CIME Summer Schools

A. Tognoli (Ed.)

Singularities of Analytic Spaces

66

Bressanone, Italy 1974

Springer

FONDAZIONE CIME ROBERTO CONTI

A. Tognoli (Ed.)

Singularities of Analytic Spaces

Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
held in Bressanone (Bolzano), Italy,
June 16-25, 1974

Springer

FONDAZIONE CIME ROBERTO CONTI

C.I.M.E. Foundation
c/o Dipartimento di Matematica “U. Dini”
Viale Morgagni n. 67/a
50134 Firenze
Italy
cime@math.unifi.it

ISBN 978-3-642-10942-3 e-ISBN: 978-3-642-10944-7
DOI:10.1007/978-3-642-10944-7
Springer Heidelberg Dordrecht London New York

Reprint of the 1st ed. C.I.M.E., Ed. Cremonese, Roma 1975
With kind permission of C.I.M.E.

Printed on acid-free paper

Springer.com

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

2° Ciclo - Bressanone dal 16 al 25 giugno 1974

SINGULARITIES OF ANALYTIC SPACES

Coordinatore: Prof. A. TOGNOLI

H. HIRONAKA	:	The text was not delivered by the Author.	
F. LAZZERI	:	Analytic singularities.	Pag. 3
V. POENARU	:	Lectures of the singularities of C^∞ mappings	" 83
A. Tognoli	:	About the set of non coherence of a real analytic variety. - Pathology and imbedding problems for real analytic spaces.	" 145

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
(C. I. M. E.)

H. HIRONAKA

SPECIAL CLASSES OF SINGULARITIES OF CURVES AND SURFACES

(The text was not delivered by the Author)

Corso tenuto a Bressanone dal 16 al 25 giugno 1974

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
(C. I. M. E.)

H. HIRONAKA

SPECIAL CLASSES OF SINGULARITIES OF CURVES AND SURFACES

(The text was not delivered by the Author)

Corso tenuto a Bressanone dal 16 al 25 giugno 1974

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

F. LAZZERI

ANALYTIC SINGULARITIES

Corso tenuto a Bressanone dal 16 al 25 giugno 1975

Introduction.

1. Let X be a "space" and $(X_t)_{t\in T}$ a "continuous family of subspaces of X" ; in general that means that one has a "total space" $\tilde{X}$, a "moduli (or parameter) space" T and morphisms $\tilde{X} \xrightarrow{\pi} T$, $\tilde{X} \xrightarrow{\varphi} X$ where the family $(X_t)_{t\in T}$ is the family of the fibres $(\pi^{-1}(t))_{t\in T}$ of π and for each $t \in T$, the restriction of φ to X_t is an embedding of X_t in X . Obviously the meaning of the words subspace, fibre, embedding has to be specified depending on the geometric context (algebraic or analytic geometry, differential topology etc...) in which one is working. What happens in general is that there exists a closed subset Δ of T (the "discriminant locus") s.th. locally on $T - \Delta$, π can be viewed as the projection map of a product space; in particular on each connected component of $T - \Delta$, the fibres X_t are equivalent to each other.

Similarly one can consider the continuous family $(X_t)_{t\in\Delta}$ (instead of $(X_t)_{t\in T}$) and one finds as before a closed subset Γ of Δ ; by repeating this process one tries to stratify the map π and to classify (i.e. to describe) the fibres of π , i.e. the elements of the continuous family .

Alternatively instead of attempting to stratify the family as above, one can examine the family $(X_t)_{t\in T-\Delta}$ more closely. Fix $t_0 \in T - \Delta$ and associate to each loop γ in $T - \Delta$ with base point t_0 a continuous family $(X_s)_{s\in S^1}$ parametrized by the circle S^1 ; by associating to such a family some (in general topological) invariant that depends only on the element represented by γ in $\pi_1(T-\Delta, t_0)$ one obtains a representation of $\pi_1(T-\Delta, t_0)$; this kind of representation goes under the general heading of "monodromy" .

