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FONDAZIONE CIME ROBERTO CONTI

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Complex Analysis

Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
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« COMPLEX ANALYSIS »

Coordinatore: Prof. F. Gherardelli

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
(C. I. M. E.)

NINE LECTURES ON COMPLES ANALYSIS

ALDO ANDREOTTI

Corso tenuto a Bressanone dal 3 al 12 giugno 1973

A. Andreotti

Preface.

In the spring of 1972 I had the opportunity to lecture at Lund University and more extensively at Amsterdam University and at the C.I.M.E. session in the summer of 1973 on some topics of complex analysis of my choice. The subject has been chosen within the limited range of my personal knowledge and is intended for a non excessively specialized audience. We have tried therefore not to obscure the ideas, in the attempt to obtain the most general statements, with an excess of technical details; for this reason, for instance, our main attention is devoted to complex manifolds, and we have recalled basic facts and definitions when needed. The purpose was not to overcome the listeners with admiration for the preacher but to share with hime the pleasure of inspecting some beautiful facets of this field. Indeed I was very grateful to receive many valuable suggestions; in particular I am indebted to L. Garding, L. Hormander, F. Oort, A.J.H.M. van de Ven and especially to P. de Paepe who undertook the heroic task of writing the notes.

The material deals with the theory of Levi convexity and its applications, with the duality theorem of Serre and Malgrange, and with the Hans Lewy problem. The limited time at our disposal may account for some conciseness that, however we hope, will turn to the advantage of the reader.

P. de Paepe has corrected several mistakes of mathematics and presentation; probably only few remained undected.

San Pellegrino al Cassero, September 1973,

Aldo Andreotti.

A. Andreotti

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A. Andreotti

Chapter I. Elementary theory of holomorphic convexity.

1.1 Preliminaries.

a) Let Ω be an open set in $\mathbb{C}^n$, the Cartesian product of n copies of the complex field $\mathbb{C}$, with coordinate functions $z_1,\dots,z_n$.

A function $f : \Omega \to \mathbb{C}$ is called holomorphic if for every point $z_o \in \Omega$ there exists a neighborhood $U(z_o)$ of z_o in Ω on which f admits an absolutely convergent power series expansion

$$f = \Sigma\, a_\alpha (z-z_o)^\alpha \quad \text{for every} \quad z \in U(z_o).$$

Here $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ ($\mathbb{N}$ = natural numbers including 0), $a_\alpha = a_{\alpha_1,\dots,\alpha_n}$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$.

A map $f = (f_1,\dots,f_m) : \Omega \to \mathbb{C}^m$ is said to be holomorphic if each component f_i, $1 \le i \le m$, is holomorphic.
The composition of two holomorphic maps is (where it is defined) a holomorphic map.

b) We will write $z = x + iy$ with $x, y \in \mathbb{R}^n$, $i = \sqrt{-1}$. Then $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$ where the bar denotes complex conjugation, and we will write $dx = \frac{1}{2}(dz+d\bar{z})$, $dy = \frac{1}{2i}(dz-d\bar{z})$.
For any function $f : \Omega \to \mathbb{C}$ of class C^1 we have

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j$$

where

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}\right)$$

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \frac{1}{i}\frac{\partial f}{\partial y_j}\right)$$

we define

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$

$$\bar\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar z_j} d\bar z_j .$$

The following theorem establishes a criterion for a function of class C^1 to be a holomorphic function.

Theorem. A function $f : \Omega \to \mathbb{C}$ of class C^1 is holomorphic iff at every point of Ω f satisfies the Cauchy-Riemann equations:

$$\bar\partial f = 0$$

(i.e. $\frac{\partial f}{\partial \bar z_1} = 0, \ldots, \frac{\partial f}{\partial \bar z_n} = 0$, i.e.

$$\frac{\partial f}{\partial x_j} = \frac{1}{i} \frac{\partial f}{\partial y_j} \quad \text{for } 1 \le j \le n) .$$

c) Let f be holomorphic in a neighborhood of the closed polycilinder

$$P = \{ z \in \mathbb{C}^n \mid |z_j| \le 1 \quad \text{for} \quad 1 \le j \le n$$

then for every $z \in \mathring{P}$, the interior of P, we have the Cauchy integral formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1|=1} \cdots \int_{|\xi_n|=1} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} d\xi_1 \ldots d\xi_n .$$

From this formula it follows easily (by expansion of the kernel of the integral in power series) that a continuous function $f : \Omega \to \mathbb{C}$ which is separately holomorphic in each variable is a holomorphic function (Osgood's lemma). This is even true if the condition that f is continuous is removed (Hartogs' theorem) but this is much more difficult to prove.

d) We recall the following result: <u>the set of points where a holomorphic function has a zero of infinite order is open and closed</u> (<u>principle of analytic continuation</u>).

In particular if f is defined in Ω, if Ω is connected and if f vanishes at some point of Ω of infinite order then f is identically zero on Ω.

e) We denote by $\mathcal{H}(\Omega)$ the set of all holomorphic functions in Ω. It is a vector space over $\mathbb{C}$.

We can provide $\mathcal{H}(\Omega)$ with a locally convex topology defined by the family of seminorms

$$||f||_K = \sup_K |f|$$

where K is a compact subset of Ω. A fundamental system of neighborhoods of the origin is then given by the sets

$$V(K, \varepsilon) = \{ f \in \mathcal{H}(\Omega) \mid ||f||_K < \varepsilon \}$$

for K compact in Ω and $\varepsilon > 0$. This topology is the topology of uniform convergence on compact subsets of Ω.

