

Visualizing
and Investigating
Big Ideas

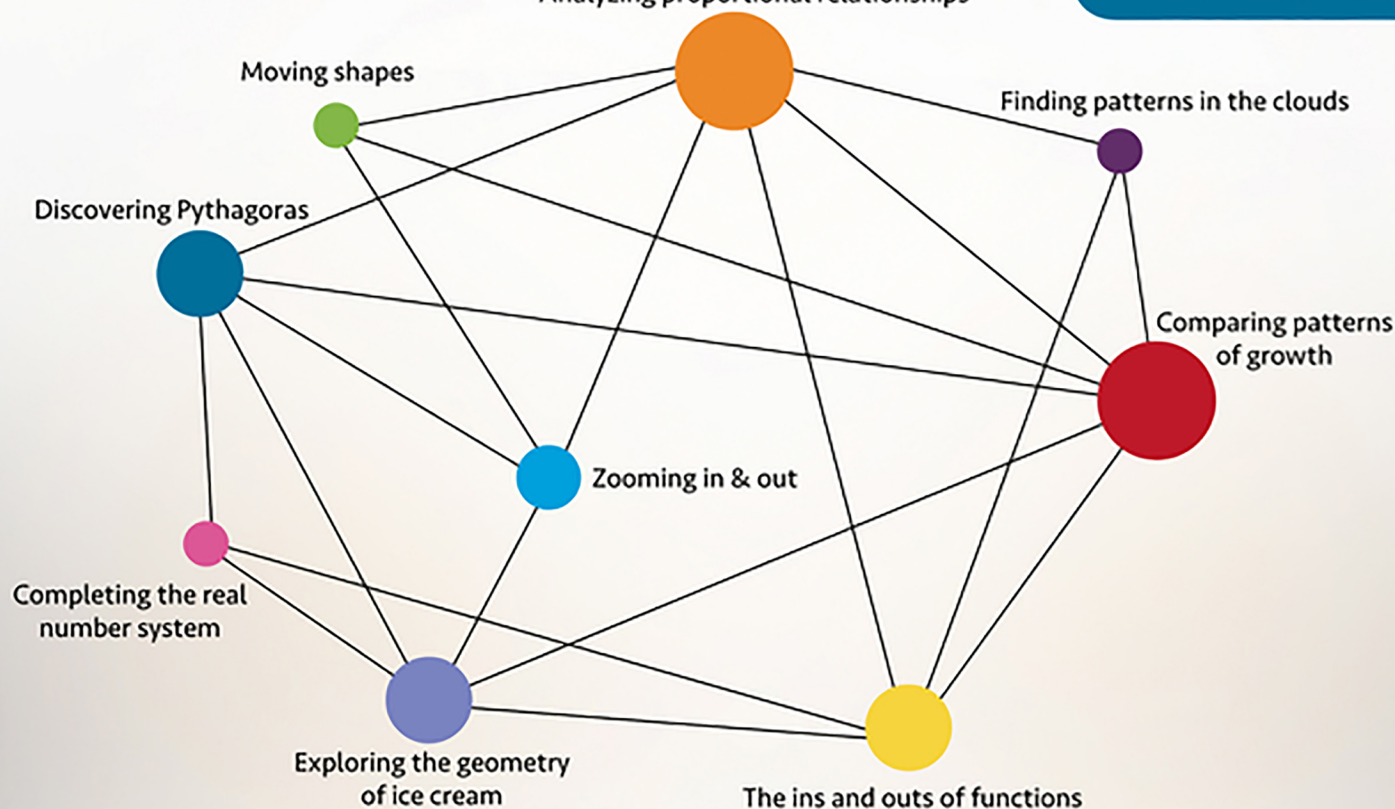


mindset

mathematics

GRADE 8

Analyzing proportional relationships



JO BOALER

JEN MUNSON

CATHY WILLIAMS

JOSSEY-BASS™
A Wiley Brand

GRADE 8

Mindset Mathematics



Visualizing and Investigating
Big Ideas

Jo Boaler

Jen Munson

Cathy Williams

JOSSEY-BASS™
A Wiley Brand

Published by Jossey-Bass
A Wiley Brand
111 River St, Hoboken, NJ 07030
www.josseybass.com

No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning, or otherwise, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400, fax 978-646-8600, or on the Web at www.copyright.com. Requests to the publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, 201-748-6011, fax 201-748-6008, or online at www.wiley.com/go/permissions.

Limit of Liability/Disclaimer of Warranty: While the publisher and author have used their best efforts in preparing this book, they make no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives or written sales materials. The advice and strategies contained herein may not be suitable for your situation. You should consult with a professional where appropriate. Neither the publisher nor author shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages. Readers should be aware that Internet Web sites offered as citations and/or sources for further information may have changed or disappeared between the time this was written and when it is read.

