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Non-linear Continuum Theories in Mechanics and Physics and their Applications

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R. S. Rivlin (Ed.)

Non-linear Continuum Theories in Mechanics and Physics and their Applications

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« NON LINEAR CONTINUUM THEORIES IN MECHANICS AND PHYSICS AND THEIR APPLICATIONS »

Coordinatore: Prof. R. S. RIVLIN

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

P. A. BLYTHE

NON-LINEAR FAR-FIELD THEORIES IN RELAXING GAS FLOWS

tenuto a Bressanone dal 3 all'11 settembre 1969

NON-LINEAR FAR-FIELD THEORIES IN RELAXING GAS FLOWS

by

P.A. Blythe

(Lehigh - University)

Summary

In the introduction the small amplitude non-linear far-field theory for one-dimensional isentropic wave propagation is briefly reviewed. The extension to non-equilibrium situations is then discussed for both high frequency and low frequency disturbances and the limitations of these classical theories are examined. It is shown that a suitable small-energy approach can be used both to remove these limitations and to provide a simplified description over the whole frequency range.

1. Introduction: isentropic far-field theory

The purpose of this lecture is to present a unified non-linear far-field theory* for relaxing or reacting gas flows. Attention will be restricted to small amplitude one-dimensional progressing waves and, for simplicity, only rate processes which involve a single internal mode or reaction will be considered.

The corresponding far-field signalling problem in an inviscid gas which is in thermodynamic equilibrium has been well understood for some time. (Whitham 1950, Lighthill 1955). It is useful first to briefly review this problem before discussing the non-equilibrium situations which are of interest here.

In general the mass-conservation, momentum and energy equations take the form (adiabatic flow)

$$\partial_t \rho + \rho u_x = 0 \tag{1.1}$$

$$\partial_t u + \rho^{-1} p_x = 0 \tag{1.2}$$

$$\partial_t e + p \partial_t (\rho^{-1}) = 0 \tag{1.3}$$

where ρ is the density, p is the pressure, u is the particle speed, e is the internal energy and ∂_t is the convective operator

$$\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \tag{1.4}$$

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*i.e., the theory must be capable of providing a valid result for 'large' time.

Here t is the time and x is a spatial co-ordinate measured from some fixed reference point.

In thermodynamic equilibrium $e=e(p,\rho)$ and (1.3) can be re-written

$$\partial_t p - a^2 \partial_t \rho = 0 \tag{1.5}$$

where

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = (p\rho^{-2} - e_\rho)\, e_p^{-1} \tag{1.6}$$

and the entropy s is defined, in equilibrium, by

$$\theta ds = de + pd(\rho^{-1}). \tag{1.7}$$

where θ is the translational temperature.

It is sometimes convenient to replace the system (1.1) to (1.3) by (1.5) and the characteristic forms

$$\partial_{\pm} p \pm \rho a \partial_{\pm} u = 0 \tag{1.8}$$

where the operators

$$\partial_{\pm} \equiv \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x} \tag{1.9}$$

are associated with the characteristic directions

$$\left(\frac{dx}{dt}\right)_{\pm} = u \pm a. \tag{1.10}$$

It is assumed that the disturbance is set up by the motion of a piston whose path is described by

$$x = x_p f(\omega t), \quad t > 0 \tag{1.11}$$

(with the origin chosen such that $f(0)=0$) and for $t \leq 0$

$$u = 0, \quad p = p_0, \quad \rho = \rho_0. \tag{1.12}$$

If, in addition,

$$f'(0) = 0, \tag{1.13}$$

though $f''(0)$ is finite, the disturbance is usually termed an acceleration wave. Unless explicitly stated otherwise these conditions on f will be assumed to hold in the subsequent analysis.

