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C. Cattaneo (Ed.)

Vedute e problemi attuali in relatività generale

Sestriere, Italy 1958

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FONDAZIONE CIME ROBERTO CONTI

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Lectures given at the
Centro Internazionale Matematico Estivo (C.I.M.E.),
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VEDUTE E PROBLEMI ATTUALI IN RELATIVITA' GENERALE

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
(C.I.M.E.)

P.G.BERGMANN

PROBLEMS OF QUANTIZATION

ROMA - Istituto Matematico dell'Università - 1958

GENERAL REMARKS. The principal subject of these lectures is to be the present status of the program of quantization of general relativity and of general-relativistic theories. Because of the unfamiliarity of many mathematicians with the physical ideas in current quantum theory, I shall attempt to give a brief summary of the pertinent ideas later on. I shall also emphasize in my lectures the classical (i.e. non-quantum) aspects of the program, in particular the concept of observables. I shall also, if time permits, give a brief account of the present status of the *theory of motion*. Perhaps it will be necessary to relegate this topic to a seminar.

REFERENCES. For relativity I suggest any standard textbook as background. A thorough grounding in Riemannian and related geometries is desirable for any study in general relativity. For the fundamental ideas in quantum theory probably Dirac's book (Oxford, 1947) is a good contemporary introduction for mathematicians, though J.v.Neumann's old book is still excellent. For a more physical slant Bohm's recent book can be recommended. For quantum field theory there are now available, in addition to G.Wentzel's old book (Vienna, 1943), a book by S.S.Schweber and a series of articles by Schwinger, Tomonaga, Feynman, and Dyson, to mention but the most important.

Turning to the program of quantization of general relativity, I mention a series of old articles by L.Rosenfeld (Annalen d. Physik, 1930, Inst,H.Poincaré, 1932), my own articles in the Physical Review (1949 to date), and various status reports in Helvetica Physica Acta (Suppl.IV,1956, which is the report of the 1955 conference at Berne), Reviews of Modern Physics (July,1957) and the separately issued proceedings of the Chapel Hill conference of January, 1957 One article by O. Klein will be found in the N. Bohr "Festschrift", which has been published as a book. Additional references may suggest themselves in the course of our lectures and seminars.

OUTLINE OF LECTURES. The following preliminary outline is meant to be flexible, in accordance with the wishes of the participants.

1. Physical motivation of the program of quantization.
2. Formal properties of general-relativistic theories with an action principle.
3. Summary of concepts of quantum theory.
4. Technical report on the status of the program of quantization.
5. Construction of observables in general relativity.
6. Theory of motion.

1. PHYSICAL MOTIVATION. At present we have two major theoretical structures in theoretical physics, which have not been fused together, quantum theory and relativity. Quantum theory represents the formal and complete codification of our recognition that it is impossible to determine simultaneously with complete accuracy any two dynamical variables of a system which are canonically conjugate (in the sense of Hamilton's mechanics). According to quantum theory there is a strong mutual interaction between a physical system and an observer that prevents the construction of a complete set of Cauchy data and their integration in the course of time, as had been envisaged by Laplace. General relativity, on the other hand, probably represents the most perfect example

of a (non-quantum) field theory now available and certainly accounts better than any other theory for all the known facts about the gravitational field.

With two such comprehensive theoretical structures available, it appears only reasonable that one should attempt to extend each into the field covered by the other, so that the attempted integration should either result in an irreconcilable clash and contradiction, or in success. Either event would have the greatest heuristic value for the development of physical theory as a whole. At present we have not yet reached that stage.

2. GENERAL-RELATIVISTIC THEORIES WITH AN ACTION PRINCIPLE.

We shall call a theory general-relativistic or generally covariant if its laws take the same form in every reasonable curvilinear coordinate system. For this definition it is not essential that this form be that of tensor equations, though tensor laws are an important example. If we consider a set of dynamical laws that may be interpreted as the Euler-Lagrange equations of a variational principle - and all proposed theories in physics possess this property - , then it is necessary and sufficient for the relativistic invariance of these laws that for any two coordinate systems chosen the action integrals of the same form are equivalent, in that they differ at most by a surface integral,

$$(2.1) \qquad \int L[y_A(x^\rho)]\, d^4x = \int L[y_B'(x^{\sigma\prime})\, d^4x' + \oint \Gamma^\rho \, d\Sigma_\rho$$

where the y' are the transforms of the y. This general principle makes no reference to the Riemannian nature of space-time, or any other assumed geometric structure.