Certain pages from this book are designed for use in a group setting and may be customized and reproduced for educational/training purposes. The reproducible pages are designated by the appearance of the following copyright notice at the foot of each page:

Mindset Mathematics, Grade 8, copyright © 2020 by Jo Boaler, Jen Munson, Cathy Williams. Reproduced by permission of John Wiley & Sons, Inc.

This notice may not be changed or deleted and it must appear on all reproductions as printed. This free permission is restricted to the paper reproduction of the materials for educational/training events. It does not allow for systematic or large-scale reproduction, distribution (more than 100 copies per page, per year), transmission, electronic reproduction, or inclusion in any publications offered for sale or used for commercial purposes—none of which may be done without prior written permission of the Publisher.

Jossey-Bass books and products are available through most bookstores. To contact Jossey-Bass directly call our Customer Care Department within the U.S. at 800-956-7739, outside the U.S. at 317-572-3986, or fax 317-572-4002.

Wiley publishes in a variety of print and electronic formats and by print-on-demand. Some material included with standard print versions of this book may not be included in e-books or in print-on-demand. If this book refers to media such as a CD or DVD that is not included in the version you purchased, you may download this material at <http://booksupport.wiley.com>. For more information about Wiley products, visit www.wiley.com.

The Visualize, Play, and Investigate icons are used under license from Shutterstock.com and the following artists: Blan-k, Marish, and SuzanaM.

Library of Congress Cataloging-in-Publication Data

Names: Boaler, Jo, 1964- author. | Munson, Jen, 1977- author. | Williams, Cathy, 1962- author.
Title: Mindset mathematics : visualizing and investigating big ideas, grade 8 / Jo Boaler, Jen Munson, Cathy Williams.
Description: San Francisco : Jossey-Bass, 2020. | Series: Mindset mathematics | Includes index.
Identifiers: LCCN 2019045094 (print) | LCCN 2019045095 (ebook) | ISBN 9781119358749 (paperback) | ISBN 9781119358855 (adobe pdf) | ISBN 9781119358848 (epub)
Subjects: LCSH: Games in mathematics education. | Mathematics—Study and teaching (Middle school)—Activity programs. | Eighth grade (Education)
Classification: LCC QA20.G35 B638 2020 (print) | LCC QA20.G35 (ebook) | DDC 510.71/2—dc23
LC record available at <https://lcn.loc.gov/2019045094>
LC ebook record available at <https://lcn.loc.gov/2019045095>

Cover design by Wiley
Cover image: © Marish/Shutterstock-Eye; © Kritchanut/iStockphoto-Background
Printed in the United States of America

FIRST EDITION

PB Printing 10 9 8 7 6 5 4 3 2 1

Contents

Introduction	1
Low-Floor, High-Ceiling Tasks	2
Youcubed Summer Camp	3
Memorization versus Conceptual Engagement	4
Mathematical Thinking, Reasoning, and Convincing	5
Big Ideas	9
Structure of the Book	9
Note on Materials	17
Manipulatives and Materials Used in This Book	18
Activities for Building Norms	21
Encouraging Good Group Work	21
Paper Folding: Learning to Reason, Convince, and Be a Skeptic	22
■ Big Idea 1: Moving Shapes	27
Visualize: What Does It Mean to Be the Same?	29
Play: Pixel Puzzles	35
Investigate: Slide It, Flip It, Turn It	45

■ Big Idea 2: Zooming In and Out	53
Visualize: What Is Similarity?	55
Play: Find the Fakes	60
Investigate: Growing on a Grid	65
■ Big Idea 3: Analyzing Proportional Relationships	71
Visualize: Dropping the Altitude	73
Play: Seeing Triangles under the Line	79
Investigate: Stairway to Eleven	85
■ Big Idea 4: Comparing Patterns of Growth	91
Visualize: Squared Squares	93
Play: Skip-Counting Arrays	100
Investigate: Stacking Pennies	107
■ Big Idea 5: The Ins and Outs of Functions	117
Visualize: Growing Functions	120
Play: Getting Warmer!	134
Investigate: The Functions of Near Squares	140
■ Big Idea 6: Finding Patterns in the Clouds	149
Visualize: What's Going On in This Graph?	151
Play: What's the Story Here?	158
Investigate: Creating Data Visualizations	166
■ Big Idea 7: Completing the Real Number System	177
Visualize: Square Sides	179
Play: Between 4×4 and 5×5	187
Investigate: The Hypotenuse Hypothesis	193
■ Big Idea 8: Discovering Pythagoras	201
Visualize: Unpacking Pythagoras	204
Play: Pythagorean Triples	209
Investigate: Approximating Square Roots	214

■ Big Idea 9: Exploring the Geometry of Ice Cream	221
Visualize: Building Intuition about Volume	223
Play: Comparing Cylinders	227
Investigate: Scooping Up Volume	233
Appendix	239
Centimeter Grid Paper	240
Grid Paper	241
Isometric Dot Paper	242
Dot Paper	243
About the Authors	245
Acknowledgments	247
Index	249

To all those teachers pursuing a mathematical mindset journey with us.