If $K_1 \subset K_2 \subset K_3 \subset \dots$ is a sequence of compact sets such that $K_m \subset \overset{\circ}{K}_{m+1}$ for $m = 1,2,\dots,$ and $\Omega = \bigcup_{m=1}^{\infty} K_m$, one easily verifies that the countable set of seminorms $|| \ ||_{K_m}$ defines the same topology. Thus $\mathcal{H}(\Omega)$ is a metrizable space, one can take for instance as a distance the function

$$d(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{||f-g||_{K_m}}{1 + ||f-g||_{K_m}}, \quad f, g \in \mathcal{H}(\Omega).$$

Since continuous functions satisfying the Cauchy integral formula are necessarily holomorphic, it follows that $\mathcal{H}(\Omega)$ is a complete metric space (i.e. a Frechet space) and therefore a Baire space.

We also note that bounded sets $B \subset \mathcal{H}(\Omega)$ are relatively compact. This is a consequence of

Vitali's theorem: If $\{f_\nu\}$ is a sequence of holomorphic functions on Ω such that for every compact set $K \subset \Omega$ there exists a constant $C(K)$ for which we have

$$\|f_\nu\|_K \leq C(K) \qquad \nu = 1,2,\ldots ,$$

then we can extract from $\{f_\nu\}$ a subsequence $\{f_\nu\}$ which converges uniformly on any compact subset of Ω.

In particular the unit ball in the norm $\| \ \|_{K_{m+1}}$ is relatively compact with respect to the norm $\| \ \|_{K_m}$ i.e. $H(\Omega)$ is a space of Frechet-Schwartz (cf. [14]). For more details we refer to [28] , [31] , [43].

1.2 Hartogs domains.

Consider the subset in $\mathbb{R}^3$ (coordinates x, y, t, $z = x + iy$)

$$T = \{|z| < b, \ 0 \leq t < c\} \cup \{a < |z| < b, \ 0 \leq t < d\}$$

where $0 < a < b$ and $0 < c < d$.

Because of its shape we call T a "top hat". A top hat (or Hartogs domain) in $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, $n \geq 2$, is the set of all points $z = (z_1,\ldots,z_n)$ in $\mathbb{C}^n$ for which $(z_1, (\sum_{j=2}^n |z_j|^2)^{\frac{1}{2}})$ is contained in a top hat in $\mathbb{R}^3$.

Theorem (1.2.1). (Hartogs). Let f be holomorphic in the top hat

$$T = \{|z_1| < b, \ \sum_{j=2}^n |z_j|^2 < c^2\} \cup \{a < |z_1| < b, \ \sum_{j=2}^n |z_j|^2 < d^2\}$$

in $\mathbb{C}^n$, $n \geq 2$.

Then f extends holomorphically to the filled up top hat

$$T = \{ |z_1| < b, \ \sum_{j=2}^n |z_j|^2 < d^2 \} .$$

Proof. Let the functions g_p, $p \in \mathbb{N}$, be defined by the Cauchy integral

$$g_p(z_1,\dots,z_n) = \frac{1}{2\pi i} \int_{|\xi| = b-\frac{1}{p}} \frac{f(\xi, z_2,\dots,z_n)}{\xi - z_1} d\xi$$

If p is large g_p is well-defined, continuous and holomorphic in each variable for $|z_1|$ b-1/p and $\sum_{j=2}^{n} |z_j|^2 < d^2$.
Therefore g_p is holomorphic. Moreover g_p is independent of p. Set $g = g_p$, then g is defined in $\hat{T}$, is holomorphic there and $g|T = f$ because $g-f$ is holomorphic in T, is zero on $\{|z_1| < b, \sum_2^n |z_j|^2 < c^2\}$ and T is connected. Q.E.D.

Let $h : \hat{T} - \mathbb{C}^n$ be a biholomorphic map onto an open set $h(\hat{T})$ of $\mathbb{C}^n$, (i.e. h is invertible and h and h^{-1} are both holomorphic). Then any holomorphic function on $h(T)$ extends holomorphically to $h(\hat{T})$.

This is a consequence of Hartogs' theorem and the fact that the composition of two holomorphic maps is a holomorphic map. It is sometimes called the "disc theorem".

We quote some simple consequences of Hartogs' theorem.

Let $n \geq 2$ and f holomorphic on the punctured ball $\{0 < \sum_1^n |z_j|^2 < r^2\}$ then f extends holomorphically to the ball $\{\sum_1^n |z_j|^2 < r^2\}$. In fact we can put a top hat T in the punctured ball so that $\hat{T}$ covers the origin. In particular it follows that a holomorphic function f in $n \geq 2$ variables cannot have an "isolated singularity" nor an isolated zero (since this would be an isolated singularity for $1/f$).

1.3 Open sets of holomorphy

a) Open sets with a smooth boundary. An open set Ω in $\mathbb{C}^n$ has a smooth boundary if for every point $z_o \in \partial\Omega = \bar{\Omega} - \Omega$ we can find a neighborhood $U(z_o)$ and a C^∞ function $\phi : U(z_o) = \mathbb{R}$ such that

$$(d\phi)_{z_o} \neq 0 \ , \ \Omega \cap U(z_o) = \{z \in U(z_o) \mid \phi(z) < \phi(z_o)\}.$$

This amounts to say that by a local diffeomorphism near z_o $\Omega \cap U(z_o)$ can be transformed into an open subset of a half space. Indeed we can select a set of real C^∞ coordinates in which $\phi - \phi(z_o) = x_1$ is the first coordinate. Thus $\Omega \cap U(z_o)$ is an open subset of the halfspace $\{x_1 < 0\}$.