If we consider in particular an infinitesimal coordinate transformation, and if we restrict ourselves to an action principle in which L is a function only of the y's and their first partial derivatives $y_{A,\rho}$, then we have the principle :

$$\partial^A L \, \bar{\delta} y_A + \partial^{A\rho} L \, \bar{\delta} y_{A,\rho} + \Gamma^\rho{}_{,\rho} \equiv 0$$

or

(2.2) $$\delta^A L \, \bar{\delta} y_A + C^\rho{}_{,\rho} \equiv 0$$

where the symbol $\delta^A L$ stands for $\partial^A L - (\partial^{A\rho} L)_{,\rho}$, the so-called variational derivative of L, and the field C^ρ, which I shall call the "generating density", is determined by the structure of the Lagrangian. $\bar{\delta}$ is the symbol for the infinitesimal transformation law, in this case of the field variables, representing the (infinitesimal) change of the field as a function of the coordinates.

Because we assume general-relativistic covariance, $\bar{\delta} y_A$ involves a set of four arbitrary functions, the "descriptors" of the infinitesimal coordinate transformation $\xi^a \equiv \delta x^a$. It follows that Eq.(2.2) involves differential identities between the field equations, whose structure depends on the assumed transformation law of the field variables. Because the (contracted) Bianchi identities are an example of such identities, we shall call the identities between the field equations that are related to their covariance Bianchi identities. Let, for instance, the transformation law be of the form

(2.3) $$\bar{\delta} y_A = c_{A\rho}{}^\sigma (y_B, y_{B,\sigma}) \, \xi^\rho{}_{,\sigma} + d_{A\rho} \xi^\rho ,$$

then we have

$$\delta^A L \, (c_{A\rho}{}^\sigma \xi^\rho{}_{,\sigma} + d_{A\rho} \xi^\rho) + C^\rho{}_{,\rho} \equiv 0$$

and thus

$$(\delta^A L \, c_{A\rho}{}^\sigma \, \xi^\rho + C^\sigma)_{,\sigma} + [\delta^A L \, d_{A\rho} - (c_{A\rho}{}^\sigma \, \delta^A L)_{,\sigma}] \xi^\rho \equiv 0$$

Because the functions ξ^ρ are arbitrary, we can, by integrating this equation over a four-dimensional domain and converting the first term into a surface integral, conclude that

(2.4) $$(c_{A\rho}^{\ \sigma}\, \delta^A L)_{,\sigma} - d_{A\rho}\, \delta^A L \equiv 0 \quad ,$$

a set of four differential identities between the field equations

(2.5) $$\delta^A L = 0$$

From the procedure that we have used in the derivation of these identities it is clear that the order of the differential identities equals the highest differential order of the ξ^ρ that occurs in the transformation law of the type (2.3), whereas the differential order of the field variables y_A, which are arguments of the coefficients c,d,..., is immaterial.

Even if the field equations cannot be interpreted as a set of Euler-Lagrange equations, they will not lend themselves to an ordinary Cauchy-type initial-value problem, provided the variables occurring in them are not all individually invariant, $\bar{\delta} y_A = 0$. Even with given initial values on a given three-dimensional hypersurface of the field variables and of a given (finite) number of their derivatives, it is always possible to change the values of the field variables elsewhere by a coordinate transformation, which is restricted to be the identity transformation on the initial hypersurface, hence the values of the field variables off the hypersurface cannot be determined by the initial values on the hypersurface.

With differential identities of the type (2.4), we can prove in detail just how the equations differ from an ordinary set. Consider the one term in Eqs.(2.4) which contains third-order derivatives of the field variables. This term is :

(2.6) $$C_{A\mu}^{\sigma}\,(\partial^{A\tau}L)_{,\tau\sigma} \equiv C_{A\mu}^{\sigma}\,\partial^{A\tau}\,\partial^{B\rho}\,Ly_{B,\rho\tau\sigma} + \ldots \equiv 0$$

Suppose, for the sake of simplicity we choose as an initial hypersurface one on which $x^0 = 0$. Then it follows that

(2.7) $$C_{A\mu}^{\;\;\circ}\,\Lambda^{AB} \equiv 0\,,$$

$$\Lambda^{AB} \equiv \partial^{A0}\,\partial^{B0}\,L\,.$$

But this matrix Λ^{AB} also represents the set of coefficients of the second-order "time" derivatives in the field equations themselves,

(2.8) $$\delta^{A}L = -\,\Lambda^{AB}\,y_{B,00} + \ldots$$

It follows from Eq.(2.8) that the matrix Λ^{AB} is singular and that it possesses (at least) four eigenvectors that belong to the eigenvalue 0

We arrive at two conclusions :

(1) (At least) four of the highest "time" derivatives of the field variables are not determined by the field equations.