Introduction

I still remember the moment when Youcubed, the Stanford center I direct, was conceived. I was at the Denver NCSM and NCTM conferences in 2013, and I had arranged to meet Cathy Williams, the director of mathematics for Vista Unified School District. Cathy and I had been working together for the past year improving mathematics teaching in her district. We had witnessed amazing changes taking place, and a filmmaker had documented some of the work. I had recently released my online teacher course, called How to Learn Math, and been overwhelmed by requests from tens of thousands of teachers to provide them with more of the same ideas. Cathy and I decided to create a website and use it to continue sharing the ideas we had used in her district and that I had shared in my online class. Soon after we started sharing ideas on the Youcubed website, we were invited to become a Stanford University center, and Cathy became the codirector of the center with me.

In the months that followed, with the help of one of my undergraduates, Montse Cordero, our first version of youcubed.org was launched. By January 2015, we had managed to raise some money and hire engineers, and we launched a revised version of the site that is close to the site you may know today. We were very excited that in the first month of that relaunch, we had five thousand visits to the site. At the time of writing this, we are now getting three million visits to the site each month. Teachers are excited to learn about the new research and to take the tools, videos, and activities that translate research ideas into practice and use them in their teaching.

Low-Floor, High-Ceiling Tasks

One of the most popular articles on our website is called “Fluency without Fear.” I wrote this with Cathy when I heard from many teachers that they were being made to use timed tests in the elementary grades. At the same time, new brain science was emerging showing that when people feel stressed—as students do when facing a timed test—part of their brain, the working memory, is restricted. The working memory is exactly the area of the brain that comes into play when students need to calculate with math facts, and this is the exact area that is impeded when students are stressed. We have evidence now that suggests strongly that timed math tests in the early grades are responsible for the early onset of math anxiety for many students. I teach an undergraduate class at Stanford, and many of the undergraduates are math traumatized. When I ask them what happened to cause this, almost all of them will recall, with startling clarity, the time in elementary school when they were given timed tests. We are really pleased that “Fluency without Fear” has now been used across the United States to pull timed tests out of school districts. It has been downloaded many thousands of times and used in state and national hearings.

One of the reasons for the amazing success of the paper is that it does not just share the brain science on the damage of timed tests but also offers an alternative to timed tests: activities that teach math facts conceptually and through activities that students and teachers enjoy. One of the activities—a game called How Close to 100—became so popular that thousands of teachers tweeted photos of their students playing the game. There was so much attention on Twitter and other media that Stanford noticed and decided to write a news story on the damage of speed to mathematics learning. This was picked up by news outlets across the United States, including *US News & World Report*, which is part of the reason the white paper has now had so many downloads and so much impact. Teachers themselves caused this mini revolution by spreading news of the activities and research.

How Close to 100 is just one of many tasks we have on youcubed.org that are extremely popular with teachers and students. All our tasks have the feature of being “low floor and high ceiling,” which I consider to be an extremely important quality for engaging all students in a class. If you are teaching only one student, then a mathematics task can be fairly narrow in terms of its content and difficulty. But whenever you have a group of students, there will be differences in their needs, and they will be challenged by different ideas. A low-floor, high-ceiling task is one in which everyone can engage, no matter what his or her prior understanding or knowledge, but also one that is open enough to extend to high levels, so that

all students can be deeply challenged. In the last two years, we have launched an introductory week of mathematics lessons on our site that are open, visual, and low floor, high ceiling. These have been extremely popular with teachers; they have had approximately four million downloads and are used in 20% of schools across the United States.

In our extensive work with teachers around the United States, we are continually asked for more tasks that are like those on our website. Most textbook publishers seem to ignore or be unaware of research on mathematics learning, and most textbook questions are narrow and insufficiently engaging for students. It is imperative that the new knowledge of the ways our brains learn mathematics is incorporated into the lessons students are given in classrooms. It is for this reason that we chose to write a series of books that are organized around a principle of active student engagement, that reflect the latest brain science on learning, and that include activities that are low floor and high ceiling.

