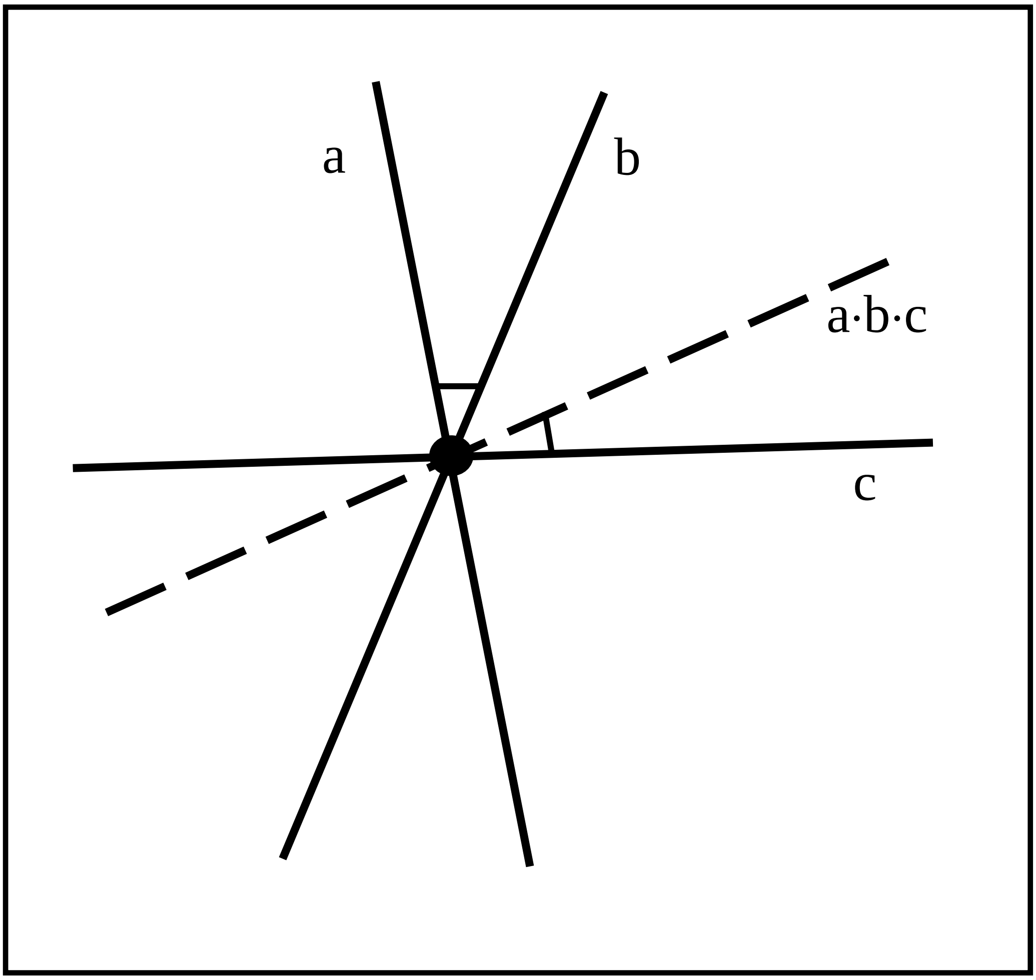
Helmut Karzel & Günter Graumann

Metric Planes
A Group Theoretical Foundation

Renewed and translated by
G. Graumann



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Metric Planes
A Group Theoretical Foundation

based
on two lectures of
Helmut Karzel
held in 1962 and 1963 at the University of Hamburg
and worked out under the leading of Prof. Dr. Karzel

Renewed, completed and translated by
G. Graumann

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Preface

After the discovery of non-Euclidean geometries in the nineteenth century and the change in the view over mathematics away from an object orientated view to a theory of structures, a foundation of classical Euclidean geometry in a modern axiomatic version was given by David Hilbert in his book "Grundlagen der Geometrie". Afterwards several mathematicians were concerned with the classical Euclidean and non-Euclidean geometries.

In 1907, J. Hjelmslev used the aid of reflections in his foundation of geometry. A further impulse while working with metric geometry - based only on incidence, perpendicularity and congruence - took place in the 1930s by K. Reidemeister, whereby the concept of straight-line reflection was chosen as the basic concept instead of congruence. In 1943, A. Schmidt first took the theorem of the three reflections under the axioms.

