagliari
Cagliari, Italy

Michela Zedda
Department of Mathematical, Physical
& Computer Sciences
University of Parma
Parma, Italy

ISSN 1862-9113 ISSN 1862-9121 (electronic)
Lecture Notes of the Unione Matematica Italiana
ISBN 978-3-319-99482-6 ISBN 978-3-319-99483-3 (eBook)
<https://doi.org/10.1007/978-3-319-99483-3>

Library of Congress Control Number: 2018954939

Mathematics Subject Classification (2010): 32Q15, 32A10

© Springer Nature Switzerland AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland



The Unione Matematica Italiana (UMI) has established a bi-annual prize, sponsored by Springer-Verlag, to honor an excellent, original monograph presenting the latest developments in an active research area of mathematics, to which the author made important contributions in the recent years.

The prize-winning monographs are published in this series.
Details about the prize can be found at:

<http://umi.dm.unibo.it/en/unione-matematica-italiana-prizes/book-prize-unione-matematicaitaliana/>

This book has been awarded the 2017 Book Prize of the Unione Matematica Italiana.

The members of the scientific committee of the 2017 prize were:

Fabrizio Catanese
University of Bayreuth, Germany

Ciro Ciliberto (Presidente of the UMI)
Università degli Studi di Roma Tor Vergata, Italy

Vittorio Coti Zelati
University of Naples Federico II, Italy

Susanna Terracini
Università degli Studi di Torino, Italy

Valentino Tosatti
Northwestern University, Evanston, USA

Preface

The study of Kähler immersions of a given real analytic Kähler manifold into a finite- or infinite-dimensional complex space form originates from the pioneering work of Eugenio Calabi [10]. With a stroke of genius, Calabi defined a powerful tool, a special (local) potential called the *diastasis function*, which allowed him to obtain necessary and sufficient conditions for a neighbourhood of a point to be locally Kähler immersed into a finite- or infinite-dimensional complex space form. As an application of this criterion, he also provided a classification of (finite-dimensional) complex space forms admitting a Kähler immersion into another. However, a complete classification of Kähler manifolds admitting a Kähler immersion into complex space forms is not known, not even when the Kähler manifolds involved are of great interest, e.g. when they are Einstein or homogeneous spaces. In fact, the diastasis function is not always explicitly given, and most of the time Calabi's criterion, although theoretically impeccable, is difficult to apply. Nevertheless, throughout the last 60 years many mathematicians have worked on the subject and many interesting results have been obtained.

The aim of this book is to describe Calabi's original work, to provide a detailed account of what is known today on the subject and to point out some open problems.

Each chapter begins with a brief summary of the topics discussed and ends with a list of exercises to test the reader's understanding.

Apart from the topics discussed in Sect. 3.1 of Chap. 3, which could be skipped without compromising the understanding of the rest of the book, the prerequisites for this book are a basic knowledge of complex and Kähler geometry (treated, e.g. in Moroianu's book [61]).

The authors are grateful to Claudio Arezzo and Fabio Zuddas for their careful reading of the text and for their valuable comments, which have greatly improved the book's exposition.

Cagliari, Italy
Parma, Italy
June 2018

Andrea Loi
Michela Zedda

Contents

1	The Diastasis Function	1
1.1	Calabi's Diastasis Function	1
1.2	Complex Space Forms	5
1.3	The Indefinite Hilbert Space	7
	Exercises	10
2	Calabi's Criterion	13
2.1	Kähler Immersions into the Complex Euclidean Space	13
2.2	Kähler Immersions into Nonflat Complex Space Forms	19
2.3	Kähler Immersions of a Complex Space Form into Another	24
	Exercises	27
3	Homogeneous Kähler Manifolds	29
3.1	A Result About Kähler Immersions of Homogeneous Bounded Domains into $\mathbb{C}P^\infty$	29
3.2	Kähler Immersions of Homogeneous Kähler Manifolds into $\mathbb{C}^{N \leq \infty}$ and $\mathbb{C}H^{N \leq \infty}$	32
3.3	Kähler Immersions of Homogeneous Kähler Manifolds into $\mathbb{C}P^{N \leq \infty}$	35
3.4	Bergman Metric and Bounded Symmetric Domains	37
3.5	Kähler Immersions of Bounded Symmetric Domains into $\mathbb{C}P^\infty$	40
	Exercises	44
4	Kähler–Einstein Manifolds	47
4.1	Kähler Immersions of Kähler–Einstein Manifolds into $\mathbb{C}H^N$ or \mathbb{C}^N	48
4.2	Kähler Immersions of KE Manifolds into $\mathbb{C}P^N$: The Einstein Constant	52

4.3	Kähler Immersions of KE Manifolds into $\mathbb{C}P^N$: Codimension 1 and 2	56
	Exercises	60
5	Hartogs Type Domains	63
5.1	Cartan–Hartogs Domains	63
5.2	Bergman–Hartogs Domains	69
5.3	Rotation Invariant Hartogs Domains	70
	Exercises	74
6	Relatives	75
6.1	Relatives Complex Space Forms	75
6.2	Homogeneous Kähler Manifolds Are Not Relative to Projective Ones	78
6.3	Bergman–Hartogs Domains Are Not Relative to a Projective Kähler Manifold	80
	Exercises	82
7	Further Examples and Open Problems	83
7.1	The Cigar Metric on \mathbb{C}	83
7.2	Calabi’s Complete and Not Locally Homogeneous Metric	89
7.3	The Taub-NUT Metric on \mathbb{C}^2	92
	Exercises	93
	References	95
	Index	99