Youcubed Summer Camp

We recently brought 81 students onto the Stanford campus for a Youcubed summer math camp, to teach them in the ways that are encouraged in this book. We used open, creative, and visual math tasks. After only 18 lessons with us, the students improved their test score performance by an average of 50%, the equivalent of 1.6 years of school. More important, they changed their relationship with mathematics and started believing in their own potential. They did this, in part, because we talked to them about the brain science showing that

- There is no such thing as a math person—anyone can learn mathematics to high levels.
- Mistakes, struggle, and challenge are critical for brain growth.
- Speed is unimportant in mathematics.
- Mathematics is a visual and beautiful subject, and our brains want to think visually about mathematics.

All of these messages were key to the students' changed mathematics relationship, but just as critical were the tasks we worked on in class. The tasks and the messages about the brain were perfect complements to each other, as we told students they could learn anything, and we showed them a mathematics that was open, creative, and engaging. This approach helped them see that they could learn

mathematics and actually do so. This book shares the kinds of tasks that we used in our summer camp, that make up our week of inspirational mathematics (WIM) lessons, and that we post on our site.

Before I outline and introduce the different sections of the book and the ways we are choosing to engage students, I will share some important ideas about how students learn mathematics.

Memorization versus Conceptual Engagement

Many students get the wrong idea about mathematics—exactly the wrong idea. Through years of mathematics classes, many students come to believe that their role in mathematics learning is to memorize methods and facts, and that mathematics success comes from memorization. I say this is exactly the wrong idea because there is actually very little to remember in mathematics. The subject is made up of a few big, linked ideas, and students who are successful in mathematics are those who see the subject as a set of ideas that they need to think deeply about. The Program for International Student Assessment (PISA) tests are international assessments of mathematics, reading, and science that are given every three years. In 2012, PISA not only assessed mathematics achievement but also collected data on students' approach to mathematics. I worked with the PISA team in Paris at the Organisation for Economic Co-operation and Development (OECD) to analyze students' mathematics approaches and their relationship to achievement. One clear result emerged from this analysis. Students approached mathematics in three distinct ways. One group approached mathematics by attempting to memorize the methods they had met; another group took a “relational” approach, relating new concepts to those they already knew; and a third group took a self-monitoring approach, thinking about what they knew and needed to know.

In every country, the memorizers were the lowest-achieving students, and countries with high numbers of memorizers were all lower achieving. In no country were memorizers in the highest-achieving group, and in some high-achieving countries such as Japan, students who combined self-monitoring and relational strategies outscored memorizing students by more than a year's worth of schooling. More detail on this finding is given in this *Scientific American* Mind article that I coauthored with a PISA analyst: <https://www.scientificamerican.com/article/why-math-education-in-the-u-s-doesn-t-add-up/>.

Mathematics is a conceptual subject, and it is important for students to be thinking slowly, deeply, and conceptually about mathematical ideas, not racing

through methods that they try to memorize. One reason that students need to think conceptually has to do with the ways the brain processes mathematics. When we learn new mathematical ideas, they take up a large space in our brain as the brain works out where they fit and what they connect with. But with time, as we move on with our understanding, the knowledge becomes compressed in the brain, taking up a very small space. For first graders, the idea of addition takes up a large space in their brains as they think about how it works and what it means, but for adults the idea of addition is compressed, and it takes up a small space. When adults are asked to add 2 and 3, for example, they can quickly and easily extract the compressed knowledge. William Thurston (1990), a mathematician who won the Field's Medal—the highest honor in mathematics—explains compression like this:

Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through the same process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.

You will probably agree with me that not many students think of mathematics as a “real joy,” and part of the reason is that they are not compressing mathematical ideas in their brain. This is because the brain only compresses concepts, not methods. So if students are thinking that mathematics is a set of methods to memorize, they are on the wrong pathway, and it is critical that we change that. It is very important that students think deeply and conceptually about ideas. We provide the activities in this book that will allow students to think deeply and conceptually, and an essential role of the teacher is to give the students time to do so.

Mathematical Thinking, Reasoning, and Convincing

When we worked with our Youcubed camp students, we gave each of them journals to record their mathematical thinking. I am a big fan of journaling—for myself and my students. For mathematics students, it helps show them that mathematics is a subject for which we should record ideas and pictures. We can use journaling to encourage students to keep organized records, which is another important part of mathematics, and help them understand that mathematical thinking can be a long and slow process. Journals also give students free space—where they can be creative,

share ideas, and feel ownership of their work. We did not write in the students' journals, as we wanted them to think of the journals as their space, not something that teachers wrote on. We gave students feedback on sticky notes that we stuck onto their work. The images in Figure I.1 show some of the mathematical records the camp students kept in their journals.