In the 1950s, F. Bachmann, who had also been involved in the area in the 1930s, followed this up. From his 1951 article „Zur Begründung der Geometrie aus dem Spiegelungsbegriff" (The Founding of Geometry from the Concept of Reflection) and his lecture on this subject, which was published in 1952/53. The book „Aufbau der Geometrie aus dem Spiegelungsbegriff" (Construction of Geometry from the Concept of Reflection), published first in 1956, contains the fundamental ideas of our subject matter in its preface.
"For the usual Euclidean plane and also for the classical non-Euclidean planes, the following facts can easily be found: The points and the straight lines clearly correspond to the reflections at the points and the reflections on the straight lines, the involution elements of the movement group. Geometric relationships such as the incidence of points and lines and the orthogonality of straight lines can be represented by group-theoretic relations between the corresponding reflections. It is therefore possible to translate geometrical sentences into sentences via reflections and reflection products.
We are thus led to make the reflections to the subject of geometrical consideration, and to operate in the movement group "Geometry of Reflections". If we consider the reflections themselves as geometric objects, namely as new "points" and "straight lines," we can define geometric relations such as "incidence" and "orthogonality" by means of group theoretic relations such that the new domain is a faithful image of the originally given points and lines with their incidence, orthogonality, etc. " (Translation G. Graumann)

H. Karzel, in the second half of the 1950s, developed the group theoretical foundation of metric planes based on the ideas of F. Bachmann and an article „Ein gruppentheoretischer Beweis des Satzes von Desargues in der absoluten Axiomatik" (A Group Theoretical Evidence of the Theorem of Desargues in the Absolute Axiomatics) of his Ph.D. supervisor E. Sperner whereat he found the so-called “Lotkerngeometrien” (Geometries with a Kernel of Perpendicularity) and geometries with a proper center. The algebraic description of these geometries is characterized by the fact that the corresponding field has the characteristic 2. In 1962/63, H. Karzel presented in a lecture followed by a proseminar a comprehensive presentation of the group-theoretic construction of all possible plane metric geometries.

In collaboration with H. Karzel, G. Graumann subsequently prepared a German lecture paper which forms the basis for this book.

In the past fifty years, generalizations have also been made on three and more dimensions, and particular types of metric geometry have been investigated more closely and related to particular properties of the corresponding algebraic structure. However, almost no changes have been made to the basic concept of the fundamental theory of plane metric geometries, so the text here is still relevant.
In the present version, a number of sets of sentences have been altered in contrast to the lecture preparation of 1963, and explanatory notes, which refer to the illustrative Euclidean geometry, have been added to support the understanding. In addition, several proofs were presented more fully.

The book is aimed at interested mathematicians as well as students of mathematics. In particular, it is suitable for teachers and teacher students of grammar schools and high schools as a mathematical background of geometry teaching concerning congruent mappings.

Bielefeld / Wessling, 2018 Günter Graumann / Helmut Karzel

Index of Content

Preface 1

Index of content 3

1 Groups with involutoric generator 5

1.1 Fundamental statements about groups with an involutoric generator 5

1.2 Mappings in groups with involutoric generator 6

2 The Group -lane (**G**,**E**) 9

2.1 Fundamental theorems for a group plane 9

2.2 Mappings in the group plane 16

2.3 Geometries with kernels of perpendicularity 23

2.4 Regular geometries 29

2.5 Survey of the different types of geometries 33

3 The group Space $\mathbf{G}(\mathbf{E}^2,\mathbf{E}^3)$ 35

4 Construction of the field of coordinates **K**(**G**,**E**) 53

5 Imbedding of the group plane into a projective plane 69

5.1 Implementation of homogeneous coordinates for the points of a plane of the group space 70

5.2 Implementation of homogeneous coordinates for the lines and planes of the bundle of the fixed point (ω) 72

6 Construktion of a quadratic form 79

6.1 Construktion of a quadratic form for Char $\mathbf{K}(\mathbf{G}, \mathbf{E}) \neq 2$ 81

6.2 Constraction of a quadratic form for Char $\mathbf{K}(\mathbf{G}, \mathbf{E}) = 2$ 85

6.3 Main theorem of the metric plane (**G**, **E**) that is imbedded into a projective plane over $\mathbf{V}_3(\mathbf{K})$ 93

1 Groups with involutoric generator

All metric planes, i.e. all plane geometrical structures with a congruence relation and a orthogonality, can be founded by its group of congruence mappings which are generates by the reflections. A fundamental theorem with this is the so-called theorem of the three reflections. Conversely for the foundation of metric planes we will start with a group which is generated by involutorisch elements.