Let Ω be an open subset of $\mathbb{C}^n$ $(= \mathbb{R}^{2n})$ with a smooth boundary and let $z_o \in \partial\Omega$ be a boundary point of Ω. Given $f \in \mathcal{H}(\Omega)$ we will say that f is <u>holomorphically extendable over</u> z_o if we can find a neighborhood $V(z_o)$ of z_o and a holomorphic function $\hat{f} \in \mathcal{H}(\Omega \cup V(z_o))$ such that

$$\hat{f}|_\Omega = f \ .$$

<u>Definition</u>. Let Ω be an open subset of $\mathbb{C}^n$ with a smooth boundary. We say that Ω is an <u>open set of holomorphy</u> if for every boundary point $z_o \in \partial\Omega = \bar{\Omega} - \Omega$ we can find a holomorphic function $f \in \mathcal{H}(\Omega)$ which cannot be extended holomorphically over z_o.

<u>Examples</u>.

1. Every open subset $\Omega \subset \mathbb{C}$ with smooth boundary is an open set of holomorphy. Indeed for every $z_o \in \partial\Omega$ $f = (z-z_o)^{-1}$ is not extendable over z_o.

2. The ball $\Omega = \{\sum_{j=1}^n |z_j|^2 < b^2\}$ in $\mathbb{C}^n$ is an open set of holomorphy. Indeed $f = (z_1 - b)^{-1}$ is not extendable over $(b,0,\dots,0)$. Since the unitary group $U(n)$ acts transitively on $\partial\Omega$ by holomorphic transformations, the assertion follows.

3. The circular shell $\Omega = \{a^2 < \sum_{j=1}^n |z_j|^2 < b^2\}$ for $0 < a < b$, if $n \geq 2$ is not an open set of holomorphy. Indeed for every point z_o of the inner boundary $\sum_{j=1}^n |z_j|^2 = a^2$ we can place a top hat T in Ω such that $z_o \in \hat{T}$.

b) Open sets with arbitrary boundary.

By a domain we mean an open connected set.

Let $\Delta \subset \hat{\Delta}$ be two domains in $\mathbb{C}^n$ and let $S \subset \mathcal{H}(\Delta)$ be a set of holomorphic functions in Δ. We say that $\hat{\Delta}$ is an S - completion of Δ if

$$\operatorname{Im}\{\mathcal{H}(\hat{\Delta}) \to \mathcal{H}(\Delta)\} \supset S$$

i.e. if every $f \in S$ extends holomorphically to $\hat{\Delta}$.
Note that by the principle of analytic continuation the extension of f to $\hat{f} \in \mathcal{H}(\hat{\Delta})$ is unique.

For instance for a top hat T, $\hat{T}$ is an $\mathcal{H}(T)$-completion of T $(n \geq 2)$.

Definition.

Let Ω be an open set in $\mathbb{C}^n$. We say that Ω is an open set of holomorphy if:

for every domain $\Delta \subset \Omega$ every $\mathcal{H}(\Omega)|_\Delta$ - completion $\hat{\Delta}$ of Δ is contained in Ω.

Remark: Open sets of holomorphy with a smooth boundary are necessarily open sets of holomorphy in the sense of this general definition. We will see later that the converse is also true. We will refer for the moment to the definition given before for open sets of holomorphy with a smooth boundary as the "provisorial definition of open sets of holomorphy".

c) Holomorphic convexity, characterization of open sets of holomorphy.

An open set Ω in $\mathbb{C}^n$ is called holomorphically convex if for every compact subset $K \subset \Omega$ the holomorphically convex envelope $\hat{K}$ of K in Ω, defined by

$$\hat{K} = \{z \in \Omega \mid |f(z)| \leq \|f\|_K \text{ for every } f \in \mathcal{H}(\Omega)\},$$

is also compact.

Theorem (1.3.1). An open set $\Omega \subset \mathbb{C}^n$ is holomorphically convex if for every divergent [1] sequence $\{x_\nu\} \subset \Omega$ there exists an $f \in \mathcal{H}(\Omega)$ such that

$$\sup_\nu |f(x_\nu)| = +\infty \text{ ("condition D").}$$

Proof. Condition D implies that Ω is holomorphically convex. Indeed, if K is compact and $\hat{K}$ is not we can find a divergent sequence $\{x_\nu\}$ in Ω , $\{x_\nu\} \subset K$. But then for every $f \in \mathcal{H}(\Omega)$, $|f(x_\nu)| \leq \|f\|_K < \infty$ which contradicts condition D.

Conversely let Ω be holomorphically convex. We want to show that condition D holds. Of this fact we will give two proofs.

1st proof: By absurdity; suppose that there exists a divergent sequence $\{x_\nu\} \subset \Omega$ such that for every $f \in \mathcal{H}(\Omega)$ $\sup_\nu |f(x_\nu)| < \infty$. By passing to a subsequence we may assume $\{x_\nu\}$ contained in a connected component of Ω. Without loss of generality we may thus assume Ω connected. Set

$$A = \{ f \in \mathcal{H}(\Omega) \mid \sup_\nu |f(x_\nu)| \leq 1 \}.$$

Then

$$\mathcal{H}(\Omega) = \bigcup_{m=1}^{\infty} m A.$$

Now A is a closed subset of $\mathcal{H}(\Omega)$. Thus by the Baire category theorem A must contain an interior point. But A is convex and symmetric (A = -A) thus A must contain a neighborhood of the origin say

$$V(K, \varepsilon) = \{ f \in \mathcal{H}(\Omega) \mid \|f\|_K < \varepsilon \} \subset A$$

(for a compact $K \subset \Omega$ and some $\varepsilon > 0$). We may assume as well that K has non-empty interior.

