Lecture Notes of the Unione Matematica Italiana

Andrea Loi · Michela Zedda

Kähler Immersions of Kähler Manifolds into Complex Space Forms

U
M
I
Unione
Matematica
Italiana

Springer

Lecture Notes of the Unione Matematica Italiana 23

More information about this series at http://www.springer.com/series/7172

Editorial Board

Unione Matematica Italiana

The Editorial Policy can be found
at the back of the volume.

Andrea Loi • Michela Zedda

Kähler Immersions of Kähler Manifolds into Complex Space Forms

Springer

U
M
Unione
Matematica
Italiana
I

Andrea Loi
Department of Mathematics & Computer Science
University of Cagliari
Cagliari, Italy

Michela Zedda
Department of Mathematical, Physical & Computer Sciences
University of Parma
Parma, Italy

ISSN 1862-9113 ISSN 1862-9121 (electronic)
Lecture Notes of the Unione Matematica Italiana
ISBN 978-3-319-99482-6 ISBN 978-3-319-99483-3 (eBook)
https://doi.org/10.1007/978-3-319-99483-3

Library of Congress Control Number: 2018954939

Mathematics Subject Classification (2010): 32Q15, 32A10

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

The Unione Matematica Italiana (UMI) has established a bi-annual prize, sponsored by Springer-Verlag, to honor an excellent, original monograph presenting the latest developments in an active research area of mathematics, to which the author made important contributions in the recent years.

The prize-winning monographs are published in this series.
Details about the prize can be found at:

http://umi.dm.unibo.it/en/unione-matematica-italiana-prizes/book-prize-unione-matematicaitaliana/

This book has been awarded the 2017 Book Prize of the Unione Matematica Italiana.

The members of the scientific committee of the 2017 prize were:

Fabrizio Catanese
University of Bayreuth, Germany

Ciro Ciliberto (Presidente of the UMI)
Università degli Studi di Roma Tor Vergata, Italy

Vittorio Coti Zelati
University of Naples Federico II, Italy

Susanna Terracini
Università degli Studi di Torino, Italy

Valentino Tosatti
Northwestern University, Evanston, USA

Preface

The study of Kähler immersions of a given real analytic Kähler manifold into a finite- or infinite-dimensional complex space form originates from the pioneering work of Eugenio Calabi [10]. With a stroke of genius, Calabi defined a powerful tool, a special (local) potential called the *diastasis function*, which allowed him to obtain necessary and sufficient conditions for a neighbourhood of a point to be locally Kähler immersed into a finite- or infinite-dimensional complex space form. As an application of this criterion, he also provided a classification of (finite-dimensional) complex space forms admitting a Kähler immersion into another. However, a complete classification of Kähler manifolds admitting a Kähler immersion into complex space forms is not known, not even when the Kähler manifolds involved are of great interest, e.g. when they are Einstein or homogeneous spaces. In fact, the diastasis function is not always explicitly given, and most of the time Calabi's criterion, although theoretically impeccable, is difficult to apply. Nevertheless, throughout the last 60 years many mathematicians have worked on the subject and many interesting results have been obtained.

The aim of this book is to describe Calabi's original work, to provide a detailed account of what is known today on the subject and to point out some open problems.

Each chapter begins with a brief summary of the topics discussed and ends with a list of exercises to test the reader's understanding.

Apart from the topics discussed in Sect. 3.1 of Chap. 3, which could be skipped without compromising the understanding of the rest of the book, the prerequisites for this book are a basic knowledge of complex and Kähler geometry (treated, e.g. in Moroianu's book [61]).

The authors are grateful to Claudio Arezzo and Fabio Zuddas for their careful reading of the text and for their valuable comments, which have greatly improved the book's exposition.

Cagliari, Italy
Parma, Italy
June 2018

Andrea Loi
Michela Zedda

Contents

Chapter 1
The Diastasis Function

Abstract In this chapter we describe the *diastasis function*, a basic tool introduced by Calabi (Ann Math 58:1–23, 1953) which is fundamental to study Kähler immersions of Kähler manifolds into complex space forms. In Sect. 1.1 we define the diastasis function and summarize its basic properties, while in Sect. 1.2 we describe the diastasis functions of complex space forms, which represent the basic examples of Kähler manifolds. Finally, in Sect. 1.3 we give the formal definition of what a *Kähler immersion* is and prove that the indefinite Hilbert space constitutes a universal Kähler manifold, in the sense that it is a space in which every real analytic Kähler manifold can be locally Kähler immersed.