Appropriate non-dimensional variables are

$$\begin{gathered} p' = p/p_0, \quad \rho' = \rho/\rho_0, \quad e' = \frac{e\rho_0}{p_0} \\ u' = u\sqrt{\frac{\rho_0}{p_0}}, \quad a' = a\sqrt{\frac{\rho_0}{p_0}} \\ t' = \omega t \text{ and } x' = x\omega\sqrt{\frac{\rho_0}{p_0}}. \end{gathered} \tag{1.14}$$

The relations (1.1) to (1.10) are invariant under this transformation and it is convenient to omit the primes and to regard (1.1) through (1.10) as dimensionless. Corresponding boundary conditions , again omitting primes, are

$$u = 0, \quad p = \rho = 1, \quad t \leq 0, \tag{1.15}$$

and

$$u = \delta f'(t) \text{ on } x = x_p f(t), \quad t > 0. \tag{1.16}$$

The dimensionless amplitude parameter δ, which is a measure of the ratio of the piston speed to the ambient sound speed, is given by

$$\delta = \omega x_p \sqrt{\rho_o/p_o} \tag{1.17}$$

and the particular aim of the present discussion is to obtain solutions which are valid in the limit $\delta \to 0$.

Substitution of the regular expansion

$$\left.\begin{aligned} u(x,t;\delta) &= \delta u_1(x,t)+.. \\ p(x,t;\delta) &= 1+\delta p_1(x,t)+.. \end{aligned}\right\} \tag{1.18}$$

etc. into (1.1) to (1.3) shows that the first order perturbation quantities satisfy the linear wave equation

$$\frac{\partial^2 A}{\partial t^2} - a_o^2 \frac{\partial^2 A}{\partial x^2} = 0. \tag{1.19}$$

and, in particular, that the piston condition on u_1 is (see 1.16)

$$u_1(0,t) = f'(t) . \tag{1.20}$$

Hence the appropriate solution, $\xi = t - x/a_o > 0$, is

$$u_1 = f'(\xi). \tag{1.21}$$

In addition

$$p_1 = a_o^2 \rho_1 = a_o u_1. \tag{1.22}$$

However, evaluation of the second order approximation shows that the solution contains secular terms of the form $tg(\xi)$. It is apparent that the expansion (1.18) is not uniformly valid as $t \to \infty$ and that difficulties arise, for $\xi = 0(1)$, when $\delta t = 0(1)$.

These secular terms are, in fact, due to the displacement of the exact characteristics from their position as predicted by linearized theory (Whitham, 1950). This is easily seen for acceleration waves since the exact solution of the full equations, over a certain time interval, is a simple wave. In the present case it is more useful, with a view to later application, to construct the solution in the small amplitude limit by means of the far field expansions,

$$\left.\begin{aligned} u(x,t;\delta) &= \delta U_1(\xi,\eta)+\ldots, \\ p(x,t;\delta) &= 1+\delta P_1(\xi,\eta)+\ldots, \\ \rho(x,t;\delta) &= 1+\delta R_1(\xi,\eta)+\ldots, \end{aligned}\right\} \qquad (1.23)$$

where

$$\eta = \delta t \quad . \qquad (1.24)$$

Substitution in (1.8) and (1.5) shows that, as in linearized theory,

$$P_1 = a_0^2 R_1 = a_0 U_1 \qquad (1.25)$$

but U_1 now satisfies

$$\frac{\partial U_1}{\partial \eta} - \frac{b}{a_0} U_1 \frac{\partial U_1}{\partial \xi} = 0 \ , \qquad (1.26)$$

where

$$b = [\frac{1}{a}\{\frac{\partial}{\partial \rho} \rho a\}_s]_o \qquad (1.27)$$

and the suffix o denotes evaluation at the initial conditions.

The appropriate solutionof (1.26), subject to the matching condition

$$U_1(\xi,0) = f'(\xi), \tag{1.28}$$

is

$$U_1 = f(\phi) \tag{1.29}$$

where the characteristic lines ϕ=constant are given by

$$\xi = \phi - \frac{b}{a_0} f'(\phi)\eta \tag{1.30}$$

(choosing $\phi=\xi$ on $\eta=0$). Obviously (1.29) and (1.30) are the small amplitude limit of the exact simple wave solution.