We shall describe now some examples to clarify this discussion.

a) The family of hypersufaces of degree d in $\mathbb{P}_n$.

Let A denote the vector space of homogeneous polynomials of degree d in $n+1$ variables (over $\mathbb{R}$ or $\mathbb{C}$). To each $f \in A - \{0\}$ one can associate its locus of zeros $F(f)$ (a hypersurface of degree d) in P^n ; the family of these hypersurfaces can be seen as a "continuous family" of subspace of $\mathbb{P}_n$ in the following way : first of all remark that $F(f)$ depends only on the "direction" of $f \in A$, so that the natural parameter space is not $A - \{0\}$ but the set of straight lines through the origin in A , i.e. the projective space $\mathbb{P}(A)$; denote by $\bar{f}$ the image of $f \in A - \{0\}$ in $\mathbb{P}(A)$. Define $X = \mathbb{P}_n$, $T = \mathbb{P}(A)$, $\tilde{X} = \{(x,\bar{f}) \in \mathbb{P}_n \times \mathbb{P}(A) \mid f(x) = 0\}$; the projections of the product space $\mathbb{P}_n \times \mathbb{P}(A)$ onto its factors induce by restriction to $\tilde{X}$ two maps $\tilde{X} \xrightarrow{\pi} T$, $\tilde{X} \xrightarrow{\varphi} X$. One can give these spaces the appropriate structure (algebraic or analytic or differentiable). In each case φ embeds every fibre $\pi^{-1}(t)$ into $X = \mathbb{P}_n$. Now we shall distinguish two cases:

a_1) $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$. Let $\Sigma = \{x \in \tilde{X} \mid \pi$ does not have maximal rank at $x\} = \{x \in \tilde{X} \mid$ the fibre of π passing through x has a singular point at $x\}$ and $\Delta = \pi(\Sigma) = \{t \in T \mid \pi^{-1}(t)$ has some singular point$\}$. One can show (by elimination theory) that Δ is an algebraic hypersurface in $\mathbb{P}(A)$, and in fact it is the locus of zeros of a homogeneous polynomial of degree $(n+1)\cdot(d-1)^n$.

Now Δ has real codimension two in T , so that $T - \Delta$ is connected . The fibres of π over $T - \Delta$ are exactly the non singular hypersurfaces of degree d in $\mathbb{P}_n(\mathbb{C})$; in general (for example $n = 2$, $d \geq 3$) there is no open set U in $T - \Delta$ such that the fibres $(\pi^{-1}(t))_{t \in U}$ are isomorphic (algebraically or complex analitically) to each other. On the other hand, if one considers the differentiable structure (or the real analytic one), since $\pi : \tilde{X} - \pi^{-1}(\Delta) \longrightarrow T - \Delta$ is a proper morphism between differentiable manifolds which has maximal rank everywhere, by a standard theorem in differential topology, every point $t_o \in T - \Delta$ has an open neighbourhood U such that there exists a diffeomorphism of $\pi^{-1}(U)$ with $U \times \pi^{-1}(t_o)$ which commutes with the projections on U . This implies that π induces a fibre bundle over $T - \Delta$ with fibre $\pi^{-1}(t_o)$ and structure group the group of diffeomorphism of $\pi^{-1}(t_o)$ onto itself.

