Victor Anandam

Harmonic Functions and Potentials on Finite or Infinite Networks

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Preface

The aim of the current work is to present an autonomous theory of harmonic functions on infinite networks akin to potential theory on locally compact spaces as developed primarily by Brelot (without sanctioning any explicit role for the derivatives of functions defined on the space). Though random walks and electrical networks are two important sources for the advancement of the present theory, neither probabilistic methods nor energy integral techniques are used here to prove the results in an infinite network. The relevance of this study is partly because in many infinite networks (like homogeneous trees, for example), any real-valued function defined on the network is a difference of two superharmonic functions.

We consider principally the classification theory of infinite networks based on the existence of Green functions, bounded harmonic functions etc., and then balayage, equilibrium principle, domination principle, Schrödinger operators, polyharmonic functions and the Riesz-Martin compactification of the network. An important feature is the study of parabolic networks. These are the networks on which no positive potentials exist or equivalently, these are the networks on which the Green function cannot be defined. On parabolic networks we investigate the properties of pseudo-potentials (analogous to logarithmic potentials on the complex plane) introduced via a development of a notion of flux.

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Chapter 1
Laplace Operators on Networks and Trees

Abstract This work is an autonomous study of functions on infinite networks reflecting potential theory on locally compact spaces, influenced by the function theory associated with random walks and electrical networks. Starting with an overview of the contents of the five chapters presented here, this chapter introduces harmonic and superharmonic functions and their basic properties in networks. A discrete version of the Green's formula is given and the Minimum Principle for superharmonic functions is proved. Infinite trees as a special case are seen to provide examples and motivations for the development of an abstract discrete function theory on infinite networks.

1.1 Introduction

A *graph* consists of a finite number of points (called vertices) and a finite number of lines (called edges) joining some of them. The graph theory studies the inter-relation between the vertices and the edges (for example, [66]). Now for some problems, the edges have to be oriented in which case the graph is called a *digraph*. It would be easier to represent a digraph by its *incidence matrix* of order $n \times m$, where m is the number of edges and n is the number of vertices, with entries $-1, 0$ or 1. The interest in graph theory comes from the fact that many real-life situations can be represented as graphs.

Take for example, the postman problem: The postman collects the post from the post office and walks through all the streets in his beat, distributing the letters and finally returns to the post office. His problem is how well to choose a route so that, if possible, he does not go through any street more than once, yet covers all the streets. To solve this, we can think of each street corner as a vertex and each street as an edge, thus getting the model of a graph; the problem now reduces to finding a path that contains all the edges once and once only. Like this, there are other problems connected with chemical bonding, bus routes, work assignments etc. In some situations like bus routes, the distance between two vertices (that is, between

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two bus stops) may be important. That is, each edge has a real number associated with it and then we have *weighted graphs*. It is interesting to study these *geometrical structures* of a graph for their own merit. But it would be more fruitful to represent a *physical problem* as a graph theory problem and try to solve it.

Though graph theory generally deals with a finite number of objects and their inter-connectedness where the geometrical aspects of graphs play a decisive role, yet there are also problems that involve functions on finite graphs. For example, consider a *finite electrical network*. This can be represented as a graph [32] provided with a voltage-current regime subject to *Ohm's law* and *Kirchhoff's voltage and current laws*. Here we are interested not only in the geometrical properties of a finite graph but also on functions defined on nodes and branches satisfying certain conditions. In this context, the incidence matrix of the graph takes care of the geometrical properties of the graph and for the function-theoretic aspects one introduces the *Laplace operator* Δ dependent on the incidence matrix and its transpose which can be considered as operators on functions defined on its edges and vertices.

There is another development which requires the study of *infinite graphs*. Consider finite difference approximations of equations in physics; some of them lead to partial difference equations [14]. The approximations to find a solution involve horizontal and vertical displacements and so can be treated as functions on an infinite grid in the context of electrical networks. Take for example, *wave equations*; the domain of existence of the solution may be unbounded, suggesting a problem in a graph with infinite vertices [73]. Another example of an infinite graph arises in the study of *Markov chains* [68]. A Markov chain consists of a countable number of states provided with a *transition probability* and the *Markov property* which says that given the present, the past and the future are independent.

The study of functions on infinite networks has thus far been carried out on the background of Markov chains and random walks or on the requirements of extending results from finite electrical networks to infinite networks. There are many common features in these two developments. Actually, a close connection has been established between the concepts like transition probabilities, transience, recurrence, hitting time etc. used in the probabilistic study of functions on infinite networks and the concepts like effective resistance, equilibrium principle, capacity etc. used in a current-voltage regime in electrical networks. The effective resistance has a close relation to the escape probability for a reversible Markov chain [59, 64] which is characterized by the transition probability from one state to another. The similarity between the conductance and the transition probability is obvious. Consequently, it is not uncommon to see a problem arising in the context of random walks being solved by electrical methods and conversely. The electrical methods make use of functional analysis techniques, starting with the Dirichlet norm (the discrete analogue of the energy integral in the classical case) and its associated inner-product. Thus, the abstract potential theory on infinite networks, as developed by Yamasaki [70], Soardi [63] and others, is a study of Dirichlet finite functions (modeled after Dirichlet functions in the classical potential theory) dealing with discrete analogues of the solution of Poisson equation, Green's function, extremal length, Royden decomposition and Royden compactification.