Another resource I always share with learners is the act of color-coding—that is, students using colors to highlight different ideas. For example, when working on an algebraic task, they may show the x in the same color in an expression, in a graph, and in a picture, as shown in Figure I.2. When adding numbers, color-coding may help show the addends (Figure I.3).

Color-coding highlights connections, which are a really critical part of mathematics.

Another important part of mathematics is the act of reasoning—explaining why methods are chosen and how steps are linked, and using logic to connect ideas.

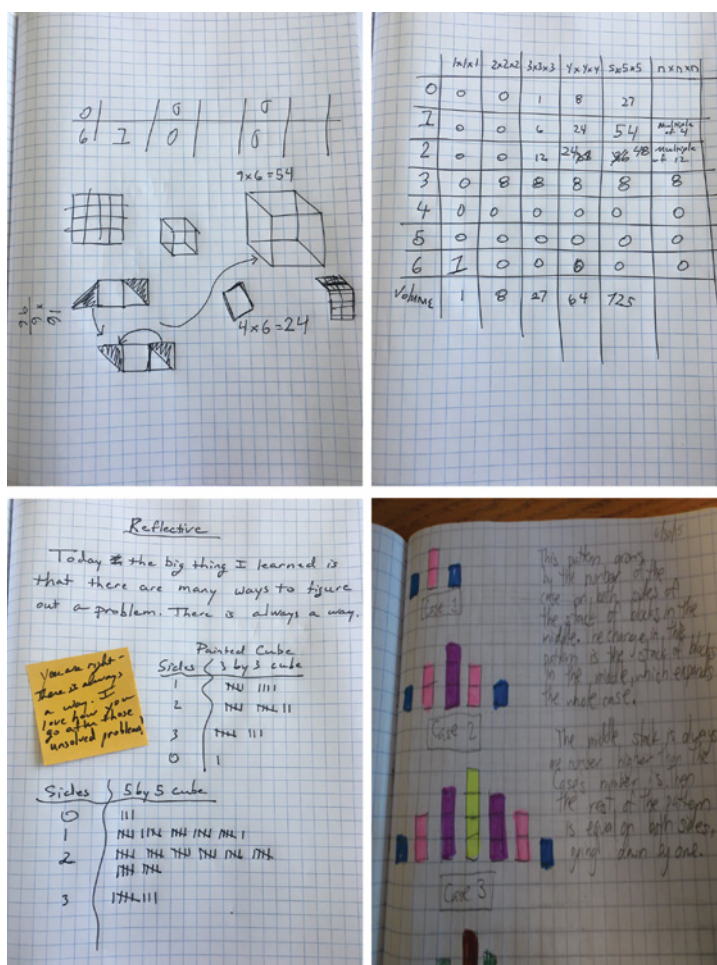


Figure I.1

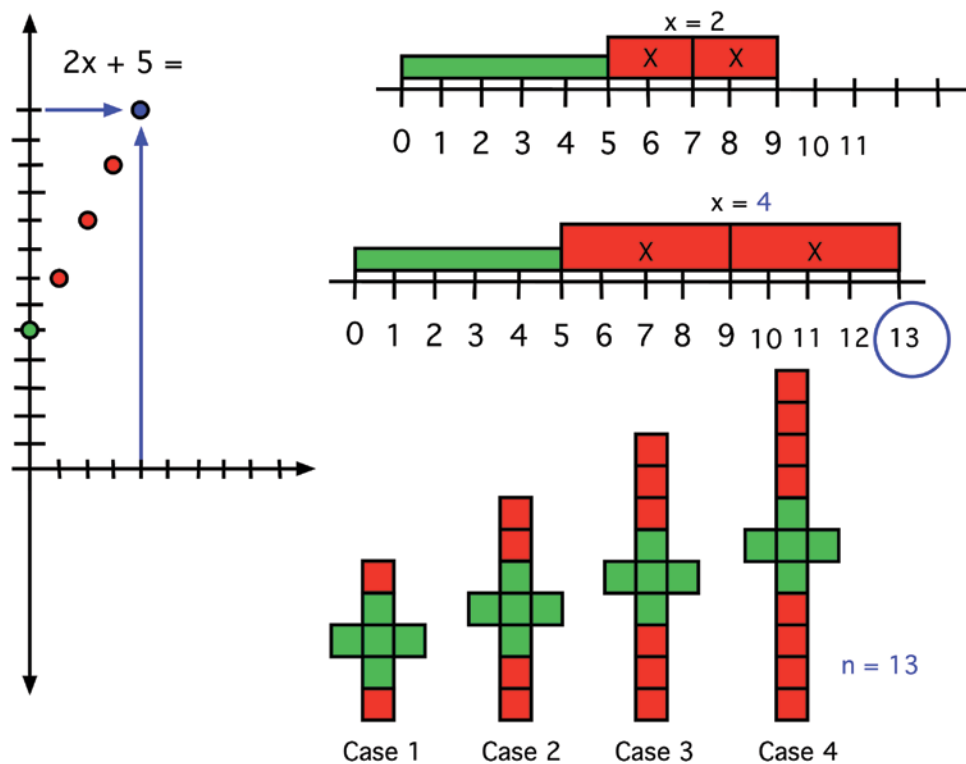


Figure I.2

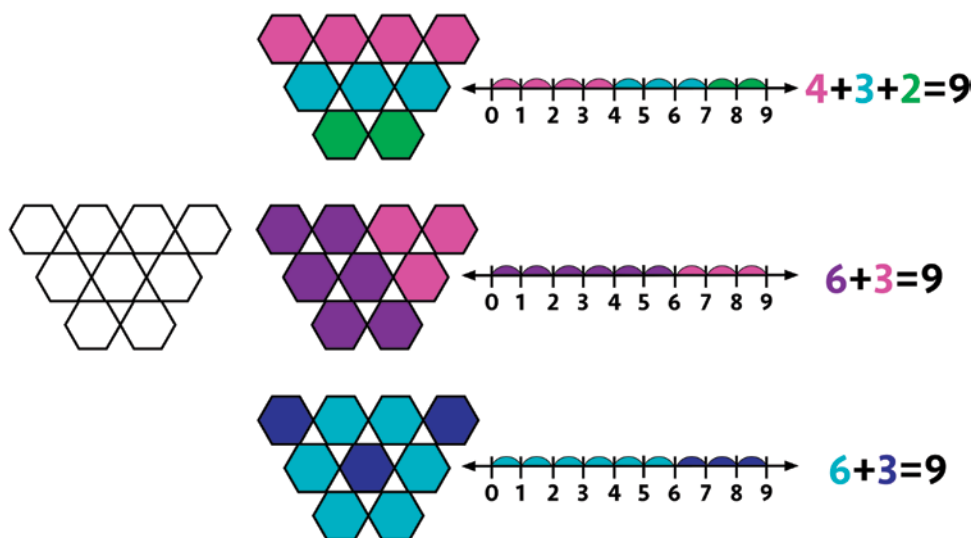


Figure I.3

Reasoning is at the heart of mathematics. Scientists prove ideas by finding more cases that fit a theory, or counterexamples that contradict a theory, but mathematicians prove their work by reasoning. If students are not reasoning, then they are not really doing mathematics. In the activities of these books, we suggest a framework that

encourages students to be convincing when they reason. We tell them that there are three levels of being convincing. The first, or easiest, level is to convince yourself of something. A higher level is to convince a friend. And the highest level of all is to convince a skeptic. We also share with students that they should be skeptics with one another, asking one another why methods were chosen and how they work. We have found this framework to be very powerful with students; they enjoy being skeptics, pushing each other to deeper levels of reasoning, and it encourages students to reason clearly, which is important for their learning.