(1) By this we mean a sequence $\{x_\nu\}$ with no accumulation point in Ω .

Now for every $f \neq 0$, $f \in \mathcal{H}(\Omega)$, $\frac{\varepsilon}{\|f\|_K} \cdot f \in A$, therefore for every $f \in \mathcal{H}(\Omega)$

$$\sup_{\nu} |f(x_\nu)| \leq \frac{1}{\varepsilon} \|f\|_K .$$

In particular, replacing f by f^m, we get

$$\sup_{\nu} |f^m(x_\nu)| \leq \frac{1}{\varepsilon} \|f^m\|_K$$

i.e.

$$\sup_{\nu} |f(x_\nu)| \leq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{m}} \|f\|_K .$$

This shows that

$$\sup |f(x_\nu)| \leq \|f\|_K$$

for every $f \in \mathcal{H}(\Omega)$. Hence $\{x_\nu\} \subset \hat{K}$. This contradicts the holomorphic convexity of Ω .

2nd <u>proof</u>: Select a sequence $\{K_m\}$ of compact subsets of such that

$$K_m \subset \overset{\circ}{K}_{m+1} , \quad \bigcup_{m=1}^{\infty} K_m = \Omega , \quad \hat{K}_m = K_m .$$

Let $\{x_\nu\}$ be a divergent sequence in Ω . Replacing $\{K_m\}$ and $\{x_\nu\}$ by subsequences we may assume that

$$x_m \notin K_m , \quad x_m \in K_{m+1} . \quad \text{for } m = 1,2,\ldots .$$

Since $x_m \notin K_m = \hat{K}_m$ we can find $g_m \in \mathcal{H}(\Omega)$ such that

$$\|g_m\|_{K_m} < 1 \quad |g_m(x_m)| .$$

Choose positive integers λ_m successively so that

$$\|g_m^{\lambda_m}\|_{K_m} < 2^{-m}$$

$$|g_m^{\lambda_m}(x_m)| > m + \sum_{l=m+1}^{\infty} \frac{1}{2^l} + |\sum_{i=1}^{m-1} g_i^{\lambda_i}(x_m)| .$$

Now $f = \Sigma g_m^{\lambda_m}$ converges uniformly on every K_m and thus on every compact subset of Ω as any such set is contained in some K_m.
Thus f is a holomorphic function in Ω. But from the last inequality we derive that $|f(x_m)| > m$.
Therefore $\sup_{\nu} |f(x_\nu)| = +\infty$.

<u>Theorem</u> (1.3.2). (Cartan - Thullen). <u>An open set $\Omega \subset \mathbb{C}^n$ is an open set of holomorphy iff Ω is holomorphically convex.</u>

<u>Proof</u>. If Ω is holomorphically convex, then condition D holds, thus clearly Ω is an open set of holomorphy.

Conversely suppose that Ω is an open set of holomorphy. We want to prove that Ω is holomorphically convex. If this is not the case, then there exists a compact subset $K \subset \Omega$ such that $\hat{K}$ is not compact.
Because for each coordinate function we have $\|z_j\|_K = \|z_j\|_{\hat{K}}$, $\hat{K}$ is bounded. Let $\{x_\nu\} \subset \hat{K}$ be a divergent sequence in Ω such that $x_\nu \to z_o \in \partial\Omega$. Let $\|z\| = \sup_j |z_j|$ denote the polycilindrical norm in $\mathbb{C}^n$, and let

ρ = polycilindrical distance of K and $\partial\Omega$.

Certainly $\rho > 0$. If $\rho = \infty$ then $\Omega = \mathbb{C}^n$ which is clearly holomorphically convex. We may assume $\rho < \infty$. Let K' be the set of points in Ω whose polycilindrical distance from K is $\leq \frac{1}{2}\rho$. Then K' is a compact subset of Ω . For every n - tuple α of non-negative integers and every $f \in \mathcal{H}(\Omega)$

$\frac{1}{\alpha !} D^{\alpha} f \in \mathcal{H}(\Omega)$ (1) ,

therefore for any $x \in \hat{K}$

(1) $\quad \left| \frac{1}{\alpha!} D^{\alpha} f(x) \right| \leq \left\| \frac{1}{\alpha!} D^{\alpha} f \right\|_K$.

For $z \in K$ we have by the Cauchy formula

$$|D^{\alpha} f(z)| = \left| \frac{\alpha !}{(2\pi i)^n} \int_{|\xi_1 - z_1| = \frac{\rho}{2}} \cdots \int_{|\xi_n - z_n| = \frac{\rho}{2}} \frac{f(\xi)}{(\xi_1 - z_1)^{\alpha_1 + 1} \cdots (\xi_n - z_n)^{\alpha_n + 1}} d\xi_1 \ldots d\xi_n \right| ,$$

and therefore

(2) $\quad |D^{\alpha} f(z)| \leq \alpha! \quad \|f\|_{K'} \dfrac{1}{(\frac{\rho}{2})^{|\alpha|}}$

From (1) and (2) it follows that the Taylor series of f at a point $x \in \hat{K}$

$$\sum \frac{1}{\alpha!} D^{\alpha} f(x) \, (z-x)^{\alpha}$$

is majorized by the series

$$\|f\|_{K'} \sum \left| \frac{z-x}{\rho/2} \right|^{\alpha}$$

and therefore is absolutely convergent in $Q(x) = \{ \|z-x\| < \rho/4 \}$.