Now we consider the family of singular hypersurfaces, $(X_t)_{t \in \Delta}$. Let $\Gamma = \{\text{singular set of } \Delta\}$. Then one can show that $\Delta - \Gamma$ is connected and that each fibre X_t , $t \in \Delta - \Gamma$ has just one singular point, which is a generic quadratic point, i.e. given locally by $\sum_{1}^{n} x_i^2 = 0$. Again π induces a differentiable (or a real analytic) fibre bundle over $\Delta - \Gamma$ where the fibre is a "space with singularities" . In the next step, taking $\Gamma' = \{$singular set of $\Gamma\}$ one finds that $\Gamma - \Gamma'$ is no longer connected; nevertheless one has again a differentiable fibre bundle. In this way one stratifies, step by step, the family $(X_t)_{t \in T}$ but in general one cannot hope to get local differentiable trivializations as in the first step. For example let $n = 2$, $d = 4$. Then the family of four distinct lines through one point, contains a continuous set of non isomorphic elements from the differentiable point of view (in fact classified by the cross ratio); what one can ask for in general is only a topological trivialization. Now we return to the fibre bundle induced by π over $T-\Delta$. If one fixes an integer $r \geq 0$, one can find a representation $\sigma : \pi_1(T-\Delta, t_0) \longrightarrow \mathrm{Aut}\, H^r(X_{t_0}, \mathbb{Z})$; as we shall see this is interesting only for $r = n - 1$. But it is important to notice that this kind of monodromy associated with this continuous family does not depend on the map φ , i.e. it "forgets" the fact that the X_t are subspaces of X . Another kind of monodromy which takes care of this can be defined in the following way: let $\mathcal{E}$ be the space of all differentiable subspaces in X diffeomorphic to X_{t_0} ; define a fundamental system of neighbourhoods of a point $E \in \mathcal{E}$ by fixing a tubulor neighbourhood U of E and a retraction (by geodesics for some Riemanian metrix on X) onto E : the neighbourhood of E is the set of all $E' \in \mathcal{E}$ contained in U and such that the restriction on E induces or diffeomorphism of E with E' . Denote by Ω the connected component of X_{t_0} in $\mathcal{E}$. Then one obtains a homomorphism $\bar{\sigma} : \pi_1(T - \Delta, t_o) \longrightarrow \pi_1(\Omega, X_{t_0})$. Obviously $\bar{\sigma}$ contains much more information than σ , but in general $\bar{\sigma}$ is very difficult to deal with. We shall return to the difference between σ and $\bar{\sigma}$ in example b) .

a_2) $\mathbb{P}_n = \mathbb{P}_n(\mathbb{R})$. If one makes an algebraic computation by elimination theory one finds a discriminant polynomial that is the same as in example a_1), and whose locus of zeros is not just the set of critical values of the map $\pi: \tilde{X} \longrightarrow X$ but a larger one (in fact it also contains all the $\bar{f} \in \mathbb{P}(A)$ such that $\{x \in \mathbb{P}_n(\mathbb{C}) \mid f(x) = 0\}$ has singularities even though these may all have strictly complex coordinates" i.e. $\{ x \in \mathbb{P}_n(\mathbb{R}) \mid f(x) = 0$ is non singular $) \}$.

Nevertheless $\Delta = \{$critical values of $\pi \}$ is a semianalytic set in T, which in general disconnects T. On each connected component of $T - \Delta$, one has a differentiable fibre bundle, the fibre in general being different (possibly empty) over different components. One can show that over the simple points of Δ, the fibre X_t has exactly one singular point, which is a generic quadratic point, i.e. of local equation $\sum_1^r x_i^2 = \sum_{r+1}^n x_i^2$ for some r.

For $n = 2$, $d = 2$, i.e. the case of real conics, one has $T \simeq \mathbb{P}_5$, Δ has degree three and $T - \Delta$ has two connected components Ω_1, Ω_2; over Ω_1 the fibre is empty, over Ω_2 the fibre is a circle. So $H_1(x_{t_0}, \mathbb{Z}) \simeq \mathbb{Z}$ and the continuous family obtained by "translating" the center of a circle along a stright line, changes the orientation of the fibre. Thus one gets a homomorphism $\pi_1(\Omega_2, t_0) \longrightarrow \mathrm{Aut}\, M_1(X_{t_0}, \mathbb{Z}) \simeq \mathbb{Z}/_{2\mathbb{Z}}$ which is in fact an isomorphism.

