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"ECONOMIA MATEMATICA"

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CENTRO INTERNATIONALE MATEMATICO ESTIVO

(C. I. M. E.)

S. N. AFRIAT

" ECONOMIC TRANSFORMATION"

Corso tenuto a Villa Falconieri (Frascati) dal 22 al 30 agosto 1966

Economic Transformation

by

S. N. Afriat

Purdue University

1: Transformation Possibility

Economics is especially concerned with possessions; and the institution which is of greatest importance to economics is the claim, or enforcible right to possession. Theories of economics deal with economic agents, their possible states, and their possible actions affecting states. Economic state is described by possessions; economic action alters state, and can be described as a transformation of possessions; and these are the two essential aspects of economic agents. An action, when it is not just a constraint, is a choice. To give account of a choice, there has to be shown the variety of possibilities, and then the motive for decision. Here to be considered is the structure of the variety of possibilities before an economic agent.

Possessions are described as compositions of goods of various kinds and amounts, in other words as stocks of goods. Possible possessions, or stocks, are thus represented by the vectors in $\Omega = \{x : x = (x_1, \ldots, x_n) \geqq 0\}$. A transformation of a given stock $x \in \Omega$ possessed by an agent results in the attainment of possession of some other stock $y \in \Omega$. Thus the possible transformations of the agent are described by a relation $T \subseteq \Omega \times \Omega$, between all possible stocks, where xTy denotes $(x,y) \in T$ and asserts the possibility of the transformation of x into y; that is, were the agent in the economic state defined by possession of x, it would be possible, by available means, to attain the state y. Those means might be the exchanges which take place between agents, or through markets, or in the input-output of industrial processes permitted by technology. Whatever the sources of possibility, they are limited, or the economic meaning of goods would vanish.

The question now is the structure which is to be assumed for transformation-possibility relation T. One special structure will arise from the Koopmans static model of production activity.[1] The same formal structure will here be established on the basis of six independent axioms. Five of these virtually are inseparable from the concept of T, and are rendered true by proper interpretation. The remaining one (Axiom 3) more has the character of a special assumption. The essential distinction between Koopman's static model and von Neumann's dynamic model[2] of production appears especially in one of the axioms (Ax.1).[3]

2: Koopman's Transformation Model

Koopmans assumes m basic activities A_i, involving n basic goods G_j. The outcome of activity A_i is that a quantity a_{ij} of good G_j is produced, or equivalently $-a_{ij}$ is consumed. Hence the vector a_i with elements a_{ij} represents the outcome of activity A_i. The activities A_i can be combined with any intensities $w_i \geq o$ to form an activity, symbolically denoted $A = \Sigma w_i A_i$, whose outcome is represented by the vector $a = \Sigma w_i a_i$. Accordingly, all the activities, thus generated by the basic activities, form a system $\mathcal{A}$, whose outcomes are represented by the vectors in the convex cone V generated by the vectors a_i.

Fig. 1 >

In order to be able to perform an activity A, it is necessary to possess in sufficient quantities the goods it consumes. That is, considering an agent whose possession of goods is given by $x \epsilon \Omega$, it is necessary that $x+a \epsilon \Omega$, where $a \epsilon V$ is the outcome of the activity A, otherwise some goods would be overexhausted by the activity. Performed in conjunction with possession x, the activity then appears as affecting the transformation of $x \epsilon \Omega$ into another possible possession $x+a \epsilon \Omega$. Hence in regard to any two possible possessions $x, y \epsilon \Omega$ of an agent who has command of the activities $\mathcal{A}$, a necessary and sufficient condition that there exists an activity $A \epsilon \mathcal{A}$ which will effect the transformation of possession from x into y is that $y-x \epsilon V$. With the system of activities $\mathcal{A}$ as the sole source of means for effecting transformations of possessions, there is established a transformation possibility relation T such that

$$xTy \Leftrightarrow y-x \epsilon V.$$

Fig. 2 >

That is, if

$$E_V = \{(x,y) : y-x \epsilon V;\ x,y \epsilon \Omega\},$$

then $T = E_V$.

Thus Koopman's production activity model leads to consideration of a transformation-possibility relation of the form $T = E_V$ where V is a closed convex cone. A finitely generated, and therefore closed cone was obtained from the model; and now a general closed convex cone V can be assumed in the construction of the relation E_V, which can be said to have V as its <u>transformation displacement cone</u>.

Any ordered couple (x,y) of n-vectors x, y can be identified with the 2n-vector z = (x, y) which it represents in partitioned form. Accordingly, any $T \subseteq \Omega \times \Omega$, as a set of ordered couples of n-vectors, is identified with a set in the space of 2n-vectors, and, as such, there is meaning to the assertion that T be a convex 2n-cone. With this understanding, there is the following proposition:

<u>If V is a convex n-cone, then E_V is a convex 2n-cone</u>.