The current work presents an autonomous study of functions on infinite networks influenced by potential theory on locally compact spaces which does not assign any direct role for the notion of derivatives of functions. Initially, finite networks provided with the Laplacian operator are taken up for the study, the development depending on algebraic methods starting with the incidence matrix. We can call this *algebraic potential theory* because of the association of the Laplacian (represented as a matrix) with electric potentials [24]. Later, infinite networks, with the Laplace operator defined as in the finite case, are taken up for consideration. In this situation, the development resembles the study of harmonic and subharmonic functions in the complex plane or more generally in $\Re^n, n \geq 2$ [10, 27, 53], and in the Brelot axiomatic potential theory [17, 28, 40]. Here the Dirichlet norm does not play a dominant role; nor are the probabilistic interpretations considered. However, in both the finite and the infinite cases, the important basic properties and significant results like the equilibrium principle, the condenser principle, the capacity etc. that are related to an electrical network come as solutions to the following *Dirichlet-Poisson problem* on the (finite or infinite) graph X, namely: Let F be a subset of the vertex set X. Suppose f and g are real-valued functions on X. Then, find u defined on the vertex set X satisfying the conditions $\Delta u = f$ at each vertex in F and $u = g$ on $X \setminus F$.

The present work is rather like a discrete version of function theory on Riemann surfaces [3, 39, 58, 65] and Riemannian manifolds [60], devoid of any attempt to connect it to any of the many important works on electrical networks and random walks. We develop a function theory on networks similar to the classical and the axiomatic potential theory on Euclidean spaces and on locally compact spaces. The basic definitions of potentials, Green's kernel, balayage etc. are introduced here as in the case of the Brelot axiomatic theory rather than as in the theory of probability ([31] for example). Yamasaki [69, 70] also has proved many potential theoretic results in an infinite network without involving the methods used in the study of random walks. However his study is based on Dirichlet norms and functional analysis methods, resembling potential theory on Dirichlet spaces studied by Deny [41, 42], Beurling and Deny [22, 23], Fukushima [46], Bouleau and Hirsch [26] and others. These methods are not convenient if we have to study potential theory on infinite networks in which the only non-negative potential is 0. Thus a deeper study of infinite networks without positive potentials as in the case of parabolic Riemann surfaces becomes cumbersome. Under these circumstances, the approach we have adopted here is easy to deal with situations in infinite networks that resemble those of Riemann surfaces that are hyperbolic or parabolic.

Chapter 1 is devoted to some preliminary remarks concerning networks and trees, the interior and the boundary of subsets in a network, inner and outer normal derivatives, Green's formulas, the definition and some properties of superharmonic functions and the minimum principle.

Chapter 2 brings into focus certain aspects of potential theory on *finite networks*. The Laplacian is represented by a matrix and the properties of this matrix lead to the minimum principle, domination principle, equilibrium principle and solutions to some mixed boundary-value problems like Poisson-Dirichlet problem and Neumann problem in a finite network. It is easy then to consider in a similar vein the Schrödinger operators in finite networks. These results in a finite network are already proved in Bendito et al. [20] by assuming the symmetry of conductance and then constructing equilibrium measures appropriate to each principle. Ours is a unified method based on the inverses of certain sub-matrices of the Laplace matrix.

Chapter 3 deals with the classification theory of infinite networks. It starts with the first broad division of networks into *hyperbolic and parabolic networks*, depending on whether it is possible or not to define the *Green kernel* on the network. A hyperbolic network is further classified based on the existence of non-constant positive and bounded harmonic functions on the network. This leads to the *Riesz-Martin representation* of positive superharmonic functions on a hyperbolic network. Later a study of parabolic networks is taken up, starting with the construction of a kernel like $\log|x-y|$ in the plane. The notion of *flux at infinity* of a superharmonic function is discussed in detail. Balayage and Dirichlet problem on arbitrary subsets of a parabolic network receive attention. Then, an introductory study of *pseudo-potentials* (similar to logarithmic potentials in the plane) follows.

Chapter 4 is devoted to the potential theory on an infinite network X associated to the *Schrödinger operator* $\Delta u(x) - q(x)u(x)$, for an arbitrary real-valued function $q(x)$ and then more specifically when $q(x) \geq \dfrac{\Delta\xi(x)}{\xi(x)}$ for some function $\xi > 0$ on X. This condition implies that q can take negative values, but ensures that there exists a positive q-superharmonic function on X. With respect to this operator, the topics like generalised Dirchlet problem, balayage, condenser principle, equilibrium principle, etc. are investigated. This example of two related harmonic structures, one from the Laplace operator and the other from the Schrödinger operator, in the same infinite network is later generalised to study the relation between a basic harmonic structure and a subordinate harmonic structure on X.