Chapter 1

The Diastasis Function



Abstract In this chapter we describe the *diastasis function*, a basic tool introduced by Calabi (Ann Math 58:1–23, 1953) which is fundamental to study Kähler immersions of Kähler manifolds into complex space forms. In Sect. 1.1 we define the diastasis function and summarize its basic properties, while in Sect. 1.2 we describe the diastasis functions of complex space forms, which represent the basic examples of Kähler manifolds. Finally, in Sect. 1.3 we give the formal definition of what a *Kähler immersion* is and prove that the indefinite Hilbert space constitutes a universal Kähler manifold, in the sense that it is a space in which every real analytic Kähler manifold can be locally Kähler immersed.

1.1 Calabi's Diastasis Function

Let M be an n -dimensional complex manifold endowed with a real analytic Kähler metric g . Recall that the Kähler metric g is real analytic if fixed a local coordinate system $z = (z_1, \dots, z_n)$ on a neighbourhood U of any point $p \in M$, it can be described on U by a real analytic Kähler potential $\Phi : U \rightarrow \mathbb{R}$. In that case the potential $\Phi(z)$ can be analytically continued to an open neighbourhood $W \subset U \times U$ of the diagonal. Denote this extension by $\Phi(z, \bar{w})$.

Definition 1.1 The *diastasis function* $D(z, w)$ on W is defined by:

$$D(z, w) = \Phi(z, \bar{z}) + \Phi(w, \bar{w}) - \Phi(z, \bar{w}) - \Phi(w, \bar{z}). \quad (1.1)$$

The following proposition describes the basic properties of $D(z, w)$.

Proposition 1.1 (Calabi [10]) *The diastasis function $D(z, w)$ given by (1.1) satisfies the following properties:*

- (i) *it is uniquely determined by the Kähler metric g and it does not depend on the choice of the local coordinate system;*
- (ii) *it is real valued in its domain of (real) analyticity;*

- (iii) it is symmetric in z and w and $D(z, z) = 0$;
- (iv) once fixed one of its two entries, it is a Kähler potential for g .

Proof

- (i) By the $\partial\bar{\partial}$ -Lemma a Kähler potential is defined up to the addition of the real part of a holomorphic function, namely, given two Kähler potentials Φ and Φ' on $U \subset M$, then $\Phi' = \Phi + f + \bar{f}$ for some holomorphic function f . Conclusions follow again by (1.1).
- (ii) Since $\Phi(z, \bar{z}) = \Phi(z)$ is real valued, then from $\overline{\Phi(z, \bar{z})} = \overline{\Phi(z, \bar{z})}$ and by uniqueness of the extension it follows $\Phi(z, \bar{w}) = \overline{\Phi(w, \bar{z})}$.
- (iii) It follows directly from (1.1).
- (iv) Fix w (the case of z fixed is totally similar). Then:

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} D(z, w) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \Phi(z, \bar{z}) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \Phi(z).$$

□

The last property justifies the following second definition.

Definition 1.2 If $w = (w_1, \dots, w_n)$ are local coordinates for a fixed point $p \in M$, the *diastasis function centered at p* is given by:

$$D_p(z) = D(z, w).$$

In particular, if p is the origin of the coordinate system chosen, we write $D_0(z)$.

The importance of the diastasis function for our purposes is expressed by the following:

Proposition 1.2 (Calabi [10]) *Let (M, g) and (S, G) be Kähler manifolds and assume G to be real analytic. Denote by ω and Ω the Kähler forms associated to g and G respectively. If there exists a holomorphic map $f : (M, g) \rightarrow (S, G)$ such that $f^*\Omega = \omega$, then the metric g is real analytic. Further, denoted by $D_p^M : U \rightarrow \mathbb{R}$ and $D_{f(p)}^S : V \rightarrow \mathbb{R}$ the diastasis functions of (M, g) and (S, G) around p and $f(p)$ respectively, we have $D_{f(p)}^S \circ f = D_p^M$ on $f^{-1}(V) \cap U$.*

Proof Observe first that the metric g on M is real analytic being the pull-back through a holomorphic map of the real analytic metric G . In order to prove the second part, fix a coordinate system $\{z\}$ around $p \in M$. From $f^*G|_{V \cap f(U)} = g|_{f^{-1}(V) \cap U}$, if Φ^S and Φ^M are Kähler potential for G and g around $f(p)$ and p respectively, we get:

$$\frac{\partial^2 \Phi^S(f(z), \overline{f(z)})}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 \Phi^M(z, \bar{z})}{\partial z_j \partial \bar{z}_k},$$

i.e. $\Phi^S(f(z), \overline{f(z)}) = \Phi^M(z, \bar{z}) + h + \bar{h}$ and conclusion follows by (1.1). □