Now for ν sufficiently large $Q(x_\nu)$ contains the point z_o . Let Δ be the connected component of $Q(x_\nu) \cap \Omega$ containing x_ν.

---

(1) As usual for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we set $D^{\alpha} = \dfrac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$, $\alpha ! = \alpha_1 ! \ldots \alpha_n !$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Then $Q(x_\nu)$ is an $\mathcal{H}(\Omega)_{|\Delta}$ -completion of Δ . But $Q(x_\nu) \not\subset \Omega$ as it contains the point z_o. Therefore Ω cannot be an open set of holomorphy.

Remark. Let Ω be an open set in $\mathbb{C}^n$ with a smooth boundary. If Ω is an open set of holomorphy then by the Cartan-Thullen theorem Ω is holomorphically convex and therefore satisfies condition D. Hence Ω is an open set of holomorphy in the provisorial sense. For open sets with a smooth boundary the provisorial and the general definition of open set of holomorphy are equivalent.

1.4 Levi (1) - convexity.

a) Let Ω be an open subset of $\mathbb{C}^n$ and let $\phi : \Omega \to \mathbb{R}$ be a C^∞ function. At a point $a \in \Omega$ we consider the Taylor expansion of ϕ ; with obvious notations for the partial derivatives, we have,

$$\begin{aligned}\phi(z) = \phi(a) &+ \Sigma\, \partial_\alpha \phi(a)(z_\alpha - a_\alpha) + \Sigma\, \partial_{\bar\alpha}\phi(a)(\bar z_\alpha - \bar a_\alpha) + \\ &+ \tfrac{1}{2}\Sigma\, \partial_{\alpha\beta}\phi(a)(z_\alpha - a_\alpha)(z_\beta - a_\beta) \\ &+ \tfrac{1}{2}\Sigma\, \partial_{\bar\alpha\bar\beta}\phi(a)(\bar z_\alpha - \bar a_\alpha)(\bar z_\beta - \bar a_\beta) \\ &+ \Sigma\, \partial_{\alpha\bar\beta}\phi(a)(z_\alpha - a_\alpha)(\bar z_\beta - \bar a_\beta) + O(\|z-a\|^3) \ .\end{aligned}$$

Because $\overline{\dfrac{\partial f}{\partial \bar z_j}} = \dfrac{\partial \bar f}{\partial z_j}$ and because ϕ is real-valued, we must have $\overline{\partial_\alpha \phi(a)} = \partial_{\bar\alpha}\phi(a)$; $\overline{\partial_{\alpha\beta}\phi(a)} = \partial_{\bar\alpha\bar\beta}\phi(a)$; $\partial_{\alpha\bar\beta}\phi(a) = \overline{\partial_{\bar\alpha\beta}\phi(a)}$.

In particular the quadratic form

$$L(\phi)_a(v) = \Sigma\, \partial_{\alpha\bar\beta}\phi(a)\ v_\alpha \bar v_\beta$$

---

(1) Eugenio Elia Levi, 1883 - 1917.

is hermitian; it is called the <u>Levi-form</u> of ϕ at a.

A biholomorphic change of coordinates near a acts on $L(\phi)_a$ with a linear change of variables

$$v \to J(a)v$$

where $J(a)$ is the Jacobian matrix of the change of variables at a.

It follows that <u>the number of positive and the number of negative eigenvalues of the Levi-form at a does not depend on the choice of local coordinates.</u>

<u>Remark</u>. If $(d\phi)_a \neq 0$ we can perform a change of coordinates in which a is at the origin and in which the new z_1 - coordinate is

$$\Sigma\ \partial_\alpha \phi(a)(z_\alpha - a_\alpha) + \tfrac{1}{2}\Sigma\ \partial_{\alpha\beta}\phi(a)(z_\alpha - a_\alpha)(z_\beta - a_\beta).$$

Then ϕ takes the following Taylor expansion:

$$\phi(z) = \phi(0) + 2\ \mathrm{Re}\ z_1 + L(\phi)_0(z) + O(\|z\|^3)\ .$$

b) Let us assume that $(d\phi)_a \neq 0$ and, for simplicity of notations that a is at the origin. Set

$$U = \{z \in \Omega \mid \phi(z) < \phi(0)\}\ .$$

Then $\partial U = \bar{U} - U$ is smooth near $a = 0$ and the real tangent plane to U at the origin is given by

$$\Sigma\ \frac{\partial \phi}{\partial x_\alpha}(0) x_\alpha + \Sigma\ \frac{\partial \phi}{\partial y_\alpha}(0)\ y_\alpha = 0\ .$$

This plane contains the (n-1) - dimensional complex plane with equation

$$\Sigma\ \partial_\alpha \phi(0)\ z_\alpha = 0\ .$$

This is called the <u>analytic tangent plane</u> to ∂U at a and will be denoted by $T_a(\partial U)$.