1.1 Calabi's Diastasis Function

Let M be an n-dimensional complex manifold endowed with a real analytic Kähler metric g. Recall that the Kähler metric g is real analytic if fixed a local coordinate system $z = (z_1, \dots, z_n)$ on a neighbourhood U of any point $p \in M$, it can be described on U by a real analytic Kähler potential $\Phi : U \to \mathbb{R}$. In that case the potential $\Phi(z)$ can be analytically continued to an open neighbourhood $W \subset U \times U$ of the diagonal. Denote this extension by $\Phi(z, \bar{w})$.

Definition 1.1 The *diastasis function* $\mathrm{D}(z, w)$ on W is defined by:

$$\mathrm{D}(z, w) = \Phi(z, \bar{z}) + \Phi(w, \bar{w}) - \Phi(z, \bar{w}) - \Phi(w, \bar{z}) . \tag{1.1}$$

The following proposition describes the basic properties of $\mathrm{D}(z, w)$.

Proposition 1.1 (Calabi [10]) *The diastasis function* $\mathrm{D}(z, w)$ *given by (1.1) satisfies the following properties:*

(i) it is uniquely determined by the Kähler metric g and it does not depend on the choice of the local coordinate system;
(ii) it is real valued in its domain of (real) analyticity;

A. Loi, M. Zedda, *Kähler Immersions of Kähler Manifolds into Complex Space Forms*, Lecture Notes of the Unione Matematica Italiana 23,
https://doi.org/10.1007/978-3-319-99483-3_1

(iii) it is symmetric in z *and* w *and* $\mathrm{D}(z, z) = 0$*;*
(iv) once fixed one of its two entries, it is a Kähler potential for g*.*

Proof

(i) By the $\partial\bar{\partial}$-Lemma a Kähler potential is defined up to the addition of the real part of a holomorphic function, namely, given two Kähler potentials Φ and Φ' on $U \subset M$, then $\Phi' = \Phi + f + \bar{f}$ for some holomorphic function f. Conclusions follow again by (1.1).
(ii) Since $\Phi(z, \bar{z}) = \Phi(z)$ is real valued, then from $\Phi(z, \bar{z}) = \overline{\Phi(z, \bar{z})}$ and by uniqueness of the extension it follows $\Phi(z, \bar{w}) = \overline{\Phi(w, \bar{z})}$.
(iii) It follows directly from (1.1).
(iv) Fix w (the case of z fixed is totally similar). Then:

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k}\mathrm{D}(z, w) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k}\Phi\,(z, \bar{z}) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k}\Phi\,(z)\,.$$

□

The last property justifies the following second definition.

Definition 1.2 If $w = (w_1, \ldots, w_n)$ are local coordinates for a fixed point $p \in M$, the *diastasis function centered at* p is given by:

$$\mathrm{D}_p(z) = \mathrm{D}(z, w).$$

In particular, if p is the origin of the coordinate system chosen, we write $\mathrm{D}_0(z)$.

The importance of the diastasis function for our purposes is expressed by the following:

Proposition 1.2 (Calabi [10]) *Let* (M, g) *and* (S, G) *be Kähler manifolds and assume* G *to be real analytic. Denote by* ω *and* Ω *the Kähler forms associated to* g *and* G *respectively. If there exists a holomorphic map* $f : (M, g) \to (S, G)$ *such that* $f^*\Omega = \omega$*, then the metric* g *is real analytic. Further, denoted by* $\mathrm{D}^M_p : U \to \mathbb{R}$ *and* $\mathrm{D}^S_{f(p)} : V \to \mathbb{R}$ *the diastasis functions of* (M, g) *and* (S, G) *around* p *and* $f(p)$ *respectively, we have* $\mathrm{D}^S_{f(p)} \circ f = \mathrm{D}^M_p$ *on* $f^{-1}(V) \cap U$*.*

Proof Observe first that the metric g on M is real analytic being the pull-back through a holomorphic map of the real analytic metric G. In order to prove the second part, fix a coordinate system $\{z\}$ around $p \in M$. From $f^*G|_{V\cap f(U)} = g|_{f^{-1}(V)\cap U}$, if Φ^S and Φ^M are Kähler potential for G and g around $f(p)$ and p respectively, we get:

$$\frac{\partial^2 \Phi^S(f(z), \overline{f(z)})}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 \Phi^M(z, \bar{z})}{\partial z_j \partial \bar{z}_k},$$

i.e. $\Phi^S(f(z), \overline{f(z)}) = \Phi^M(z, \bar{z}) + h + \bar{h}$ and conclusion follows by (1.1). □