(2) (At least) four linear combinations of the field equations are free of second-order time derivatives and thus represent restrictions on the choice of the field variables and their first-order derivatives on an initial hypersurface. Such relationships are often called constraints, an expression that was originally used in connection with the Hamiltonian formulation of the theory.

In passing, I should like to note that relationships of the form (2.2) play a role in the theory of motion, a topic to which I hope to come back toward the end of these lectures.

The differential identities, and in particular the relations (2.7), lead to complications if we attempt to pass over from the

Lagrangian to the Hamiltonian form of the theory, a step that is often considered preliminary to quantization. Ordinarily, in a field theory, one introduces the so-called canonical momentum densities by the definition

(2.9)
$$\pi^A = \partial^{A0} L \quad .$$

With their help, one then defines the Hamiltonian density

(2.10)
$$H = y_{A,0}\, \pi^A - L$$

where all "time" derivatives have been expressed in terms of the new canonical field variables, the y_A (and possibly their "spatial" derivatives, $y_{A,m}$) and the π^A. The complete set of canonical field equations is

(2.11)
$$y_{A,0} = \partial_A H, \quad \pi^A{}_{,0} = -\delta^A H \,,$$
$$\partial_A \equiv \frac{\partial}{\delta \pi^A}, \quad \delta^A H \equiv \partial^A H - (\partial^{Am} H)_{,m} \quad .$$

Moreover, given some functional of the canonical field variables on an initial hypersurface x^0 = constant and of the coordinates x^a, say Γ, we have the general dynamical law

(2.12)
$$\frac{d\Gamma}{dx^0} = (\Gamma, H) + \frac{\partial \Gamma}{\partial x^0} \,,$$

where the symbol H represents the Hamiltonian, i.e. the integral $\int H d^3x$, and the symbol (,) is a Poisson bracket, defined with the help of the "functional derivatives"

(2.13)(,)
$$(A, B) = \int \Big[\frac{\partial A}{\partial y_A(x^m)} \frac{\partial B}{\partial \pi^A(x^m)} - \frac{\partial A}{\partial \pi^A(x)} \frac{\partial B}{\partial y_A(x)} \Big] d^3x \quad .$$

The functional derivatives of a functional are defined (if they exist) by the relationship

(2.14) $$\delta A = \int [\frac{\partial A}{\partial y_A(x)} \delta y_A(x) + \frac{\partial A}{\partial \pi^A(x)} \delta \pi^A(x)] \, d^3x .$$

where $\delta y_A(x)$, $\delta\pi^A(x)$ are arbitrary infinitesimal variations of the arguments of the functional. The definitions (2.10) through (2.14) are the natural analogs of the corresponding definitions in classical mechanics. The Hamiltonian formalism, when it works, enables us to replace the Euler-Lagrange field equations (2:5) by a set of first-order equations, solved with respect to their "time" derivatives. The Hamiltonian formalism is thus ideally suited to the formulation of initial-value problems in field theory.

The success of the procedure just sketched depends on our ability to express the quantities $y_{A,0}$ wholly in terms of the canonical variables, and this is possible only if the Jacobian of the transformation $y_{A,0} \to \pi^A$ is non-zero. However, we see immediately that the matrix of the partial derivatives,

(2.15) $$\frac{\delta \pi^A}{\delta y_{A,0}} \equiv \Lambda^{AB} ,$$

is singular. Hence, though the "velocities $y_{A,0}$ determine the momentum densities π^A uniquely, the reverse does not hold. Furthermore, the π^A as functions of the "velocities" are not algebraically independent of each other, but satisfy (at least) four relations not involving any "time" derivatives. These relations are called primary constraints. They are satisfied solely as the result of the defining equations (2.9) and bear no relation to the field equations.

The further development of the Hamiltonian theory has shown that it is possible to construct a Hamiltonian density of the type (2.10), which however is not unique but involves four arbitrary functions, multiplied by the four primary constraints. Fixing these arbitrary functions is equivalent to introducing coordinate conditions. Without such conditions, the formal Cauchy problem cannot be uniquely defined, hence the arbitrary functions in the Hamiltonian density.

If the primary constraints are satisfied on one hypersurface x^0 = const., we must require that they remain satisfied, i.e. that their Poisson brackets with the Hamiltonian vanish. This requirement leads to four additional conditions on the canonical field variables, the so-called secondary constraints. Iteration, i.e. the construction of higher time derivatives of the primary constraints, does not lead to additional conditions. The total number of constraints in general relativity and in similar theories is eight at each point of the initial hypersurface. These constraints and the Hamiltonian form a function group.