Consider the Levi-form of ϕ at a restricted to $T_a(\partial U)$,

$$L(\phi)_a|_{T_a(\partial U)} \equiv \begin{cases} \sum \partial_{\alpha\bar{\beta}}\phi(a) v_\alpha \bar{v}_\beta \\ \sum \partial_\alpha \phi(a) v_\alpha = 0 \end{cases}$$

We obtain in this way a hermitian form in n-1 variables and again we realize that the number of positive and negative eigenvalues is independent of the choice of local holomorphic coordinates.

Suppose now that U is defined in a neighborhood of a by a nother C^∞ function ψ with $(d\psi)_a \neq 0$:

$$U = \{ z \in \Omega \mid \psi(z) < \psi(a) \}.$$

By subtracting constants from ϕ and ψ we may assume that $\phi(a) = \psi(a) = 0$. Then either ϕ or ψ can be taken among a set of C^∞ real local coordinates (cf. 1.3. a)). Applying the Taylor formula with the rest in integral form we realize that in a neighborhood of a $\phi = h\psi$ with h a C^∞ function and invertible (i.e. $h(a) \neq 0$). Since $\phi > 0$ where $\psi > 0$ we must have $h(a) > 0$.

Now

$$\begin{aligned} \partial . \bar{\partial}\phi &= \partial(h\,\bar{\partial}\psi + \bar{\partial}h.\psi) \\ &= h\,\partial\bar{\partial}\psi + \partial h.\bar{\partial}\psi + \bar{\partial}h.\partial\psi + \partial\bar{\partial}h.\psi \end{aligned}$$

and therefore

$$L(\phi)_a|_{T_a(\partial U)} = h(a)\, L(\psi)_a|_{T_a(\partial U)} .$$

This shows that the signature (i.e. the number of positive and negative eigenvalues) of the Levi-form restricted to the analytic tangent plane to ∂U at a is independent also of the choice of the defining function ϕ for U near a.

Proposition (1.4.1). Let U be an open subset of $\mathbb{C}^n$ with a smooth boundary. At any point $a \in \partial U$ the Levi-form of any defining function for ∂U restricted to the analytic tangent plane to ∂U at a has a signature which is independent of local holomorphic coordinates and of the choice of the defining function.

Let $p(a)$ $(q(a))$ be the number of strictly positive (strictly negative) eigenvalues of $L(\phi)_a|_{T_a(\partial U)}$. These are biholomorphic invariants of the triple $(U, \partial U, a)$. Note that we must have

$$p(a) + q(a) \leq n - 1 .$$

As an exercise we can show that there is an analytic disc of dimension p

$$\tau : D^p \to \mathbb{C}^n$$

(i.e. the biholomorphic image of the unit ball $D^p = \{t \in \mathbb{C}^p \mid \sum_1^p |t_i|^2 < 1$ in $\mathbb{C}^p)$ such that

$$\tau(0) = a$$

$$\tau(D^p) - \{a\} \subset \Omega - \bar{U} .$$

Analogously there is an analytic disc $\sigma : D^q \to \mathbb{C}^n$ of dimension q such that

$$\sigma(0) = a$$

$$\sigma(D^q) - \{a\} \subset U .$$

Indeed we can choose coordinates at the origin such that

$$\phi(z) = \operatorname{Re} z_1 + L(\phi)_0(z) + O(\|z\|^3)$$

with

$$L(\phi)_0\Big|_{T_0(\partial U)} = \sum_{2}^{p+1} \lambda_j \,|z_j|^2 - \sum_{p+2}^{p+q+1} \lambda_j \,|z_j|^2$$

where the λ_j's are >0.
Therefore near 0, for $\varepsilon > 0$ sufficiently small, if

$$0 < |z_2|^2 + \dots + |z_{p+1}|^2 < \varepsilon \quad \text{and} \quad z_1 = z_{p+2} = \dots = z_n = 0$$

then $\phi(z) > 0$.
This proves the first statement. The second one is proved with a similar argument.

c) Theorem (1.4.2). (E.E.Levi [36]). <u>Let Ω be an open set of holomorphy with a smooth boundary. Then the Levi-form at each boundary point restricted to the analytic tangent plane is positive semidefinite.</u>

<u>Proof.</u> Assume, if possible, that $L(\phi)_0\big|_{T_0(\partial\Omega)}$ has a negative eigenvalue at the point $0 \in \partial\Omega$, ϕ being a defining function for $\partial\Omega$ with $\phi(0) = 0$. By suitable choice of the holomorphic coordinates we may write near 0

$$\phi(z) = 2 \operatorname{Re} z_1(1 + \sum_{j=1}^{n} a_j \bar{z}_j) - z_2 \bar{z}_2 + \sum_{j=3}^{n} \lambda_j z_j \bar{z}_j + O(\|z\|^3).$$

First restrict ϕ to $\mathbb{R}^3 = \{\operatorname{Im} z_1 = 0,\ z_3 = \dots = z_n = 0\}$.
There exists $\varepsilon > 0$ such that for $\|z\| < 2\varepsilon$ on the region $\mathbb{R}^3 \cap \{x_1 \leq 0\}$, $(z_1 = x_1 + iy_1)$, we have

$$|O(\|z\|^3)| < \tfrac{1}{2}\,|\,2 \operatorname{Re} x_1(1 + a_1 x_1 + a_2 \bar{z}_2) - z_2 \bar{z}_2 \,|.$$

Therefore : for ε sufficiently small, $\phi < 0$ on the discs

$$D_r = \{z_1 = r,\ |z_2| < \varepsilon,\ z_3 = \dots = z_n = 0\} \ ; \quad -\varepsilon < r < 0 .$$

i.e. $D_r \subset \Omega$.