We start each book in our series with an activity that invites students to reason about mathematics and be convincing. I first met an activity like this when reading Mark Driscoll's teaching ideas in his book *Fostering Algebraic Thinking*. I thought it was a perfect activity for introducing the skeptics framework that I had learned from a wonderful teacher, Cathy Humphreys. She had learned about and adapted the framework from two of my inspirational teachers from England: mathematician John Mason and mathematics educator Leone Burton. As well as encouraging students to be convincing, in a number of activities we ask students to prove an idea. Some people think of proof as a formal set of steps that they learned in geometry class. But the act of proving is really about connecting ideas, and as students enter the learning journey of proving, it is worthwhile celebrating their steps toward formal proof. Mathematician Paul Lockhart (2012) rejects the idea that proving is about following a set of formal steps, instead proposing that proving is "abstract art, pure and simple. And art is always a struggle. There is no systematic way of creating beautiful and meaningful paintings or sculptures, and there is also no method for producing beautiful and meaningful mathematical arguments" (p. 8). Instead of suggesting that students follow formal steps, we invite them to think deeply about mathematical concepts and make connections. Students will be given many ways to be creative when they prove and justify, and for reasons I discuss later, we always encourage and celebrate visual as well as numerical and algebraic justifications. Ideally, students will create visual, numerical, and algebraic representations and connect their ideas through color-coding and through verbal explanations. Students are excited to experience mathematics in these ways, and they benefit from the opportunity to bring their individual ideas and creativity to the problem-solving and learning space. As students develop in their mathematical understanding, we can encourage them to extend and generalize their ideas through reasoning, justifying, and proving. This process deepens their understanding and helps them compress their learning.