3. CONCEPTS OF QUANTUM THEORY. Historically, quantum theory began with Schrödinger's celebrated equation. Subsequent developments have shown, however, that there exist many equivalent formulations, of which the "Schrödinger representation" is but one, and I shall attempt to give a fairly general description.

In classical mechanics the "state" of a physical system is determined uniquely by the location of its representative point in phase space, i.e. by the numerical values of all its canonical coordinates q^k, p_k. In quantum mechanics the state is a unit vector in a Hilbert space. Whereas the appropriate group of transformations in classical mechanics is the group of canonical transformations, the analogous group in quantum theory is the group of all unitary transxormations. In classical mechanics every physical variable is capable of generating an infinitesimal transformation. In quantum theory every "observable" A generates an infinitesimal unitary transformation in Hilbert space,

$$\delta U = -\frac{i}{\hbar} A \tag{3.1}$$

All physically meaningful quantities are, therefore, represented as Hermitian linear operators in Hilbert space. The symbol $\hbar$ stands for Planck's original quantum of action, h, divided by 2π and equals $1,05444 \times 10^{-27}$ erg sec. We can construct a complete set of base vectors in Hilbert space, so-to-speak a coordinate system, if we construct the joint eigen vectors of a complete set of commuting operators. By this expression we mean the following. One operator, say q_1, may be highly degenerate. To identify its eigenfunctions uniquely, we take a set of commuting operators $q_1, \ldots, q_n$, so that a set of eigenvalues $q'_k (k = 1, \ldots, n)$ identifies exactly one joint eigenvector. The complete set of commuting operators corresponds approximately to the set of configuration variables in classical mechanics, which also generate a set of commuting infinitesimal canonical transformations.

All other operators will either commute with all q_k (in which case they may be considered functions of the q_k), or they will have non-vanishing commutators. In particular there will be operators p_k such that their commutators with the q_k are

$$[p_k, q_e] = \frac{\hbar}{i} \delta_{kl} \tag{3.2}$$

These will be assumed to be the quantum analogs of the canonical momentum components. Commutators of the type (3.2) are generally the analogs of the correspoding Poisson brackets of classical mechanics, which also are representatives of the commutators of infinitesimal canonical transformations.

The formal scheme of quantum theory is related to physics by two sets of rules. One refers to the outcome of observations, the other sets forth a dynamical law. If an experiment is performed to measure the value of a physical quantity A, then the only possible outcomes of the measurement can be the eigenvalues of the operator $\underline{A}$. If the system is in a state described by the Hilbert vector $|\rangle$, then the average of many measurements of A will be given by the "bracket" (i.e. scalar product)

(3.3) $$\langle A \rangle_{AV} = \langle | \underline{A} | \rangle$$

If $|\rangle$ happens to be an eigen vector of $\underline{A}$, belonging to the eigenvalue a', then the "expectation value" of the measurement will be a', and moreover the expectation value of A^2 will be a'^2, hence the scatter of observations will be zero, the outcome of the measurement will invariably be a'. In all other cases the results of a measurement, repeated many times, will scatter.

The other rule introduces a dynamical law. Let $|1\rangle$ and $|2\rangle$ be two different states of which the physical system is capable. Then for an observable A we have the general rule

(3.4) $$\frac{d}{dt}\langle 1|A|2\rangle = \frac{i}{\hbar}\langle 1|[H,A]|2\rangle + \langle 1|\frac{\delta A}{\delta t}|2\rangle \ .$$

This dynamical law is the precise analog to the law of motion in Hamiltonian classical mechanics.

The formulation of these two rules is "representation-invariant", that is to say, if we perform the following unitary (and possibly time-dependent) transformations

(3.5) $$|\rangle' = U|\rangle \ , \quad \langle|' = \langle|U^+ \ ,$$

$$A' = UAU^+ \ , \quad U\,U^+ = 1 \ ,$$

nothing will change. By means of such unitary transformations we may distribute the time-dependence in any desired manner between the Hilbert vectors and the Hermitian operators, the observables. In particular we speak of a "Schrodinger representation" if

$$\frac{d}{dt}|\rangle = -\frac{i}{\hbar}H|\rangle \ , \quad |\rangle = e^{-\frac{i}{\hbar}Ht}|\rangle_0 \ , \quad \frac{dq_k}{dt} = 0 \tag{3.6}$$

and of a "Heisenberg representation" if

$$\frac{d}{dt}|\rangle = 0 \ , \quad \frac{dq_k}{dt} = \frac{i}{\hbar}[H, q_k] \ , \quad \frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}[H, A]. \tag{3.7}$$