Also if ε is sufficiently small,

$$\{z_1 = 0, \ \tfrac{\varepsilon}{2} < |z_2| < \varepsilon, \ z_3 = \dots = z_n = 0\} \subset \Omega .$$

Hence there exists δ, $0 < \delta < \varepsilon$, such that

$$A = \{\tfrac{\varepsilon}{2} < |z_2| < \varepsilon, \ |z_1|^2 + |z_3|^2 + \dots + |z_n|^2 < \delta\} \subset \Omega,$$

and there exists η, $0 < \eta < \delta$, such that

$$B = \{|z_1 + \tfrac{\delta}{2}| < \eta, \ |z_2| < \varepsilon, \ |z_3|^2 + \dots + |z_n|^2 < \eta\} \subset \Omega.$$

Let $A \cup B = \Delta$ and let

$$\hat{\Delta} = \{|z_2| < \varepsilon, \ |z_1|^2 + |z_3|^2 + \dots + z_n^2 < \delta\}.$$

By the disc-theorem $\hat{\Delta}$ is an $\mathcal{H}(\Omega)|_{\Delta}$ -completion of Δ. But $\hat{\Delta}$ contains the origin $0 \notin \Omega$, thus Ω is not an open set of holomorphy.

It is natural to ask if the above necessary condition for an open set Ω in $\mathbb{C}^n$ with a smooth boundary to be an open set of holomorphy is also sufficient (Levi-problem). The answer is affirmative for open sets in $\mathbb{C}^n$ but not for open sets on complex manifolds. We will return later to this question.

Exercises.

1. Prove that every convex domain in $\mathbb{C}^n$ is a domain of holomorphy.

2. Suppose that Ω has a smooth boundary and that at a point $a \in \partial\Omega$ the Levi-form restricted to the analytic tangent plane at a to $\partial\Omega$ is strictly positive. Prove that we can choose local holomorphic coordinates at a such that Ω is locally elementary convex at a.

Hint: we can replace the defining function ϕ by an increasing convex function ψ of ϕ so that $L(\psi)_a$ is strictly positive (for instance take $\psi = e^{c\phi}$ with $c >> 0$, see [28], p. 263) and then use the remark in a).

3. Under the same assumption of the previous exercise, prove that there is a fundamental system of neighborhoods B(a) of a which are domains of holomorphy such that $B(a) \cap \Omega$ is an open set of holomorphy.

Hint: in the above specified local coordinates take for B(a) any small coordinate ball with center in a, then apply the first exercise.

The material of this chapter is covered in all standard books on complex analysis as [28], [31].

Chapter II. Pseudoconcave manifolds.

2.1 Preliminaries.

a) Presheaves. A presheaf on a topological space X is a contravariant functor from the category of open subsets U of X to the category of abelian groups i.e.

for every U an abelian group S(U) is given and

for every inclusion of open sets $V \subset U$ a homomorphism

$$r^U_V \; : \; S(U) \to S(V)$$

is given such that for every chain of inclusions $W \subset V \subset U$ of open subsets of X we have

$$r^V_W \circ r^U_V = r^U_W \; .$$

A presheaf $\mathcal{S} = S(U) \; ; \; r^U_V$ is called a sheaf if for every open set $\Omega \subset X$ and every open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of the following sequence is exact

$$0 \to S(\Omega) \overset{\varepsilon}{\to} \prod_{i\in I} S(U_i) \overset{\delta}{\to} \prod_{(i,j)\in I^2} S(U_i \cap U_j)$$

where ε is defined by

$$\varepsilon(f)_{U_i} = r^{\Omega}_{U_i}(f) \quad , \quad f \in S(\Omega)$$

and where δ is defined by

$$\delta(f)_{U_i \cap U_j} = r^{U_j}_{U_i \cap U_j} f_{U_j} - r^{U_j}_{U_i \cap U_j} f_{U_i} \quad , \; f = \{f_{U_i}\} \in \prod_{i\in I} S(U_i).$$

Example:

$\mathcal{S} = \{\text{Homcont}\,(U, \mathbb{C}), \; r^U_V\}$, where Homcont $(U, \mathbb{C})$ denotes the space of continuous functions on U with values in $\mathbb{C}$ and where r^U_V are the natural restriction maps, is a presheaf and also a sheaf.

In a similar way one defines sheaves of rings and also sheaves of modules over a sheaf of rings.

b) a **stack** $\mathcal{F}$ over X of abelian groups is the data of a topological space $\mathcal{F}$, a continuous surjective map $\pi : \mathcal{F} \to X$ such that

α) π is a local homeomorphism i.e. every point f $\mathcal{F}$ has an open neighborhood $s = s(f)$ such that $\pi|_s$ is a homeomorphism of s onto an open subset of X ;

β) for each point $x \in X$, $\mathcal{F}_x = \pi^{-1}(x)$ has the structure of an abelian group in such a way that the map

$$\mathcal{F} \times_X \mathcal{F} \text{ (1) } \to \mathcal{F}$$

given by

$$(\alpha, \beta) \to \alpha - \beta \qquad \text{is continuous.}$$

Given a stack $(\mathcal{F}, \pi, X)$ of abelian groups, for every open set $U \subset X$ we can consider the abelian group

$$\Gamma(U, \mathcal{F}) = \{ s : U \to \mathcal{F} \mid s \text{ continuous, } \pi \circ s = \text{identity on } U \}$$

of all "**sections**" s of $\mathcal{F}$ over U. If $V \subset U$, the natural restriction map $r^U_V : \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ is defined and one obtains in this way a presheaf which is also a sheaf.