If this solution is unique in x-t space then it does represent a uniformly valid result for all η. However, in general the solution will not be single valued where

$$\xi_\phi = 0$$

or

$$\eta = a_0/bf''(\phi). \tag{1.31}$$

Since, for a gas, $b>0$ equation (1.31) is satisfied for some $\eta>0$ if $f''>0$. It is then necessary to insert a discontinuity or shock in order to make the solution unique. The jump conditions across the shock are defined by the Rankine-Hugoniot relations for the conservation of mass, momentum and energy.

It is convenient to note here the form that these relations take for weak shocks. Correct to first order in δ it follows that

$$[p] = a_0^2[\rho] = a_0[u] \tag{1.32}$$

and the shock path bisects the characteristics that meet on the shock. This latter condition, in the current notation, becomes

$$U_S = a_0 + \frac{b}{2}\delta[U_1^+ + U_1^-] \tag{1.33}$$

where the superscripts -,+ correspond to conditions ahead of and behind the shock respectively.

These relations can be used to evaluate the shock path and they become particularly simple when the shock propagates into an undisturbed region for which $U_1^- = 0$. In that case it follows from (1.33) that if $\xi = \xi_S(\phi_S, \eta)$ on the shock, then ξ_S satisfies the differential equation

$$\frac{d\xi_S}{d\eta} = -\frac{1}{2}\frac{b}{a_0}U_1(\phi_S) \tag{1.34}$$

from which, together with (1.30), the solution is easily found. This solution is defined parametrically by

$$\left.\begin{aligned} \eta &= \frac{a_0}{b} f(\phi)/f'^2(\phi) \\ \xi &= \phi - 2f(\phi)/f'(\phi) \end{aligned}\right\} \tag{1.35}$$

The relations (1.29), (1.30) and (1.35) summarize the main results in the small amplitude non-linear far-field limit for equilibrium isentropic flows.

2. Relaxation processes

In general the excitation of any of the internal degrees of freedom, e.g. vibration, molecular dissociation etc., will take a certain finite time (number of collisions) in which the mode adjusts to some new equilibrium state, although the excitation (relaxation) times for the various modes may differ considerably from each other. In fact, it is known that the time scales for the adjustment of the translational and rotational degrees of freedom are usually much less than those for the other internal modes (Herzfeld Litovitz, 1959) and it will be implicitly assumed in the subsequent analysis that the translational and rotational degrees of freedom remain in a local equilibrium state.

It is further assumed that in any situation of interest only one rate dependent process will be of significance. Hence

$$e = e(p,\rho,\sigma) \tag{2.1}$$

where σ is some relaxation variable. For convenience σ can be identified as a measure of the internal energy in the lagging mode. For vibrational excitation in a pure diatomic gas $e(p,\rho,\sigma)$ depends linearly on σ, but in more complex situations this is not necessarily true.

It is supposed that the rate of adjustment of σ is described by an equation of the form

$$\partial_t \sigma = \Lambda F(p,\rho,\sigma) \qquad (2.2)$$

where the rate function F depends only on the local values of p,ρ and σ and perhaps some initial parameters. The dimensionless rate parameter Λ is the ratio of the time scale defined by the piston to some characteristic relaxation time τ, i.e.

$$\Lambda = (\omega\tau)^{-1} \qquad (2.3)$$

[Equation (2.2) is to be regarded as dimensionless with σ and F both normalized by $p_0\rho_0^{-1}$.]

In an equilibrium state, which is identified by the singular limit $\Lambda \to \infty$, $F \equiv 0$. The corresponding equilibrium path is denoted by

$$\sigma = \bar{\sigma}(p,\rho) \ . \qquad (2.4)$$

In this limit the problem reduces to the isentropic case discussed earlier.

A second isentropic limit is defined by $\Lambda=0$. For this case the internal energy σ remains frozen at its initial value. Obviously this limit is also included in the analysis of §1 .