It is remarkable how much can be accomplished with this bare skeleton of rules. For instance it is a fairly easy task to show that if two operators p and x satisfy commutation relations of the kind (3.2) and if we assume for the Hamiltonian H the form

$$H = \frac{1}{2}(x^2 + p^2) \ , \tag{3.8}$$

(the one-dimensional harmonic oscillator), then the only eigenvalues of H are

$$\epsilon_n = (n+\tfrac{1}{2})\hbar \quad , \quad n = 0,1,2,\ldots \tag{3.9}$$

Another simple example, which bears a close relationship to the possible representations of the three-dimensional orthogonal group, is the following. Let L_x, L_y, and L_z be three operators which satisfy the cyclic commutation relations

$$[L_x, L_y] = \frac{\hbar}{i} L_z \ , \text{ etc.} \tag{3.10}$$

and let the Hamiltonian be

$$H = \frac{1}{2}(L_x^2 + L_y^2 + L_z^2).$$

Then the eigenvalues of H are

(3.11) $$\epsilon_j = \frac{1}{2}\hbar^2 j(j+1) \quad , \quad j = 0,\frac{1}{2},1,\frac{3}{2},2,\ldots,$$

and the individual operators $L_x,\ldots$ have the following eigenvalues :

(3.12) $$L_x^m = m\hbar \ , \quad m = -j,-j+1,\ldots,+j \ .$$

In this system (the three-dimensional rotator) H and L_x form a complete set of commuting operators.

I have said that the unitary transformations in Hilbert space are the analog of the canonical transformations in classical physics. This analogy is not perfect, insofar as for a given classical system and its quantum analog it cannot be said that the group of canonical transformations and the group of unitary transformations are isomorphic; they are not. However, these transformations that determine the invariance properties and the symmetry character of the physical system, and their commutator algebra, are generally the same. And these invariant transformations are generated by corresponding constants of the motion. Thus the Hamiltonian operator generates the evolution of the system in the course of time, the components of the linear momentum generate displacements of the coordinate origin, the components of the angular momentum generate orthogonal transformations, etc. Because the transformations of quantum theory are linear transformations it is proper to speak of representations of certain groups. For instance, in the example of a quantum system given in Eqs. (3.10), (3.11), the operators $L_x,\ldots$ form all the representations of the (proper) orthogonal group. The irreducibile representations are characterized by the quantum number j, which takes all integral and half-odd values. The rank of each irreducibile representation equals (2j+1).

In the modern development of quantum theory, the process of quantization has been extended from mechanics to field theories. The axiomatics of quantum field theory has been developed much less well than that of quantum mechanics. Roughly speaking one may conceive of a field theory as of a mechanical system with an infinite number of degrees of freedom. For instance, if we assign to each degree of freedom of a physical system a Hilbert space and if we form the Hilbert space of the whole system as the Kronecker product of the partial Hilbert spaces, then we obtain a space with a non-denumerable number of dimensions, i.e. no Hilbert space at all. This difficulty has been met, in part, by the specification that only those states of a system are to be admitted which differ from the state of lowest energy, the ground state, only with respect to a finite number of degrees of freedom (which ones is not specified). This restriction is, however, not invariant with respect to some very important canonical transformations; there are many other problems of this type that have been met only partially. Although apparently "formal", many of these difficulties have their physical implications. Physicists have worked out a number of working rules that enable them to perform the quantization of some very simple field theories successfully. The only realistic theory with which we are well satisfied is quantum electrodynamics, that is the theory of the electromagnetic field coupled to the field of electrons and positrons according to Dirac's theory. The extension to nuclear forces and meson fields has been only partially successful; we do not know whether we do not understand the dynamical laws imperfectly, whether our procedure of quantization is defective, or whether the principal blame attaches to our methods of approximation.

4. QUANTIZATION PROCEDURES IN GENERAL RELATIVITY. It might appear that with the Hamiltonian formulation of general relativity the groundwork has been laid for a successful quantization. One would hope to replace the classical dynamical variables (the canonical field variables) by quantum operators obeying the canonical commutation relations, and to admit as physical only states which permit the constraints to be satisfied. One obvious difficulty is that there are dynamical variables that are canonically conjugate to the constraints. Now it is very easy to show that if two operators A, B satisfy a commutation relation of the form

$$[A,B] = i\, c\, I \quad , \tag{4.1}$$

where c is an ordinary number and I stands for the identity operator, then neither A nor B possesses proper eigen vectors. For if, e.g. $|a'\rangle$ were an eigenvector of A, so that

$$A|a'\rangle = a'|a'\rangle, \quad \langle a'|A = a' \langle a'|, \tag{4.2}$$

then

$$\langle a'|[A,B]|a'\rangle = 0 \; , \quad \langle a'|icI|a'\rangle = ic \; , \tag{4.3}$$

an obvious contradiction. The only other possibility is that the operation $B|a'\rangle$ does not lead to a Hilbert vector.