Big Ideas

The books in the Mindset Mathematics Series are all organized around mathematical “big ideas.” Mathematics is not a set of methods; it is a set of connected ideas that need to be understood. When students understand the big ideas in mathematics, the methods and rules fall into place. One of the reasons any set of curriculum standards is flawed is that standards take the beautiful subject of mathematics and its many connections, and divide it into small pieces that make the connections disappear. Instead of starting with the small pieces, we have started with the big ideas and important connections, and have listed the relevant Common Core curriculum standards within the activities. Our activities invite students to engage in the mathematical acts that are listed in the imperative Common Core practice standards, and they also teach many of the Common Core content standards, which emerge from the rich activities. Student activity pages are noted with a ☹️ and teacher activity pages are noted with a ☺️.

Although we have chapters for each big idea, as though they are separate from each other, they are all intrinsically linked. Figure I.4 shows some of the connections between the ideas, and you may be able to see others. It is very important to share with students that mathematics is a subject of connections and to highlight the connections as students work. You may want to print the color visual of the different connections for students to see as they work. To see the maps of big ideas for all of the grades K through 8, find our paper “What Is Mathematical Beauty?” at youcubed.org.

Structure of the Book

Visualize. Play. Investigate. These three words provide the structure for each book in the series. They also pave the way for open student thinking, for powerful brain connections, for engagement, and for deep understanding. How do they do that? And why is this book so different from other mathematics curriculum books?

Visualize

For the past few years, I have been working with a neuroscience group at Stanford, under the direction of Vinod Menon, which specializes in mathematics learning. We have been working together to think about the ways that findings from brain science can be used to help learners of mathematics. One of the exciting discoveries that has

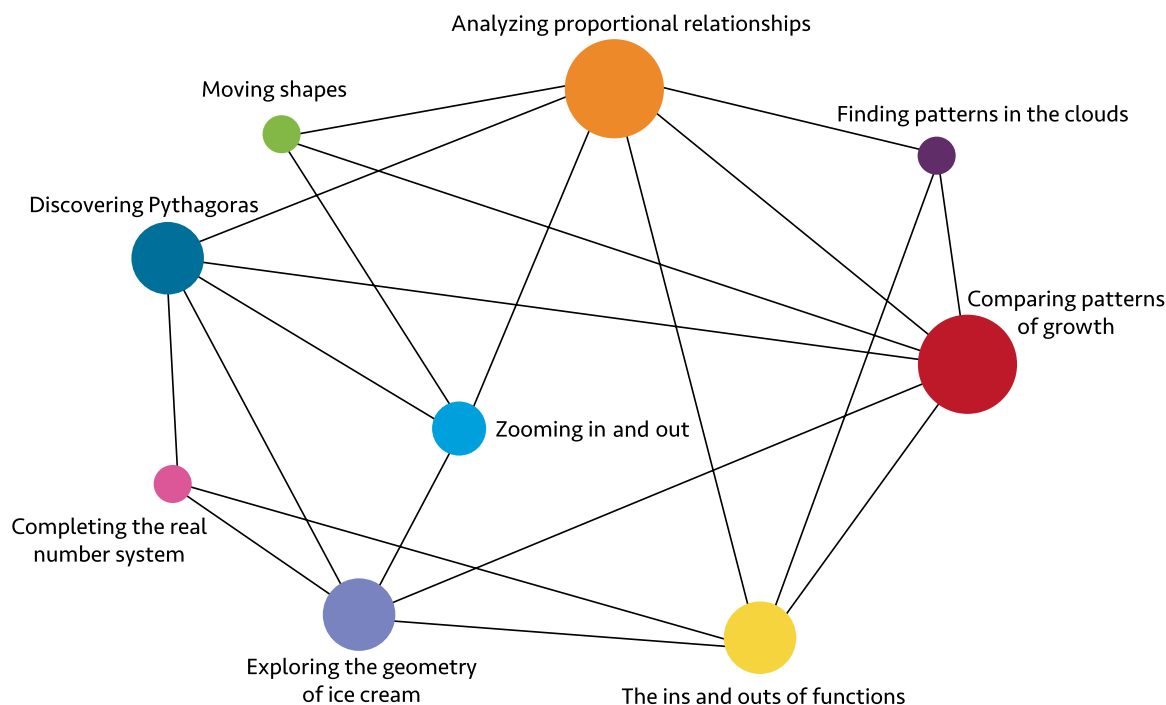


Figure I.4

been emerging over the last few years is the importance of visualizing for the brain and our learning of mathematics. Brain scientists now know that when we work on mathematics, even when we perform a bare number calculation, five areas of the brain are involved, as shown in Figure I.5.