Chapter 5 takes up the study of polyharmonic functions on an infinite tree T. A real-valued function s on X is said to be polysuperharmonic of order m or simply m-superharmonic (for an integer $m \geq 1$) if $(-\Delta)^m s \geq 0$. Actually, to characterize m-superharmonic and m-harmonic functions, we build up on the solutions u to the Poisson equation $\Delta u = f$ for an arbitrary real-valued function f on X. Since the solutions to this equation are not easily available in an arbitrary network, we have to confine our study of polyharmonic functions defined on a tree T only. For polysuperharmonic functions on T, Laurent decomposition, m-harmonic Green functions, domination principle, balayage etc. are obtained. Finally, the Riesz-Martin representation for positive m-superharmonic functions is exhibited.

1.2 Preliminaries

Let X be a countable (finite or infinite) set of points, called vertices, some of them pair-wise joined by *edges*; we say that the edge $[x, y]$ joins the vertices x and y. Let Y denote the set of edges which are assumed to be countable. Denote $x \sim y$ to mean that there is an edge $[x, y]$ joining x and y, in which case the vertices x and y are said to be *neighbours*. A vertex e is named *terimal* if it has only one neighbour. A *walk* from x to y is a collection of vertices $\{x = x_0, x_1, ..., x_n = y\}$ where $x_i \sim x_{i+1}$ if $0 \leq i \leq n-1$; for this walk, the *length* is n. If the vertices in the walk are distinct, the walk is referred to as a *path*. The shortest length $d(x, y)$ between x and y is called *the distance between* x *and* y. We also assume that given any two vertices x and y, there exists *an associated non-negative number*, called *conductance*, $t(x, y) \geq 0$ such that t(x, y) > 0 if and only if $x \sim y$. Then $N = \{X, Y, t\}$ is called *a network* if the following conditions are also satisfied:

1. There is no *self-loop* in N, that is no edge of the form $[x, x]$ in Y.
2. Given any vertices x and y in X, there is a path connecting x and y. (That is, X is *connected*.)
3. Every $x \in X$ has only a finite number of neighbours. (That is, X is *locally finite*.)

Instead of writing $N = \{X, Y, t\}$, we simply write X to refer to a network. If $t(x, y) = t(y, x)$ for every pair of vertices x and y, then we say that X is a network with *symmetric conductance*. A network X is called a *tree* if there is no *cycle* in X, that is there is no closed path $\{x_1, x_2, ..., x_n, x_1\}$ with $n \geq 3$. An infinite tree T is said to be *homogeneous* of degree $q + 1$, if each vertex in T has $(q + 1)$ neighbours. A tree T is said to be a *standard homogenous tree of degree* $q + 1$ if every vertex in T has exactly $(q + 1)$ neighbours and $t(x, y) = (q + 1)^{-1}$ if $x \sim y$. If a tree T is considered in the context of probability, we denote the conductance as $p(x, y)$ instead of $t(x, y)$, so that $\sum\limits_{y \sim x} p(x, y) = 1$ for any $x \in T$. We refer to $p(x, y)$ as the *transition probability* from x to y. It is important to note that in a tree T, if x and y are any two vertices, then there exists a unique path joining x and y.

For any subset E of a network X, we write $\overset{0}{E} = \{x : x$ and all its neighbours are in $E\}$ and $\partial E = E \setminus \overset{0}{E}$. $\overset{0}{E}$ is referred to as the *interior* of E and ∂E is referred to as the *boundary* of E. This definition of boundary differs from the one used by Chung and Yau [35] and Bendito et al. [18]. According to them, $y \notin E$ is a boundary point of E if and only if there exists a vertex x in E such that $x \sim y$ and the collection of these boundary points is the boundary δE of E. However, the definition of the boundary ∂E given here is preferable in the case of infinite networks, since for many boundary-value problems like the Dirichlet problem the boundary function will be defined on E only. So it is convenient to define the boundary ∂E as a subset of E rather than as a subset lying outside E. Note that for a non-empty subset E, we have $E = \overset{0}{E}$ if and only if $E = X$. An arbitrary set E in X is said to be *circled* if every vertex in ∂E has at least one neighbour in $\overset{0}{E}$. That is, E is circled if and only if $E = \overset{0}{E} \cup \delta \overset{\circ}{E}$, if we use the notation of Bendito et al. [20].

Example: Let e be a fixed vertex. For any vertex x, let $|x|$ denote the distance between e and x. Then $B_m = \{x : |x| \leq m\}$ is circled. For an example of a non-circled set, we can consider in a homogeneous tree of degree $q + 1$, $q \geq 2$, the set