Consider now a constraint of the theory, C. Then the only admissible Hilbert vectors are those for which $C|\rangle = 0$, i.e. eigenvectors of C . It follows that for this whole set of quantum states an operator D which is canonically conjugate to C leads outside Hilbert space and thus can have no expectation value or other sensible physical property. In fact, because the eigenvalue $c' = 0$, the same holds true for any operator which does not commute with all the constraints. Hence, because general relativity in the Hamiltonian formulation has twenty canonical field variables, $g_{\mu\nu}$, $\pi^{\mu\nu}$, there are only four algebraically independent observables per space point. There are eight constraints, i.e. combinations of variables required to have the value 0, and eight additional variables conjugate to the constraints. This result would not be unsatisfactory in itself; the electromagnetic field has the same number of independent variables. But unfortunately the structure of the constraints in general relativity in so complicated that so far no one has succeded in ascertaining those combinations of canonical variables that commute with all the constraints. Formally, we can define the observables as the solutions of a set of partial differential equations, but that is not much help. Dirac has made some progress in separating the constraints from the remainder of the variables through a canonical transformation. But he has so far succeded only with the primary constraints. The insulation of the secondary constraints is a much more formidable, and as yet quite unsolved problem.

The discussion sketched out in the preceding paragraphs leads us to a new definition of "*observables*" both in classical and in quantum theory : Instead of considering every dynamical variable as observable, we define as observables those variables that commute (or have vanishing Poisson brackets) with all the constraints. Classically, one can show that the constraints are the generators of coordinate transformations, so that the observables as defined here are coordinate-invariant quantities (not scalars). They are also the only quantities that can be subject to prediction from initial data, that is to say, any formulation of a Cauchy problem in general relativity must be in terms of the observables. The discovery of the observables of general relativity would also have physical interest quite aside from the program of quantization: this discovery (or construction) would also permit us to cast all statements of the theory into manifestly coordinate-invariant form. In the following section I shall report on the construction of observables without reference to the Hamiltonian theory. But first, I shall report briefly on two other approaches to the problem of quantization, through the Lagrangian formalism and with the help of coordinate conditions.

The Hamiltonian formulation of a field theory is well suited to the formulation of continuation but tends to disguise its essentially four-dimensional, covariant nature. For the invariant-theoretical examination the original Lagrangian formulation is preferable. I shall now discuss how one can construct, within the Lagrangian formalism, a group of transformations that permits us to construct commutators between observables. We begin again with an action integral of the form

$$S = \int L[y_A(x),\ y_{A,\rho}(x)]\, d^4x \quad , \tag{4.4}$$

which transforms in accordance with Eq. (2.1); the covariance of the theory is thereby assured. We shall now consider transformations of the variables of the form

$$\begin{aligned} \delta y_A &= f_A\ (y_B, y_{B,\rho}, x^{\alpha}) \\ \delta y_{A,\rho} &= f_{A,\rho} = \partial^B f_A \cdot y_{B,\rho} + \partial^{B\sigma} f_A \cdot y_{B,\sigma\rho} + \partial_\rho f_A \ . \end{aligned} \tag{4.5}$$

In other words, the new variables are to depend on the values of the old variables at the same world point and on the values of their first partial derivatives. The resulting change in the Lagrangian density <u>as a function of its arguments</u> will be the following :

$$\delta' L = D^{\rho}_{,\rho} - \partial^A L \, f_A - \partial^{A\rho} L \, f_{A,\rho}$$

(4.6)

$$= -f_A \, \delta^A L - C^{\rho}_{,\rho}, \quad C^{\rho} = f_A \, \partial^{A\rho} L - D^{\rho}.$$

For the time being, the field D^{ρ}, and hence the field C^{ρ}, is arbitrary. Unless we restrict somehow the functions f_A, the infinitesimal transformation (4.5) will lead to the appareance of second-order derivatives in the (originally first-order) Lagrangian density. We shall call the transformation (4.5) *canonical* if one can find a field C^{ρ} which prevents the appearance of such second-order derivatives of the field variables, and we shall call the transformation *invariant (canonical)* if a C^{ρ}-field can be found so that $\delta' L$ vanishes altogether. In what follows we shall be concerned with invariant transformations. We shall call C^{ρ} the generating density, and an integral of the form $\int C^{\rho} d\Sigma_{\rho}$ the generator of the canonical (or invariant) transformation. Generators of invariant transformations are defined by the identity