Conversely, given a presheaf $\mathcal{S} = \{ S(U) ; r^U_V \}$ one can associate to it a stack $(\mathcal{F}, \pi, X)$ as follows.

We set for every $x \in X$:

$$\mathcal{F}_x = \varinjlim_{U \ni x} S(U) \text{ , i.e. an element of}$$

$\mathcal{F}_x$ is a class of equivalence of couples (U, f) with

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(1) the "fibered product" $\mathcal{F} \times_X \mathcal{F}$ is defined as the part of $\mathcal{F} \times \mathcal{F}$ lying above the diagonal Δ of $X \times X$ by projection $\pi \times \pi : \mathcal{F} \times \mathcal{F} \to X \times X$;

$\mathcal{F} \times_X \mathcal{F} = (\pi \times \pi)^{-1} (\Delta)$.

$x \in U$, $f \in S(U)$ under the relation

$$(U_1, f_1) \sim (U_2, f_2)$$

if there exists $U_3 \ni x$, $U_3 \subset U_1 \cap U_2$ such that

$$r^{U_1}_{U_3} f_1 = r^{U_2}_{U_3} f_2 .$$

The equivalence class in $\mathcal{F}_x$ of (U, f) is denoted by f_x and it is called the germ of f at x.

We then define $\mathcal{F} = \bigcup_{x \in X} \mathcal{F}_x$ and π by $\pi(\mathcal{F}_x) = x$.

If we take on $\mathcal{F}$ as a basis for open sets the sets of the form $\bigcup_{x \in U} f_x$ for all $f \in S(U)$, we obtain, as one verifies, a stack of abelian groups $(\mathcal{F}, \pi, X)$.

Starting in this construction with a sheaf, constructing the corresponding stack and then the corresponding sheaf of sections we get back the original sheaf. We thus have a one-to-one correspondence between sheaves of abelian groups and stacks of abelian groups. Although this could generate some confusion it is customary to represent a sheaf by the associated stack (see for instance [25] and [30] or [18]).

c) <u>Meromorphic functions</u>. Let now X be a complex manifold and let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on X. For every open set $U \subset X$ it is defined by the space $\mathcal{H}(U)$ and the natural restriction maps. The space $\mathcal{H}(U)$ is a ring. Let $\mathcal{D}(U)$ be the subset of $\mathcal{H}(U)$ of divisors of zero, i.e. $\mathcal{D}(U)$ is the set of those holomorphic functions on U vanishing on some connected component of U. Let $\mathcal{Q}(U)$ be the quotient ring of $\mathcal{H}(U)$ with respect to $\mathcal{D}(U)$ i.e. $\mathcal{Q}(U)$ is the set of quotients $\frac{f}{g}$ with $f \in \mathcal{H}(U)$, $g \in \mathcal{H}(U) - \mathcal{D}(U)$ with obvious identifications:

$$\frac{f}{g} = \frac{f'}{g'} \quad \text{iff} \quad fg' = f'g .$$

If $V \subset U$ is an inclusion of open sets, the restriction map $r^U_V: \mathcal{H}(U) \to \mathcal{H}(V)$ sends $\mathcal{H}(U) - \mathcal{D}(U)$ into $\mathcal{H}(V) - \mathcal{D}(V)$ and thus induces a homomorphism of rings

$$r^U_V: \mathcal{Q}(U) \to \mathcal{Q}(V)$$

We obtain in this way a presheaf. The corresponding sheaf $\mathcal{M}$ is called the <u>sheaf of germs of meromorphic functions</u> on X. the ring $\mathcal{K}(X) = \Gamma(X, \mathcal{M})$ is called the <u>ring of meromorphic functions</u> on on X. Note that $\mathcal{Q}(X) \subset \mathcal{K}(X)$ but $\mathcal{Q}(X)$ may be actually smaller than $\mathcal{K}(X)$.

<u>Example</u>: Take $X = \mathbb{P}_1(\mathbb{C})$, the Riemann sphere. Then $\mathcal{H}(X) = \mathbb{C}$ thus $\mathcal{Q}(X) = \mathbb{C}$ while $\mathcal{K}(X)$ is isomorphic to the field of all rational functions in one variable t, $\mathcal{K}(X) \simeq \mathbb{C}(t)$.

If X is connected then $\mathcal{K}(X)$ and $\mathcal{Q}(X)$ are <u>fields</u>.

In the sequel <u>we will always assume that</u> X <u>is a connected manifold.</u>

2.2. <u>Mermorphic functions and holomorphic line bundles.</u>

a) <u>Holomorphic line bundles.</u> Let X be a complex manifold; by a holomorphic line bundle on X we mean a triple (F, π, X) where F is a complex manifold, $\pi : F \to X$ a holomorphic surjective map such that

i) π is of maximal rank

ii) for every $x \in X$ $\pi^{-1}(x)$ is isomorphic to the complex field $\mathbb{C}$ in such a way that

α) the map

$$F \times_X F \to F$$

given by $(u, v) \to v+v$ is holomorphic

β) the map

$$\mathbb{C} \times F \to F$$

given by $(\lambda, v) \to \lambda v$ is holomorphic.

Given two holomorphic line bundles (F, π, X), (E, ω, X) over X a <u>morphism</u> (or bundle map) is a holomorphic map $f : F \to E$ such that

i) $\pi = \omega \circ f$