Thus, assume V is a convex cone, that is

$$a_o, a_1 \epsilon V \text{ and } \lambda \geq 0 \Rightarrow a_o\lambda,\ a_o + a_1 \epsilon V.$$

It will be shown that E_V is a convex cone, that is

$$(x_o,y_o),\ (x_1,y_1) \epsilon E_V \ \&\ \lambda \geq 0 \Rightarrow (x_o,y_o)\lambda,\ (x_o,y_o) + (x_1,y_1) \epsilon E_V$$

The hypothesis here is

$$y_o - x_o = a_o,\ y_1 - x_1 = a_1 \text{ where } a_o,\ a_1 \epsilon V,$$

and the conclusion is

$$(x_o\lambda, y_o\lambda),\quad (x_o+x_1, y_o+y_1) \in E_V ,$$

that is

$$y_o\lambda - x_o\lambda,\quad (y_o+y_1)-(x_o+x_1) \in V ,$$

that is

$$(y_o-x_o)\lambda,\quad (y_o-x_o)+(y_1-x_1) \in V,$$

that is

$$a_o\lambda,\quad a_o+a_1 \in V .$$

But this follows from the hypothesis, since V is a convex cone.

However, it is not true that if T is a convex 2n-cone, then it is of the form E_V where V is some convex n-cone, as is obvious.

A relation $T \subseteq \Omega x \Omega$ may be called translatable if

$$xTy \ \&\ z \geqq 0 \Rightarrow (x+z)T(y+z).$$

<u>Any relation of the form E_V, where V is any set of vectors, is translatable</u>.

For, if xTy and $z \geqq 0$, then x, $y \in \Omega$ so that x+z, $y+z \in \Omega$ and $y-x \in V$ so that $(y+z)-(x+z) \in V$, whence $(x+z)T(y+z)$.

Any relation $T \subseteq \Omega x \Omega$ is to be called uniform if

$$xTy \ \&\ \lambda \geqq 0 \Rightarrow (x\lambda)T(y\lambda);$$

and it is called reflexive if xTx, and transitive if

$$xTy \ \&\ yTz \Rightarrow xTz.$$

<u>If V if a convex cone then E_V is uniform, reflexive and transitive</u>.

Thus, assume V is a convex cone. Clearly $xE_V x$, since $x-x\epsilon V$, that is $o\epsilon V$, whence E_V is reflexive. If $xE_V y$, that is $y-x\epsilon V$, then $(y-x)\lambda\epsilon V$ for $\lambda \geq 0$, that is $y\lambda - x\lambda\epsilon V$, that is $(x\lambda)E_V(y\lambda)$, whence E_V is homogeneous. Again, if $xE_V y$ and $yE_V z$, that is $y-x$, $z-y\epsilon V$, then $(y-x)+(z-y)\epsilon V$, that is $z-x\epsilon V$, that is $xE_V z$, whence E_V is transitive.

While if T is a convex cone of 2n-vectors[4] it must be homogeneous, it clearly need not be reflexive or transitive, and therefore, by the proposition just proved, need not be of the form E_V where V is some convex cone of n-vectors, as was remarked previously.

To any convex cone V there corresponds a dual convex cone, defined by

$$U = \{u : u'a \leq 0 \text{ for all } a\epsilon V\},$$

provided this set is non-empty. By the duality theorem[4-α] for closed convex cones, the dual of the dual of V then exists, and is again V, that is

$$V = \{a : u'a \leq 0 \text{ for all } u\epsilon U\},$$

or equivalently,

$$a\epsilon V \Leftrightarrow u'a \leq 0 \text{ for all } u\epsilon U.$$

Since

$$xE_V y \Leftrightarrow y-x\epsilon V,$$

it follows that

$$xE_V y \Leftrightarrow u'x \geq u'y \quad \text{for all } u\epsilon U.$$

Thus if

$$I_U = \{(x,y) : u'x \geq u'y \text{ for all } u\epsilon U;\ x,y\epsilon\Omega\}$$

defines a relation between the elements of Ω corresponding to a convex cone U, then

$$I_U = E_V$$

where V is the dual of U.

The formulae E_V, I_U give dual, equivalent forms of definition of a relation $T \subseteq \Omega \times \Omega$ associated with a cone V and its dual U. They may be distinguished as the extensional and intensional forms of definition.[5]

If V is finitely generated then, by a familiar theorem,[5a] so is its dual U. In this case let $u_1, \ldots, u_k$ denote a set of generators of U, so

$$U = \{\Sigma u_r \lambda_r : \lambda_r \geq 0\}.$$

Then clearly

$$xTy \Leftrightarrow u_r'x \geq u_r'y \quad \text{for } r=1,\ldots,k ,$$

that is, xTy holds on condition that x, y satisfy a system of k homogeneous linear inequalities. For if these are satisfied, then it follows that

$$(\Sigma u_r \lambda_r)'x \geq (\Sigma u_r \lambda_r)'y \quad \text{for all } \lambda_r \geq 0,$$

that is,

$$u'x \geq u'y \quad \text{for all } u \in U;$$

and conversely. Hence the following proposition:

<u>If $T = I_U$ where U is finitely generated, then there exists a finite set of vectors $u_1, \ldots, u_k$ such that</u>

$$xTy \Leftrightarrow u_r'x \geq u_r'y \quad \text{for } r=1,\ldots,k.$$

There are two further properties which are generally going to be considered in regard to a transformation-possibility relation T. One, to be called the Axiom of Annihilation, and which means that any possession can be annihilated, that is, transformed into the null possession o, which has only zero quantities of goods, is stated xTo. For a relation of the form $T = E_V$, this requires $o - x \in V$ for all $x \in \Omega$, that is $-\Omega \subseteq V$. The other condition, to be called the Axiom of Economy, and which means that in all possible transformations there is no gain without some loss,[6] is stated $x \leq y \;\Rightarrow\; x\bar{T}y$, or what is the same, $x \leqq y$ & $xTy \Rightarrow x=y$. Applied to $T = E_V$, and taking $z=y-x$, this means that if $z \in \Omega$, that is $z \geqq o$, and if $z \in V$ then $z=o$; that is $\Omega \cap V = O$, where $O = \{o\}$. Thus the axioms of annihilation and economy applied to $T = E_V$ are equivalent to the conditions

Fig. 3)

$$-\Omega \subseteq V, \quad \Omega \cap V = O.$$

It appears from these that $-\Omega$, V are two convex sets whose interiors are non-empty and disjoint and therefore, by a general theorem on convex sets,[6a] are separated by some hyperplane through their intersection, that is through the point o. Accordingly, there exists at least one vector $u \neq o$ such that $z \in V \Rightarrow u'z \leq o$, and $z \in \Omega \Rightarrow u'z \geq o$, in which case $u > o$, since $z > o \Rightarrow u'z > o$. It follows that the dual cone U of the cone V exists, and moreover that $U \subseteq \Omega$.[6b] Thus it appears that if T is a transformation-possibility relation which satisfies the axioms of annihilation and of economy, and which is of the form $T = E_V$, where V is a closed convex cone, then the dual cone U of V must exist, and be non-negative, and give $T = I_U$.

But any cone $U \subseteq \Omega$ has a dual V, moreover such that $-\Omega \subseteq V$, $\Omega \cap V = 0$, whence the following now appears:

Given any relation $T \subset \Omega \times \Omega$, it is of the form $T = E_V$ where V is a closed convex cone, and further it satisfies the Axioms of Annihilation and Economy, if and only if it is of the form $T = I_U$ where U is a closed convex cone, and further $U \subseteq \Omega$,^{6c} and in this case U, V are duals.

So far there has been formulation of the concept of a transformation-possibility relation $T \subseteq \Omega \times \Omega$, with indication of the universality of its scope in analytical economics. Then Koopmans' static production activity model was shown to lead to the form $T = E_V$ where V is a closed convex cone. Then the requirement that T should have this form and satisfy the Axioms of Annihilation and of Economy led to the form $T = I_U$ where U is a non-negative closed convex cone. This last form is the one which is important in this investigation, and which is going to be subjected to an axiomatic analysis. But first von Neumann's dynamic production model will be reviewed; and remark will be made on the similarity and contrast between the models of Koopmans and von Neumann, and between static and dynamic transformation-possibility relations in general.

In conclusion it can be noted that, with $u_1, \ldots, u_k$ as generators of the dual U of the activity cone V, if $F(z)$ is the non-decreasing homogeneous convex function given by $F(z) = \max\{u_r z : r = 1, \ldots, k\}$, then $V = \{z : F(z) \leqq o\}$. Efficient activities are characterized by the condition $F(z) = o$.

3: von Neumann's Transformation Model

Transformations which take place through a certain span of time, say some N unit periods, give a transformation-possibility relation T_N defined for that time span.[7] While in the concept of a static transformation-possibility relation T, the transitivity condition $xTy \;\&\; yTz \Rightarrow xTz$ is inseparable, since a passage from x to y, and then from y to z gives a passage from x to z, by the nature of what is meant. the same is not the ease for a dynamic relation such as T_N. Instead,

$$xT_My \;\&\; yT_Nz \Rightarrow xT_{M+N}z,$$

that is $T_MT_N \subseteq T_{M+N}$,[8] is the natural property. That is, if there is a passage from x to y through M periods, and from y to z through N periods, then there is, at least, a passage from x to z through M+N periods. The transitivity condition is the essential distinction of a relation T which is independent of time from a dynamic relation such as T_N. This is reflected in the forms of the transformation-possibility relations which arise from the production-activity model of Koopmans, and the production-process model of von Neumann, one dealing with activities without explicit reference to time, and the other with processes applying specifically through a unit time-period.

von Neumann[9] assumes m basic processes P_i involving n basic goods G_j, carried out over a time-span of unit duration. The process P_i transforms possession of quantities $a_{ij} \geq o$ of the goods into possession of quantities $b_{ij} \geq o$, from the beginning to the end of the duration. That is, if a_i, b_i are the n-vectors with elements a_{ij}, b_{ij} then P_i will