In this chaper we collect some theorems which do not need an axiom besides the well-known axioms for a group.

Fundamental structure: We start with a **group G(·) generated by a not-empty set E of involutoric[1] elements**

Notation remark: The elements of **E** are denoted with little latin letters and for the elements of **G** we use little greak letters; often the multiplication sygn in **G**(·) we delete and only write the elements directly successively.

The starting conditions therefore are the following:

(G 0) a) **G**(·) is a group.[2]
b) **E** is a generator set of **G**,
i.e. each $\gamma \in$ **G** can be written as finite products of elements of **E**[3].
c) **E** $\neq \varnothing$.
d) Each element of **E** is involutoric,
i.e. $x \in$ **E** $\Rightarrow$ x inv (that means $x^2 = 1$ und $x \neq 1$).

1.1 Fundamental statements about groups with an involutoric generator set which require only (G 0)

Satz 1.1: **G** does contains at least two elements.

Proof: **G** as group contains the neutral element 1 and there exists at least one $e \in$ **E**. This is different from 1 because it is involutoric ($e \neq 1$). Thust **G** does contain at least the two elements 1 and e.

Theorem 1.2: Let $x_1, x_2, \ldots, x_n$ inv and $\xi = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, then
$\xi^{-1} = x_n \cdot x_{n-1} \cdot \ldots \cdot x_1$.

---

[1] An element γ of a group is called **involotoric**, if and only if $\boldsymbol{\gamma^2 = 1}$ **und** $\boldsymbol{\gamma \neq 1}$. One also says the element does have the order 2. This is equivalent with $\gamma = \gamma^{-1} \neq 1$.
If an element γ of **G** is involutoric so with denote it shortly by „**γ inv**“.
[2] We remark that a group is an algebraic structure which holds the closeness, associative law, existence of a neutral element and an iverse element for each element (see normal literature for Algebra).
[3] We also remark that a group can have different generators; thus our fundamental structure is the pair (**G**,**E**).

Proof: $(x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot x_n) \cdot (x_n \cdot x_{n-1} \cdot \ldots \cdot x_1) = (x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1}) \cdot (x_n \cdot x_n) \cdot x_{n-1} \cdot \ldots \cdot x_1) = (x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1}) \cdot 1 \cdot (x_{n-1} \cdot \ldots \cdot x_1) = (x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1}) \cdot (x_{n-1} \cdot \ldots \cdot x_1)$ because of the associative law and it is $x_n \cdot x_n = 1$ because „x_n inv“. In the same way the product can be reduced until $x_1 \cdot x_1$ which is equal to 1.

<u>Theorem 1.3</u>: Is $x, y \in \mathbf{E}$, then: xy inv $\Leftrightarrow$ $xy = yx$ and $x \neq y$.

Proof: Because of theorem 1.2 we have $(xy)^{-1} = yx$. With this we get: xy inv $\Leftrightarrow$ $xy = yx$ und $xy \neq 1$. By multiplication with y of the right side of the unequality we get $xyy = x \neq 1 \cdot y = y$ (knowing $y^2 = 1$ because $y \in \mathbf{E}$).

<u>Theorem 1.4</u>: Ist $x, y, w \in \mathbf{E}$, dann gilt: xyw inv $\Leftrightarrow$ wxy inv $\Leftrightarrow$ ywx inv
$\Leftrightarrow$ xwy inv $\Leftrightarrow$ yxw inv $\Leftrightarrow$ wyx inv.

Proof: xyw inv $\Leftrightarrow$ $xyw \cdot xyw = 1$ und $xyw \neq 1$ $\Leftrightarrow$ $w(xywxyw)w = ww = 1$ and $w(xyw)w \neq 1$, d.h. $wxywxyww = wxywxy = 1$ und $wxyww = wxy \neq 1$, so that we have: wxy inv. With this we get: xyw inv $\Leftrightarrow$ wxy inv. Moreover by multiplicating the equalitie $(wxy)^2 = 1$ resp. the inequality $wxy \neq 1$ from the right and left side with y we get the equivalent equality resp. inequality $ywxywx = 1$ and $ywx \neq 1$ (i.e. ywx inv). The statements xwy, yxw, wyx inv are equivalent with upper statements because $xwy = (ywx)^{-1} = ywx$, $yxw = (wxy)^{-1} = wxy$, $wyx = (xyw)^{-1} = xyw$ using theorem 1.2 so that we get the rest of the statements of theorem 1.4.

