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Hilbert Functions of Filtered Modules

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Preface

Hilbert Functions play major roles in Algebraic Geometry and Commutative Algebra, and are becoming increasingly important also in Computational Algebra. They capture many useful numerical characters associated to a projective variety or to a filtered module over a local ring.

Starting from the pioneering work of D.G. Northcott and J. Sally, we aim to gather together in one place many new developments of this theory by using a unifying approach which gives self-contained and easier proofs.

The extension of the theory to the case of general filtrations on a module, and its application to the study of certain graded algebras which are not associated to a filtration are two of the main features of the monograph.

The material is intended for graduate students and researchers who are interested in Commutative Algebra, in particular in the theory of the Hilbert functions and related topics.

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Acknowledgements

We would like to take this opportunity to thank sincerely Judith Sally because her work has had such a strong influence on our research into these subjects. In particular, several problems, techniques and ideas presented in this text came from a careful reading of her papers, which are always rich in examples and motivating applications.

Let us also not forget the many other colleagues who over the years have shared their ideas on these topics with us. Some of them were directly involved as co-authors in joint research reported here, while others gave a substantial contribution via their publications and discussions.

Contents

Introduction

The notion of Hilbert function is central in commutative algebra and is becoming increasingly important in algebraic geometry and in computational algebra. In this presentation we shall deal with some aspects of the theory of Hilbert functions of modules over local rings, and we intend to guide the reader along one of the possible routes through the last three decades of progress in this area of dynamic mathematical activity.

Motivated by the ever increasing interest in this field, our aim is to gather together many new developments of this theory in one place, and to present them using a unifying approach which gives self-contained and easier proofs. In this text we shall discuss many results by different authors, following essentially the direction typified by the pioneering work of J. Sally (see [86–93]). Our personal view of the subject is most visibly expressed by the presentation of Chaps. 1 and 2 in which we discuss the use of the superficial elements and related devices.

Basic techniques will be stressed with the aim of reproving recent results by using a more elementary approach. This choice was made at the expense of certain results and various interesting aspects of the topic that, in this presentation, must remain peripheral. We apologize to those whose work we may have failed to cite properly.

The material is intended for graduate students and researchers who are interested in Commutative Algebra, in particular in results on the Hilbert function and the Hilbert polynomial of a local ring, and applications of these. The aim was not to write a book on the subject, but rather to collect results and problems inspired by specialized lecture courses and schools recently delivered by the authors. We hope the reader will appreciate the large number of examples and the rich bibliography.

Starting from classical results of D. Northcott, P. Samuel, S. Abhyankar, E. Matlis and J. Sally, many papers have been written on this topic which is considered an important part of the theory of blowing-up rings. This is because the Hilbert function of the local ring $(A,\mathfrak{m})$ is by definition the numerical function $H_A(t) := \dim_k(\mathfrak{m}^t/\mathfrak{m}^{t+1})$, hence it coincides with the classical Hilbert function of the standard graded algebra $gr_{\mathfrak{m}}(A) := \oplus_{t\geq 0}\mathfrak{m}^t/\mathfrak{m}^{t+1}$, the so-called *tangent cone* of A for the reason that we shall explain later. The problems arise because, in passing from

A to $gr_{\mathfrak{m}}(A)$, we may lose many good properties, such as being a complete intersection, being Cohen–Macaulay or Gorenstein.

Despite the fact that the Hilbert function of a standard graded algebra A is well understood when A is Cohen–Macaulay, very little is known when it is a local Cohen–Macaulay ring. One of the main problems is whether geometric and homological properties of the local ring A can be carried on the corresponding tangent cone $gr_{\mathfrak{m}}(A)$. For example if a given local domain has fairly good properties, such as normality or Cohen–Macaulayness, its depth provides in general no information on the depth of the associated graded ring. It could be interesting to remind that an open problem is to characterize the Hilbert function of an affine curve in $\mathbf{A}^3$ whose defining ideal is a complete intersection, while a well known formula gives the Hilbert function of any complete intersection of homogeneous forms in terms of their degrees.