(4.7) $$f_A \delta^A L + C^{\rho}_{,\rho} \equiv 0$$

only up to a curl,

(4.8) $$C^{\rho\prime} = C^{\rho} + A^{[\rho\nu]}_{,\nu} \quad .$$

However, with suitable boundary conditions the addition of such a curl does not change the value of the generating integral over a three-dimensional hypersurface , or changes it, at any rate, only by a two-dimensional surface integral,

(4.9) $$\Gamma' \equiv \int C^{\rho\prime} \, d\Sigma_{\rho} = \Gamma + \oint A^{\rho\sigma} \, d\Sigma_{\rho\sigma} \; .$$

Furthermore, the Bianchi identities (2.4) show that there are choices of f_A,

(4.10) $$f_A = d_{A\rho}\, \xi^\rho ,$$

with arbitrary ξ^ρ, whose generating density,

(4.11) $$C^\rho = -c_{A\sigma}{}^{\rho}\, \xi^\sigma\, \delta^A L ,$$

vanishes. In general relativity the transformations (4.10) are the infinitesimal coordinate transformations, which are certainly invariant, and whose generating density, we see, vanishes.

Without proof we shall state the following :

(a) The invariant transformations form a group (this is obvious).

(b) The transformations (4.10) form an invariant subgroup.

(c) There is a one-to-one relation between the members of the factor group (with respect to the invariant subgroup) and the non-trivial generators. By defining as the (modified) Poisson brackets of the generators those generators corresponding to the commutators of the factor group, we obtain a commutator algebra of the possible generators Γ . It remains to establish the nature of the possible generators.

It follows from the defining equation (4.7) that the generating densities of the invariant transformations satisfy equations of continuity if the field equations are satisfied. With suitable boundary conditions the integrals, the generators, are therefore *constants of the motion*. We have constructed a commutator algebra between the constants of the motion. Because we include all constants of the motion, including those that depend explicitly on the coordinates (cf. Eq. (4.5)), we have obtained a set that is equivalent to the quantities that we have previously called observables. To any observable quantity it is possible to construct that constant of the motion which equals the observable at a chosen coordinate time. Conversely, no constants of the motion can be correlated to quantities that are not observables, because only of the observables can one predict the values at one time from initial data given at a different time.

A careful analysis has shown two further statements to be true :

(d) The generator Γ is related to the transformation law of another observable Δ under the transformation law (4.5) by the relationship

$$\delta\Delta = (\Delta, \Gamma) , \tag{4.12}$$

where the symbol (,) denotes the commutator bracket defined by (c).

(e) Whenever a Hamiltonian formalism is available, then the commutator brackets defined by (c) ara analogous to the commutator brackets introduced by Dirac. In the case of general relativity the Dirac brackets are Poisson brackets, restricted to observables.

Finally, I shall consider a theory which is relativistically invariant but which has been cast into a restricted coordinate frame. We assume that the coordinate frame has been restricted by four conditions of the form

$$\alpha^{A\sigma}{}_{\rho}\, y_{A,\sigma} + \beta_{\rho} = 0 , \tag{4.13}$$

where the coefficients α and β are functions of the undifferentiated field variables y_A. For the coordinate conditions to be effective, it is also necessary that the coefficients $\alpha^{Ao}{}_{\rho}$

be linearly independent of each other and of the rows (or columns) of the matrix Λ^{AB}, so that the determinant of the 4x4 matrix

(4.14) $$\gamma_{\mu\nu} = a^{Ao}{}_{\mu}\, C_{A\nu}{}^{o}$$

does not vanish. In that case it is possible to produce either Lagrangian or Hamiltonian equations by adding a quadratic form of the coordinate conditions (4.13) with non-singular coefficients to the original Lagrangian. The resulting differential equations of the modified theory will be equivalent to those of the original theory if we require that on an initial hypersurface the conditions (4.13) themselves are satisfied and their first time derivatives vanish. This scheme is the natural extension of one first proposed by E.Fermi in electrodynamics. However, if we proceed to develop the Hamiltonian formalism of the modified theory, then we find that again we have a theory with (eight) constraints. Those variables whose Poisson brackets with the constraints (i.e. with the coordinate conditions and their first time derivatives) vanish are the observables of the unmodified theory.

It is also possible to construct an equivalent commutator within the Lagrangian formalism. We consider the set of all those transformations which do not change the Lagrangian and which change the coordinate conditions at most by a linear combination of themselves. Again there will be an invariant subgroup consisting of the transformations generated by coordinate conditions themselves, and the factor group will be represented by the non-trivial observables and their commutator algebra.