1.2 Mappings in groups with involutoric generator

<u>Definition</u>: Is $\alpha \in \mathbf{G}$, so we define a mapping $\bar{\alpha} : \mathbf{G} \to \mathbf{G}$ by
$\bar{\alpha}$ **(ξ) = $\alpha \cdot \xi \cdot \alpha^{-1}$** for all $\xi \in \mathbf{G}$.
The set of all these mappings $\bar{\alpha}$ with $\alpha \in \mathbf{G}$ we denote by $\bar{\mathbf{G}}$.

<u>Theorem 1.5</u>: Each mapping $\bar{\alpha}$ is an automorphism of the group $\mathbf{G}(\cdot)$,
i.e. it is bijectiv from $\mathbf{G}(\cdot)$ into itself and $\bar{\alpha}(\gamma \cdot \delta) = \bar{\alpha}(\gamma) \cdot \bar{\alpha}(\delta)$.

Proof: First of all $\alpha \cdot \xi \cdot \alpha^{-1}$ is a unique defined element of $\mathbf{G}$. Is $\zeta \in \mathbf{G}$ any element of $\mathbf{G}$ then we take $\alpha^{-1} \cdot \zeta \cdot \alpha = \gamma$ with $\zeta = \alpha \cdot \gamma \cdot \alpha^{-1} = \bar{\alpha}(\gamma)$. Therefore we get one unique original for any element of $\mathbf{G}$. Thus the mapping $\bar{\alpha}$ is bijectiv. Moreover we find $\bar{\alpha}(\gamma \cdot \delta) = \alpha \cdot \gamma \delta \cdot \alpha^{-1} = \alpha \cdot \gamma \cdot (\alpha^{-1} \alpha) \cdot \delta \cdot \alpha^{-1} = (\alpha \cdot \gamma \cdot \alpha^{-1}) \cdot (\alpha \cdot \delta \cdot \alpha^{-1}) = \bar{\alpha}(\gamma) \cdot \bar{\alpha}(\delta)$.

<u>Theorem 1.6</u>: a) The function $\alpha \to \bar{\alpha}$, $\mathbf{G}(\cdot) \to \bar{\mathbf{G}}(\circ)$ is a homomorphism whereat „ $\circ$ “ means the application of one after mapping after the other as $\bar{\alpha} \circ \bar{\beta}(\xi) = \bar{\alpha}(\bar{\beta}(\xi))$. So the statement means: $(\alpha \cdot \beta) \to \bar{\alpha} \circ \bar{\beta}$ resp. $\overline{\alpha \cdot \beta} = \bar{\alpha} \circ \bar{\beta}$ for any $\alpha, \beta \in \mathbf{G}$.

b) We have: $\bar{\alpha} = \mathrm{Id}$ (identical mapping) $\Leftrightarrow$ $\alpha \in \mathbf{Z}$ (center of $\mathbf{G}$) where $\alpha \in \mathbf{Z}$ means $\alpha \cdot \xi = \xi \cdot \alpha$ for all $\xi \in \mathbf{G}$.

c) $\bar{\mathbf{G}}(\circ)$ is isomorphic to the factor group $\mathbf{G}(\cdot)/\mathbf{Z}(\cdot)$.

Proof: a) For any $\xi \in \mathbf{G}$ we get: $\overline{\alpha \cdot \beta}(\xi) = (\alpha\cdot\beta)\cdot\xi\cdot(\alpha\cdot\beta)^{-1} = (\alpha\cdot\beta)\cdot\xi\cdot(\beta^{-1}\cdot\alpha^{-1}) = \alpha\cdot(\beta\cdot\xi\cdot\beta^{-1})\cdot\alpha^{-1}$ $= \overline{\alpha}(\beta\cdot\xi\cdot\beta^{-1}) = \overline{\alpha}(\overline{\beta}(\xi)) = (\overline{\alpha}\circ\overline{\beta})(\xi)$, i.e. $\overline{\alpha\cdot\beta} = \overline{\alpha}\circ\overline{\beta}$.
b) $\alpha\cdot\xi\cdot\alpha^{-1} = \xi \Leftrightarrow \alpha\cdot\xi = \xi\cdot\alpha$ for all $\xi \in \mathbf{G}$ by multiplication of the right equation with α.
c) Let γ any element of **G**. For each element δ of the coset $\gamma\cdot\mathbf{Z}$ (i.e. $\delta = \gamma\cdot\zeta$ with $\zeta \in \mathbf{Z}$) we get $\overline{\delta}(\xi) = \overline{\gamma\cdot\varsigma}(\xi) = \overline{\gamma}(\overline{\varsigma}(\xi)) = \overline{\gamma}(\xi)$ for each $\xi \in \mathbf{G}$ by theorem 1.6a and 1.6b. And if $\delta \notin \gamma\cdot\mathbf{Z}$, so $\overline{\delta}(\xi) \neq \overline{\gamma}(\xi)$ because if we suppose $\overline{\delta}(\xi) = \overline{\gamma}(\xi)$ so $\overline{\gamma}^{-1}\cdot\overline{\delta}(\xi) = \xi$ for each $\xi \in \mathbf{G}$ resulting $\overline{\gamma}^{-1}\cdot\overline{\delta} = \mathrm{Id}$ so that by theorem 1.6b we get $\gamma^{-1}\cdot\delta \in \mathbf{Z}$ cross to $\delta \notin \gamma\cdot\mathbf{Z}$.