2. Germs of analytic spaces.

Let (X,x) be a germ of an analytic space, i.e. an analytic space X (with structure sheaf say $\mathcal{O}_X$) with a point $x \in X$, where one is interested only in the behaviour of X in an "arbitrary small neighbourhood" of x . One knows that the germ (X,x) is completely determined by the local ring $\mathcal{O}_{X,x}$; in fact the category of analytic spaces and analytic morphisms is equivalent (via the contravariant functor $(X,x) \rightsquigarrow \mathcal{O}_{X,x}$) to the category of $\mathbb{C}$ - analytic algebras (i.e. $\mathbb{C}$ - algebras which are isomorphic with some quotient $\mathbb{C}\{x_1,\ldots,x_n\}/I$, where I is an ideal in the convergent power series ring in n variables $\mathbb{C}\{x_1,\ldots,x_n\}$) and local homomorphisms.

The Zariski tangent space to (X,x) is defined as the dual $T(X,x)$ of the vector space (over $\mathbb{C}$) $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{X,x}$; its dimension coincides with the minimal integer n such that some neighbourhood of x in X can be embedded in $\mathbb{C}^n$, as one can see by the implicit function theorem. In fact let

(X,x) be embedded in $(\mathbb{C}^N,0)$ and denote by I its defining ideal in $\mathbb{C}\{x_1,\dots,x_N\}$; then the Zariski tangent space to (X,x) can be identified with the set $\{z \in \mathbb{C}^N \mid df(z) = 0$ for all $f \in I\}$. If for some $f \in I$, df is not identically zero on $\mathbb{C}^N$, that means that $f = 0$ is a smooth manifold of dimension $N - 1$ that contains X. Here is another way to describe $T(X,x)$. Let T_1 be the space associated to the analytic algebra $\mathbb{C}\{t\}/(t^2)$. T_1 is a unreduced space consisting of one point and can be considered as the "first order neighbourhood" of 0 in $\mathbb{C}$ or as a "point with a direction".

Let x be a point in the analytic space X ; there is a canonical bijection between morphisms $\sigma : T_1 \to X$ s.th. $\sigma(0) = x$ and local homomorphism $\sigma^* : \mathcal{O}_{X,x} \to \mathcal{O}_{T_1} = \mathbb{C}\{t\}/(t^2)$. Moreover every morphism $\tau : \mathcal{O}_{X,x} \to \mathbb{C}\{t\}/(t^2)$ induces a $\mathbb{C}$ - linear map $\tau' : \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \to \mathfrak{m}_{T_1}/\mathfrak{m}^2_{T_1} \simeq \mathbb{C}$, i.e. an element in $T(X,x)$; it is easy to see that $\tau \to \tau'$ is in fact a bijection so that $T(X,x)$ can be thought of as the set of all morphisms $\sigma : T_1 \to X$ s.th. $\sigma(0) = x$.

If $\varphi : (X,x) \to (Y,y)$ is an analytic morphism, to each $\sigma : T_1 \to (X,x)$ one associates $\varphi \circ \sigma : T_1 \to (Y,y)$; the linear map $d\varphi : T(X,x) \to T(Y,y)$ obtained in this way is called the differential of φ.

Let $\varphi : X \to T$ be a morphism between analytic spaces, $t \in T$. The set $X_t = \varphi^{-1}(t)$ has a natural structure of an analytic space, defined in the following way : for $x \in \varphi^{-1}(t)$ define I_x as the ideal in $\mathcal{O}_{X,x}$ generated by $\varphi_x^*(\mathfrak{m}_{T,t})$ where $\varphi_x^* : \mathcal{O}_{T,t} \to \mathcal{O}_{X,x}$ is the local homomorphism induced by φ ; for $x \in X - X_t$ define $I_x = \mathcal{O}_{X,x}$. The collection of all the I_x forms a coherent ideal sheaf in $\mathcal{O}_X$ defining on X_t the structure of an analytic space, which will be called the fibre of φ over t. In general, even if X and T are reduced, the fibre X_t may not be reduced.

Remark that if $\varphi : (X,x) \to (T,t)$ is a morphism between germs, then only the fibre X_t over t can be defined and it is a germ of a space. One has

$$\mathcal{O}_{X_t,x} = \mathcal{O}_{X,x} / \varphi^*(\mathfrak{m}_{T,t}) \cdot \mathcal{O}_{X,x} \quad .$$