There is, however, an important distinction that must be drawn between the two limits. In the former equilibrium case the appropriate sound speed is defined by

$$\bar{a}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\sigma=\bar{\sigma}} = (p\rho^{-2} - \bar{e}_\rho)\,\bar{e}_p^{-1} \qquad (2.5)$$

where $\bar{e}=e(p,\rho,\bar{\sigma})$, whereas in the latter frozen case

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\sigma} = (p\rho^{-2}-e_\rho)e_p^{-1} \tag{2.6}$$

with $e = e(p,\rho,\sigma)$. It can be shown that in general

$$a^2 > \bar{a}^2. \tag{2.7}$$

For the general non-equilibrium situation the relation (2.1) implies that (1.5) becomes

$$\partial_t p - a^2 \partial_t \rho = -c\Lambda F \tag{2.8}$$

where a is the frozen sound speed,

$$c = -\left(\frac{\partial p}{\partial \sigma}\right)_{e,\rho} \tag{2.9}$$

and (2.2) has been used to replace $\partial_t \sigma$. Moreover, the characteristic relations (1.8) become

$$\partial_{\pm} p \pm \rho a \partial_{\pm} u = -c\Lambda F \tag{2.10}$$

and the characteristic operators are defined by (1.9) with a interpreted as the frozen sound speed. The influence of the rate process on the energy equation and the characteristic relations introduces a source term, $-c\Lambda F$, which depends on the local values of p,ρ and σ.

The linearized signalling problem associated with this system of equations has been considered several times in the literature (Chu, 1957). The regular expansion

$$\left.\begin{aligned} u(x,t;\delta) &= \delta u_1(x,t)+.. \\ p(x,t;\delta) &= 1+\delta p_1(x,t)+.. \\ \sigma(x,t;\delta) &= \bar{\sigma}_0+\delta\sigma_1(x,t)+.. \end{aligned}\right\} \tag{2.11}$$

yields

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial t^2} - a_0^2 \frac{\partial^2 u_1}{\partial x^2}\right) + \lambda\left(\frac{\partial^2 u_1}{\partial t^2} - \bar{a}_0^2 \frac{\partial^2 u_1}{\partial x^2}\right) = 0 \qquad (2.12)$$

where

$$\lambda = \Lambda(-F_\sigma c\bar{c}^{-1})_0 \qquad (2.13)$$

is a modified rate parameter. (1.20) again defines the boundary condition on x=0 [Note that it is assumed in (2.11) that the initial conditions correspond to an equilibrium state.]

(2.12) obviously reduces to the standard linearized result in both the frozen (high frequency) limit $\lambda \to 0$ and the equilibrium (low frequency) limit $\lambda \to \infty$. For arbitrary values of λ (2.12) suggests that for $t\lambda \ll 1$ the effective propagation speed is a_0, but for $t\lambda \gg 1$ it is $\bar{a}_0$. This latter statement can be made more precise. The formal solution of (2.12), subject to (1.20) and (1.15), can be obtained by Laplace transforms. An asymptotic evaluation, $t, x \to \infty$ but sufficiently far behind the front, shows that (Clarke, 1965)

$$u_1 \sim \sqrt{\frac{D}{2\pi t}} \int_0^\infty f'(y)\exp\{-Dt^{-1}(\xi - y)^2\}dy \qquad (2.14)$$

where

$$D = \lambda(\alpha^2 - 1)^{-1}, \qquad (2.15)$$

$$\alpha = a_0/\bar{a}_0 \qquad (2.16)$$

and

$$\bar{\xi} = t - x/\bar{a}_0 \tag{2.17}$$

is the linearized characteristic associated with the low frequency (equilibrium) signal. The main disturbance is now apparently centered on these latter wavelets.

It is easily verified that according to (2.12) any plane wave is distorted both by dispersion, so that the wave speed depends on the frequency, and by absorption in which the amplitudes of the high frequency components are much more rapidly attenuated than those of the low frequency ones.

However, as in the isentropic case, it can be shown that the regular expansion (2.11) is not necessarily uniformly valid in the far field, and secular terms may again appear in higher order solutions.

The remainder of the lecture will be devoted to a discussion of the modifications that are required in order to obtain a valid far-field result.