With this we see that each mapping $\overline{\alpha} \in \overline{G}$ bijectiv corresponds to a coset $\alpha\cdot\mathbf{Z}$ and each coset $\alpha\cdot\mathbf{Z}$ bijective corresponds to a mapping $\overline{\alpha}$. Because of theorem 1.6a with this correspondance the conjunction of the mappings is transmitted to the multiplication of the cosets. Thus $\overline{G}(\circ)$ is isomorphic to the set of costs $\gamma\cdot\mathbf{Z}$ (i.e. the factor group $\mathbf{G}/\mathbf{Z}$).

<u>Theorem 1.7</u>: For all x, y $\in \mathbf{E}$ we get:
$(xy)^2 = 1 \quad xy = yx \Leftrightarrow \overline{x}(y) = y \Leftrightarrow \overline{y}(x) = x$.

Proof: Because of x,y $\in \mathbf{E}$ we have xx = 1 (i.e. $x = x^{-1}$) and yy = 1 (i.e. $y = y^{-1}$). (i.e. xx = 1 or $x^{-1} = x$ bzw. yy = 1 oder $y^{-1} = y$). By multiplication first with y and then with x from the right we get xyxy = 1 $\Leftrightarrow$ xy = yx. By multiplicating this equation from the right side with x resp. from the left side with y we get: xy = yx $\Leftrightarrow$ xyx = y and xy = yx $\Leftrightarrow$ yxy = x . With xyx = $\overline{x}(y)$ and yxy = $\overline{y}(x)$ we then have the wished statement.
In addition we remark that because of theorem 1.3 for $x \neq y$ the equation xy = yx is equivalent with “xy inv”.

<u>Theorem 1.8</u>: For any $\alpha, \gamma \in \mathbf{G}$ we have: γ inv $\Leftrightarrow$ $\overline{\alpha}(\gamma)$ inv.

Proof: γ inv $\Leftrightarrow$ $\gamma^2 = 1$ and $\gamma \neq 1$ $\Leftrightarrow$ $\alpha\gamma^2\alpha^{-1} = 1$ and $\alpha\gamma\alpha^{-1} \neq 1$ $\Leftrightarrow$ $\alpha\gamma\alpha^{-1}\cdot\alpha\gamma\alpha^{-1} = 1$ and $\alpha\gamma\alpha^{-1} \neq 1$, i.e. $\overline{\alpha}(y)\cdot\overline{\alpha}(y) = 1$ and $\overline{\alpha}(y) \neq 1$. That means: $\overline{\alpha}(y)$ inv.

<u>Theorem 1.9</u>: For any a $\in \mathbf{E}$ mit $a \notin Z$ the mapping $\overline{a}$ is involutoric,
i.e. $\overline{a}^2 = \mathrm{Id}$ and $\overline{a} \neq \mathrm{Id}$.

Proofs: First $\overline{a}^2(\xi) = a^2\xi a^2 = \xi$ because of $a^2 = 1$ for all $\xi \in \mathbf{G}$, i.e. $\overline{a}^2 = \mathrm{Id}$. If we suppose $\overline{a} = \mathrm{Id}$ so we would get $\overline{a}(\xi) = a\xi a = \xi$, i.e. $a\xi = \xi a$ for all $\xi \in \mathbf{G}$ which does mean $a \in \mathbf{Z}$ in contradiction to the supposition.

<u>Theorem 1.10</u>: For any $\alpha \in \mathbf{G}$ and any a $\in \mathbf{E}$ we have $\overline{\alpha}(a)$ inv.

Proof: Because of a $\in \mathbf{E}$ we have a inv. With theorem Satz 1.8 we then get the proposition.