The Hilbert function of a local ring $(A,\mathfrak{m})$ is a classical invariant which gives information on the corresponding singularity. The reason is that the graded algebra $gr_{\mathfrak{m}}(A)$ corresponds to an important geometric construction: namely, if A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the tangent cone of V, that is the cone composed of all lines that are limiting positions of secant lines to V in 0. The *Proj* of this algebra can also be seen as the exceptional set of the blowing-up of V in 0.

Other graded algebras come into the picture for different reasons, for example the Rees algebra, the Symmetric algebra, the Sally module and the Fiber Cone. All these algebras are doubly interesting because on one side they have a deep geometrical meaning, on the other side they are employed for detecting basic numerical characters of the ideals in the local ring $(A,\mathfrak{m})$. Therefore, much attention has been paid in the past to determining under which circumstances these objects have a good structure.

In some cases the natural extension of these results to $\mathfrak{m}$-primary ideals has been achieved, starting from the fundamental work of P. Samuel on multiplicities. More recently the generalization to the case of a descending multiplicative filtration of ideals of the local ring A has now become of crucial importance. For example, the Ratliff–Rush filtration (cf. papers by J. Elias, W. Heinzer, S. Huchaba, S. Itoh, T. Marley, T. Puthenpurakal, L.J. Ratliff–D. Rush, M.E. Rossi, J. Sally, G. Valla) and the filtration given by the integral closure of the powers of an ideal (cf. papers by A. Corso, S. Itoh, C. Huneke, C. Polini, B. Ulrich, W. Vasconcelos, J. Verma) are fundamental tools in much of the recent work on blowing-up rings.

Even though of intrinsic interest, the extension to modules has been largely overlooked, probably because, even in the classical case, many problems were already so difficult. Nevertheless, a number of results have been obtained in this direction: some of the work done by D. Northcott, J. Fillmore, C. Rhodes, D. Kirby, H. Meheran and, more recently, T. Cortadellas and S. Zarzuela, A.V. Jayanthan and J. Verma, T. Puthenpurakal, V. Trivedi has been carried over to the general setting.

We remark that the graded algebra $gr_{\mathfrak{m}}(A)$ can also be seen as the graded algebra associated to an ideal filtration of the ring itself, namely the $\mathfrak{m}$-adic filtration

$\{\mathfrak{m}^j\}_{j\geq 0}$. This gives an indication of a possible natural extension of the theory to general filtrations of a finite module over the local ring $(A,\mathfrak{m})$.

Let A be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. Let $\mathfrak{q}$ be an ideal of A; a $\mathfrak{q}$-filtration $\mathbb{M}$ of M is a collection of submodules M_j such that

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_j \supseteq \cdots.$$

with the property that $\mathfrak{q}M_j \subseteq M_{j+1}$ for each $j \geq 0$. In the present work we consider only *good* $\mathfrak{q}$*-filtrations* of M : this means that $M_{j+1} = \mathfrak{q}M_j$ for all sufficiently large j. A good $\mathfrak{q}$-filtration is also called a *stable* $\mathfrak{q}$*-filtration.* For example, the $\mathfrak{q}$-adic filtration on M defined by $M_j := \mathfrak{q}^j M$ is clearly a good $\mathfrak{q}$-filtration.

We define the *associated graded ring* of A with respect to $\mathfrak{q}$ to be the graded ring

$$gr_{\mathfrak{q}}(A) = \bigoplus_{j\geq 0}(\mathfrak{q}^j/\mathfrak{q}^{j+1}).$$

Given a $\mathfrak{q}$-filtration $\mathbb{M} = \{M_j\}$ on the module M, we consider the *associated graded module* of M with respect to $\mathbb{M}$

$$gr_{\mathbb{M}}(M) := \bigoplus_{j\geq 0}(M_j/M_{j+1})$$

and for any $\overline{a} \in \mathfrak{q}^n/\mathfrak{q}^{n+1}$, $\overline{m} \in M_j/M_{j+1}$ we define $\overline{a}\,\overline{m} := \overline{am} \in M_{n+j}/M_{n+j+1}$. The assumption that $\mathbb{M}$ is a $\mathfrak{q}$-filtration ensures that this is well defined so that $gr_{\mathbb{M}}(M)$ has a natural structure as a graded module over the graded ring $gr_{\mathfrak{q}}(A)$.