3. The high frequency limit

A simple extension of the classical isentropic far-field approach can be used in the high-frequency (near-frozen) limit $\Lambda \to 0$ (see Varley and Rogers, 1967). For ease of discussion it will be assumed that Λ and δ are of a similar magnitude. The corresponding expansion is

$$\left.\begin{aligned} u(x,t;\delta) &= \delta U_1(\xi,\eta) + .. \\ p(x,t;\delta) &= 1 + \delta P_1(\xi,\eta) + .. \\ \sigma(x,t;\delta) &= \bar{\sigma}_0 + \delta^2 \Sigma_2(\xi,\eta) + .. \end{aligned}\right\} \tag{3.1}$$

etc. Note that $\sigma-\bar{\sigma}_0$ is a second order quantity (more strictly its magnitude is $O(\Lambda\delta)$). This expansion procedure would appear to be appropriate for large times at distances behind the front which are comparable with the length scale defined by the piston signal but which are much less than the relaxation length. Substitution in (2.10), (2.8) and (2.2) shows that

$$\frac{\partial U_1}{\partial \eta} - \frac{b}{a_0} U_1 \frac{\partial U_1}{\partial \xi} + kU_1 = 0 \tag{3.2}$$

where

$$k = (1 - \frac{1}{\alpha^2})\frac{\lambda}{2\delta} = O(1), \tag{3.3}$$

and b corresponds to (1.27) with the derivative evaluated both at constant S and σ.

The first order perturbation quantities are again related, as in frozen linearized theory, by

$$P_1 = a_0^2 R_1 = a_0 U_1. \tag{3.4}$$

Equations (3.4) and (3.2) ahould be compared with (1.25) and (1.26) respectively. The attenuation factor kU_1 plays a dominant role in the asymptotic behavior of (3.2) as $\eta \to \infty$.

The inner near-field solution for $\Lambda = O(\delta)$, with $x, t = O(1)$, is given by the usual frozen linearized result

$$u_1 = f'(\xi) \tag{3.5}$$

which defines the inner matching condition for U_1.

Subject to (3.5) and the condition at the front, the solution of (3.2) is defined by

$$U_1 = f'(\phi)e^{-k\eta} \tag{3.6}$$

with

$$\xi = \phi - \frac{b}{a_0 k} f'(\phi)[1-e^{-k\eta}]. \tag{3.7}$$

Again this solution is not single valued in physical space at points where

$$\xi_\phi = 0$$

or

$$\eta = -\frac{1}{k}\log\left(1 - \frac{ka}{bf''(\phi)}\right) \tag{3.8}$$

However, in contrast to the isentropic solution shocks will not form even for compressive piston motions if

$$\frac{ka_0}{b(f'')_{max}} > 1 \tag{3.9}$$

(Varley and Rogers 1967, Rarity 1967).

If a shock does form its path can be determined, in principle, by the approach outlined in §1. Conditions (1.32) and (1.33) again hold for a weak shock, with a interpreted as the frozen sound speed, together with the additional statement

$$[\sigma] = 0 \tag{3.10}$$

In writing down (3.10) it is implicitly assumed that the shock thickness, across which the translational mode adjusts to a new equilibrium state, is neglibibly thin in comparison with the relaxation length $a_0\tau$.

For a shock propagating into an undisturbed region it can be shown that its path is described by

$$\eta = -\frac{1}{k}\log\left(1 - \frac{2a_0 k}{b} f(\phi)/f'^2(\phi)\right)$$

$$\xi = \phi - 2f(\phi)/f'(\phi) \tag{3.11}$$

Although (3.11) reduces to (1.35) as $k\to 0$, it follows from (3.6) and (3.11) that for any finite k the amplitude of the shock is exponentially weak as $\eta\to\infty$, even for pistons whose speed is asymptotically constant.

Moreover, it is apparent, both from physical reasoning and by directly computing higher order terms in (3.1), that this high frequency expansion will break down as $\xi\to\infty$, or, more precisely, at distances behind the front which are comparable with the relaxation length. It is easily shown that for $\xi=0(\delta^{-1})$, $\eta=0(1)$ the dominant behavior is described by the linear equation (2.12) (Blythe, 1969) though this result does not necessarily in itself give a uniformly valid description of the limiting asymptotic behavior. Before discussing further this particular difficulty for high frequency disturbances, it is relevant to return to the asymptotic description for $\Lambda=0(1)$.