The introduction of the coordinate conditions has apparently modified the situation insofar as all the original field variables can be made the subyect of a Cauchy problem, with the initial data being restricted by the coordinate conditions, and the field equations augmented by the time derivatives of the coordinate conditions. However, even with coordinate conditions added, the problem of finding the observables of the original theory is not facilited, and the remaining variables, though their time-dependance is now fixed by the coordinate conditions, cannot be included naturally in the commutator algebra that we can hope will lead to quantum theory.

5. OBSERVABLES IN GENERAL RELATIVITY. As I have mentioned before, the construction of true observables in general relativity by means of the defining equations in any of the formalism described in Section 4 is extremely difficult, if at all possible. Newman has proposed a scheme that permits the construction of observables by means of a power series expansion, which starts with the so-called linearized theory of gravitation and then improves systematically. The lowest non-trivial order begins with plane gravitational waves and works with these normal modes, their amplitudes and phases. The resulting observables are highly non-local, and it is not known whether the method converges.

Arnowitt, Deser, and Misner have proposed a scheme for the construction of a special coordinate system on the assumption that the metric of a Riemannian manifold satisfies reasonable boundary conditions at spatial infinity. By means of a set of non-linear partial differential equations they want to construct a special coordinate system that is uniquely determined except for Lorentz transformations (or a set of transformations isomorphic to the Lorentz group). In that special coordinate system the components of the metric tensor would all become observables. They propose to complete the program of quantization by using methods due to J. Schwinger. Their papers will be submitted to the Physical Review in the course of this summer and fall.

Komar also constructs a special coordinate system, but by means of local conditions and without reference to boundary conditions. Gèhèniau and Debevers, and independently Komar, discovered about 1955 that if Einstein's field equations are satisfied then there exist four algebraically independent scalars of the Riemann-Christoffel curvature tensor, which may, for instance, be obtained as the solutions of a characteristic-value problem. This problem may be posed most simply in the form

$$[R_{\iota k\lambda\mu} - \Lambda(g_{\iota\lambda}\, g_{k\mu} - g_{\iota\mu}\, g_{k\lambda})]\ V^{[\lambda\mu]} = 0 \quad . \tag{5.1}$$

The skew-symmetric characteristic tensor V is of no further interest in this connection. There are, however, two independent pairs of conjugate complex eigen-values Λ in general, corresponding to four real numbers at each world point. We shall denote these four numbers by the symbols $A^{\mu}(\mu = 1,\dots,4)$. Except for cases possessing special symmetry, the four functions $A^{\mu}(x^{\rho})$ are algebraically independent of each other, that is to say, the determinant

$$J \equiv \det|A^{\mu}{}_{,\rho}| \not\equiv 0 \quad . \tag{5.2}$$

does not vanish identically.

We shall now define a set of ten new functions $\gamma^{\mu\nu}$,

$$\gamma^{\mu\nu} = g^{\rho\sigma}\, A^{\mu}{}_{,\rho}\, A^{\nu}{}_{,\sigma} \quad . \tag{5.3}$$

These ten functions may be interpreted as the scalar products of the gradient fields of the four scalars A^{μ}, but also as the components of the contravariant metric tensor in the special coordinate system of the "intrinsic coordinates A^{μ}".

We now claim that the ten functions $\gamma^{\mu\nu}(A^{\rho})$ are observables in the sense that for any chosen values of the four arguments A^{ρ} and for any choice of the superscripts μ, ν the value of that quantity is completely determined by the intrinsic properties of the Riemann-Einsteinian manifold and independent of the choice of the original coordinate system in which the calculation was carried out.

Because these observables may be considered as the components of the metric in a particular coordinate system, the set of the observables $\gamma^{\mu\nu}(A^{\rho})$ is ***complete*** : Knowledge of all these quantities gives us total information about the properties of the manifold. However, these observables are ***redundant*** : They are connected by a system of differential equations that reduces the actual number of degrees of freedom. Let us consider the new observables in terms of the intrinsic coordinate system A^{ρ}, which in their capacity of coordinates we shall denote by ξ^{ρ}, reserving the designation A^{ρ} for the set of quantities that are determined as the eigenvalues of the Riemann tensor, Eq. (5.1). Then we have two sets of equations that must be satisfied, Einstein's field equations,

(5.4) $$G^{k\lambda}[\gamma^{\mu\nu}(\xi^{\rho})] = 0 \quad ,$$

and the coordinate conditions

(5.5) $$A^{\rho}(\xi^{\sigma}) = \xi^{\rho} \quad .$$

Komar has shown that the totality of these conditions reduces the number of initial data that may be chosen on a hypersurface $\Sigma(A^{\rho})=0$ to four data per point. Then the Riemann-Einstein manifold (i.e. a Riemannian manifold that satisfies the field equations) is com-