Denote by $\lambda(*)$ the length of an A-module. If $\lambda(M/\mathfrak{q}M)$ is finite, then we can define the *Hilbert function* of the filtration $\mathbb{M}$, or of the filtered module M with respect to the filtration $\mathbb{M}$. It is the numerical function

$$H_{\mathbb{M}}(j) := \lambda(M_j/M_{j+1}).$$

In the classical case of the $\mathfrak{m}$-adic filtration on a local ring $(A,\mathfrak{m},k)$ we write $H_A(n)$ and remark that it coincides with $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$.

Its generating function is the power series

$$P_{\mathbb{M}}(z) := \sum_{j\geq 0} H_{\mathbb{M}}(j) z^j.$$

which is called the *Hilbert series* of the filtration $\mathbb{M}$. By the Hilbert–Serre theorem we know that the series is of the form

$$P_{\mathbb{M}}(z) = \frac{h_{\mathbb{M}}(z)}{(1-z)^r}$$

where $h_{\mathbb{M}}(z) \in \mathbb{Z}[z]$, $h_{\mathbb{M}}(1) \neq 0$ and r is the Krull dimension of M. The polynomial $h_{\mathbb{M}}(z)$ is called the *h-polynomial* of $\mathbb{M}$.

This implies that, for $n \gg 0$

$$H_{\mathbb{M}}(n) = p_{\mathbb{M}}(n)$$

where the polynomial $p_{\mathbb{M}}(z)$ has rational coefficients, degree $r-1$ and is called the *Hilbert polynomial* of $\mathbb{M}$.

We can write

$$p_{\mathbb{M}}(X) := \sum_{i=0}^{r-1} (-1)^i e_i(\mathbb{M}) \binom{X+r-i-1}{r-i-1}$$

where we denote for every integer $q \geq 0$

$$\binom{X+q}{q} := \frac{(X+q)(X+q-1)\ldots(X+1)}{q!}.$$

The coefficients $e_i(\mathbb{M})$ are integers which will be called the *Hilbert coefficients* of $\mathbb{M}$. In particular $e_0 = e_0(\mathbb{M}) = h_{\mathbb{M}}(1)$ is the *multiplicity* and it depends on M and on the ideal $\mathfrak{q}$.

When we consider the $\mathfrak{m}$-adic filtration in the local ring $(A, \mathfrak{m})$, the Hilbert function of A measures the minimal number of generators (denote $\mu(\quad)$) of the powers of the maximal ideal. In the one-dimensional case the asymptotic value is the multiplicity e_0. It is a natural question to ask whether the Hilbert function of a one-dimensional Cohen–Macaulay ring is not decreasing. Clearly, this is the case if $gr_{\mathfrak{m}}(A)$ is Cohen–Macaulay, but this is not a necessary requirement.
Unfortunately, it can happen that $H_A(2) = \mu(\mathfrak{m}^2) < H_A(1) = \mu(\mathfrak{m})$. The first example was given by J. Herzog and R. Waldi in 1975. In 1980 F. Orecchia proved that, for all embedding dimension $v = \mu(\mathfrak{m}) \geq 5$, there exists a reduced one-dimensional local ring of embedding dimension v and decreasing Hilbert function. L. Roberts in 1982 built ordinary singularities with decreasing Hilbert function and embedding dimension at least 7. J. Sally conjectured, and J. Elias proved, that the Hilbert function of one-dimensional Cohen–Macaulay local rings of embedding dimension three is not decreasing (see [21] and [77]). Interesting problems are still open if we consider local domains. S. Kleiman proved that there is a finite number of admissible Hilbert functions for graded domains with fixed multiplicity and dimension. The analogous of Kleiman's result does not hold in the local case.

Nevertheless, V. Srinivas and V. Trivedi in [98] proved that the number of Hilbert functions of Cohen–Macaulay local rings with given multiplicity and dimension is finite (a different proof was given by M.E. Rossi, N.V. Trung and G. Valla in [79]). This is a very interesting result and it produces upper bounds on the Hilbert coefficients. If $(A, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension r and multiplicity e_0, then

$$e_i \leq e_0^{3i!-i} - 1 \text{ for all } i \geq 1.$$

(see [98, Theorem 1], [79, Corollary 4.2]). These bounds are far from being sharp, but they have some interest because very little is known about e_i with $i > 2$.