4. The low frequency far-field limit, $\Delta=0(1)$.

The dominant asymptotic signal according to linear (near-field) theory is defined by (2.14). If $u=0(\Delta(\delta))$ in this region it appears that the only non-trivial stretching of the independent variables is

$$T = \Delta^2(\delta)t, \quad \bar{\chi} = \Delta(\delta)\bar{\xi} \tag{4.1}$$

together with

$$\left.\begin{aligned} u &= \Delta(\delta)V_1(\bar{\chi},T)+.. \\ p &= 1+\Delta(\delta)\pi_1(\bar{\chi},T)+.. \\ \varepsilon = \sigma-\bar{\sigma} &= \Delta^2(\delta)E_2(\bar{\chi},T)+.. \end{aligned}\right\} \tag{4.2}$$

This last relation, which follows directly from the rate equation , implies that the departure from an equilibrium state is small. In this sense the expansion (4.1) and (4.2) defines a low frequency far-field limit. The magnitude of $\Delta(\delta)$ is defined implicitly by (2.14) (see below).

Before substituting these expansions into (2.5), (2.10) and (2.2) it is better to replace σ by ε as a basic dependent variable.

It can be shown that V_1 satisfies

$$\frac{\partial V_1}{\partial T} - \frac{\bar{b}}{\bar{a}_0} V_1 \frac{\partial V_1}{\partial \bar{\chi}} = \mu \frac{\partial^2 V_1}{\partial \bar{\chi}^2} \tag{4.3}$$

where

$$\mu = \frac{1}{2\lambda}(\alpha^2-1) = \frac{1}{2} D^{-1} \tag{4.4}$$

(4.3) is Burger's equation. It has been suggested many times that this provides a satisfactory asymptotic description of the flow field (Lighthill 1956, Jones 1964, Lick 1967). This equation can be transformed into the diffusion equation and it is easily verified that its solution will match with the outer behavior of (2.12) given in (2.14).

In deriving (4.3) it has been assumed that $V_1 = 0(1)$: the magnitude of $\Delta(\delta)$, as noted above, is defined by (2.14). However, it appears that this stretching is not permissible for all piston motions. In fact, if $tf'(t)\to 0$ as $t\to\infty$, $V_1=0(\delta)$ and the non-linear term in (4.3) is negligible in this particular far field region. For piston paths whose decay is slower, e.g.

$$f' \sim t^{-n}, \quad 0 \leq n < 1,$$

$\Delta = 0(\delta^{\frac{1}{1-n}})$.

In the high frequency limit discussed in §3 it is apparent that the solution in the intermediate linearized regime, where $xt=0(\delta^{-1})$, will break down in the same way. Appropriate far field (low-frequency) variables are then

$$T_1 = \delta\Delta^2 t, \quad \overline{\chi}_1 = \delta\Delta\overline{\xi} \qquad (4.5)$$

However, this asymptotic solution is always shock free. (Even if any shock forms at the front its strength will become exponentially weak for all bounded piston speeds.) In particular, when the piston speed attains a constant limiting value the associated steady state profile is fully-dispersed: all convective steepening can be balanced solely by the dissipative nature of the rate process. Yet, it is well known that stable steady partly dispersed wave forms, in which the relaxation region is preceeded by a Rankine-Hugoniot shock, do exist and it is informative to discuss this limitation in these asymptotic solutions.

Throughout the analysis so far it has been assumed that the energy σ is of a similar magnitude to the total internal energy e, or equivalently that

$$a/\bar{a}_0-1 = O(1). \tag{4.6}$$

This latter restriction, for steady state waves, always implies that $U_w-\bar{a}_0 = o(1)$, where U_w is the wave speed, but for partly dispersed waves to exist

$$U_w>a_0 \, . \tag{4.7}$$

This latter condition cannot hold for small amplitude waves $(\delta\to 0)$ if (4.6) is satisfied.

5. The small energy limit

Situations in which both α-1 and u are "small" are obviously of some interest. In this limit it is possible to obtain a simplified description of the far field in which both fully-dispersed and partly-dispersed wave-profiles can be discussed in a unified manner.

For ease of discussion, the magnitude parameter δ will also be used as a characteristic measure of σ.

This statement should not be taken to imply any relation between the internal energy and the piston speed. If necessary a second parameter δ_1, with $\sigma = 0(\delta_1)$, can be introduced and the subsequent analysis will hold provided terms $0(\delta, \delta_1)$ etc. are retained.

The appropriate far-field expansion is again of the type outlined in §2, with a slight modification in the energy term. ξ and η are used as independent variables and

$$\left.\begin{aligned} u &= \delta U_1(\xi,\eta)+.. \\ p &= 1+\delta P_1(\xi,\eta)+.. \\ \sigma &= \delta(e_0+\delta e_1(\xi,\eta)+.. \end{aligned}\right\} \tag{5.1}$$

Note that

$$e = \sigma\delta^{-1} = 0(1) \quad . \tag{5.2}$$

Substitution in (2.2), (2.8) and (2.10) gives

$$P_1 = a_0^2 R_1 = a_0 U_1 \tag{5.3}$$

which are the usual linearized relations but, U_1 and e_1 now satisfy

$$\frac{\partial U_1}{\partial \eta} - \frac{b}{a_0} U_1 \frac{\partial U_1}{\partial \xi} = -\frac{c_0}{2a_0} \frac{\partial e_1}{\partial \xi} \tag{5.4}$$

$$\frac{\partial e_1}{\partial \xi} = 2 \frac{a_0}{c_0} kU_1 - \lambda e_1 \tag{5.5}$$

Here

$$k = (1 - \frac{1}{\alpha^2})\frac{\lambda}{2\delta}$$

is to be regarded as $0(1)$.

In this first order approximation the rate equation (5.5) is linear, though it now contains both 'forward' and 'backward' terms. The only non-linear convective term occurs in (5.4).

In the near-frozen limit $\Lambda \to 0$ $(\lambda, k \to 0)$ equation (5.4) reduces to the expected result (1.26), and iteration using (5.5) gives the Varley-Rogers limit (3.2). In the low frequency or near-equilibrium limit, $\Lambda \to \infty (\lambda, k \to \infty)$, equations (5.4) and (5.5) give

$$\frac{\partial U_1}{\partial \eta} - \left(\frac{b}{a_0} U_1 - \frac{k}{\lambda}\right) \frac{\partial U_1}{\partial \xi} = 0.$$

Since

$$\frac{k}{\lambda} = \frac{\alpha - 1}{\delta} + 0(\delta), \quad b = \bar{b} + 0(\delta)$$

this last result reduces to

$$\frac{\partial U_1}{\partial \eta} - \frac{\bar{b}}{\bar{a}_0} U_1 \frac{\partial U_1}{\partial \bar{\xi}} = 0 \qquad (5.6)$$

neglecting terms $0(\delta)$. (5.6) is the classical equilibrium result also defined by (1.26). By including terms $0(\Lambda^{-1})$ it can be shown that U_1 satisfies Burger's equation (4.3) when only the dominant terms with respect to δ are retained.

Under the transformation

$$\left.\begin{aligned} U_1 &= \frac{k}{\lambda} \frac{a_0}{\bar{b}} w, \quad e_1 = \frac{k^2}{\lambda^2} \frac{2a_0^2}{c_0 \bar{b}} E \\ \xi &= \psi/\lambda, \quad \eta = Y/k \end{aligned}\right\} \qquad (5.7)$$

(5.4) and (5.5) reduce to

$$\frac{\partial w}{\partial Y} - w\,\frac{\partial w}{\partial \psi} = -\,\frac{\partial E}{\partial \psi} \tag{5.8}$$

$$\frac{\partial E}{\partial \psi} = w - E \tag{5.9}$$

and are free of parameters. The piston condition on Y=0 becomes

$$w = \frac{\lambda b}{ka_0}\, f'\,(\frac{\psi}{\lambda})\ . \tag{5.10}$$

Although for geometrically similar paths the solutions will in general be similar only for fixed values of the parameters λ and ka_0/b, a considerable simplification occurs in one particular case. For a centered expansion wave the condition at the origin is

$$u \sim -\,\frac{a_0}{b}\,(1 - \frac{x}{a_0 t})$$

which re-expressed in far field variables gives

$$w \sim -\psi/Y. \tag{5.11}$$

The differential equations (5.8) and (5.9), the front condition and the initial condition (5.11) are now independent of all parameters. This similarity form has been discussed in Blythe (1969) where a numerical solution, using a characteristics method, was presented

It is sometimes convenient to eliminate E from (5.8) and (5.9). The resulting second order equation is

$$\frac{\partial}{\partial \psi}(\frac{\partial w}{\partial Y} - w\frac{\partial w}{\partial \psi}) + \frac{\partial w}{\partial Y} - (w-1)\frac{\partial w}{\partial \psi} = 0. \tag{5.12}$$

The structure of this equation should be compared with that of the classical linearized result (2.12). Here the linear operators of (2.12), associated with the high and low frequency sound speeds respectively, are replaced by corresponding non-linear convective operators. The linearized form of (5.12),

$$w_{\psi Y}+w_Y+w_\psi = 0, \qquad (5.13)$$

is the telegraph equation. Moore and Gibson (1960) deduced (5.13) from the usual linearized form (2.12) in the limit $\alpha-1<<1$. In Moore and Gibson's derivation $t=0((\alpha-1)^{-1})$ but it is apparent that in order for this equation to be applicable in this domain

$$\delta^{-1}(\alpha-1)>>1.$$

The simplest solutions of (5.8) and (5.9) are those of steady state form

$$\begin{aligned} w &= w(\psi+CY), \\ E &= E(\psi+CY), \end{aligned} \qquad (5.14)$$

where the wave speed associated with C, in (x,t) space, is

$$U = a_0[1+\delta\frac{Ck}{\lambda}]\approx a_0[1+(\alpha-1)C] \qquad (5.15)$$

Solutions of this form correspond to the asymptotic state due to a compressive piston moving at constant speed.

The differential equations satisfied by w and E are

$$(C-w)w' = -E' = E-w . \qquad (5.16)$$

whose non-trivial solution is defined by

$$w' = \frac{E-w}{C-w} = \frac{\frac{1}{2}w^2-(C+1)w+K}{C-w} \tag{5.17}$$

Since $w'=w=0$ at upstream infinity apparently

$$K=0 \tag{5.18}$$

However, solutions of (5.17) are unique only if

$$C<0 \tag{5.19}$$

with the piston speed given by

$$w_p = 2(C+1)>0. \tag{5.20}$$

((5.17) cannot be used to study expansion waves with $w_p<0$. It is easily shown that the overall entropy change would be negative for this case).

Note from (5.15), that the restrictions (5.19) and (5.20) imply

$$a_o>U_w>\bar{a}_o \tag{5.21}$$

which is the usual condition for a fully-dispersed wave (Lighthill, 1956).

If $C>0$, (5.17), with $K=0$, does not represent a single valued solution. For compression waves a Rankine-Hugoniot shock must be inserted at the front. From the weak shock relations it follows that

$$w = 2C \tag{5.22}$$

immediately behind the shock. Hence from (5.17), with $E=0$, it again follows that K=0.

Apart from an arbitrary constant, (5.17) integrates to give

$$\psi+CY = \frac{C}{1+C}\log\left\{\frac{2C}{w}\left(C+1-\frac{w}{2}\right)^{-(1+\frac{2}{C})}\right\}. \tag{5.23}$$

For centered expansion waves it is expected that the asymptotic disturbance will be the equilibrium solution

$$w = -\overline{\psi}/Y = 1-\psi/Y+O(\delta). \tag{5.24}$$

It is easily verified, neglecting terms $O(\delta)$, that (5.24) is an exact solution of the full equation (5.12).

Although other exact analytical solutions of (5.4) and (5.5) are not readily found, it is apparent that these equations do provide a uniform small amplitude far-field limit with respect to the rate parameter Λ. In addition, they will describe the structure of both partly-dispersed and fully-dispersed wave forms. Some further discussion of the properties of these equations can be found in Blythe (1969) (see also Spence & Ockendon 1969).