Two of the five brain pathways—the dorsal and ventral pathways—are visual. The dorsal visual pathway is the main brain region for representing quantity. This may seem surprising, as so many of us have sat through hundreds of hours of mathematics classes working with numbers, while barely ever engaging visually with mathematics. Now brain scientists know that our brains “see” fingers when we calculate, and knowing fingers well—what they call finger perception—is critical for the development of an understanding of number. If you would like to read more about the importance of finger work in mathematics, look at the visual mathematics section of youcubed.org. Number lines are really helpful, as they provide the brain with a visual representation of number order. In one study, a mere four 15-minute sessions of students playing with a number line completely eradicated the differences between students from low-income and middle-income backgrounds coming into school (Siegler & Ramani, 2008).

Our brain wants to think visually about mathematics, yet few curriculum materials engage students in visual thinking. Some mathematics books show pictures, but they rarely ever invite students to do their own visualizing and

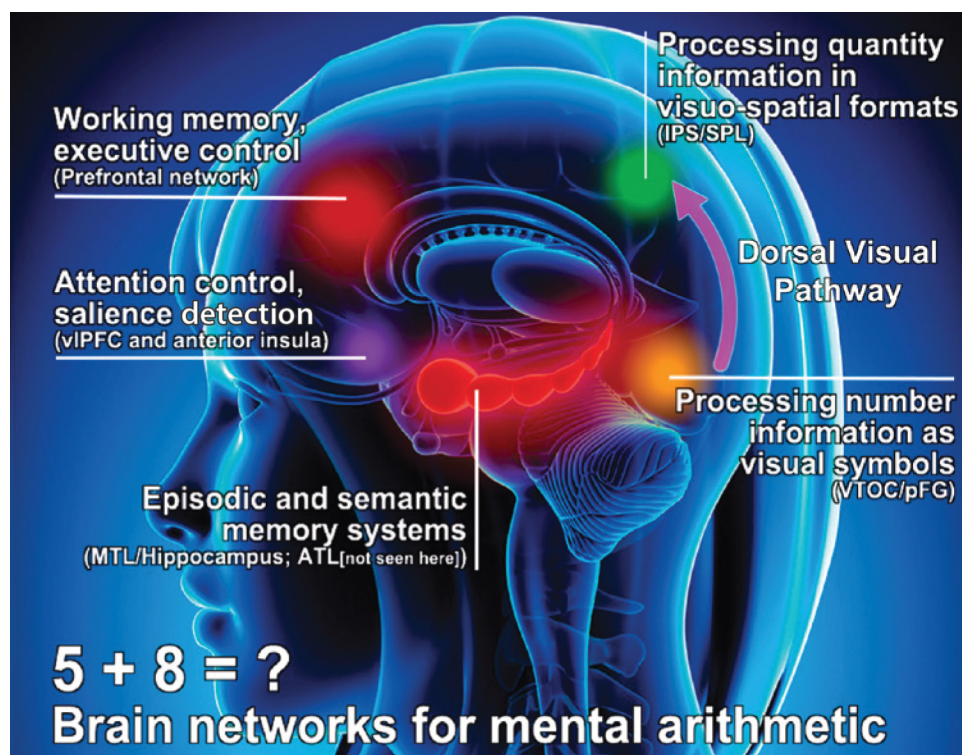


Figure I.5

drawing. The neuroscientists' research shows the importance not only of visual thinking but also of students' connecting different areas of their brains as they work on mathematics. The scientists now know that as children learn and develop, they increase the connections between different parts of the brain, and they particularly develop connections between symbolic and visual representations of numbers. Increased mathematics achievement comes about when students are developing those connections. For so long, our emphasis in mathematics education has been on symbolic representations of numbers, with students developing one area of the brain that is concerned with symbolic number representation. A more productive and engaging approach is to develop all areas of the brain that are involved in mathematical thinking, and visual connections are critical to this development.

In addition to the brain development that occurs when students think visually, we have found that visual activities are really engaging for students. Even students who think they are “not visual learners” (an incorrect idea) become fascinated and think deeply about mathematics that is shown visually—such as the visual representations of the calculation 18×5 shown in Figure I.6.

In our Youcubed teaching of summer school to sixth- and seventh-grade students and in our trialing of Youcubed's WIM materials, we have found

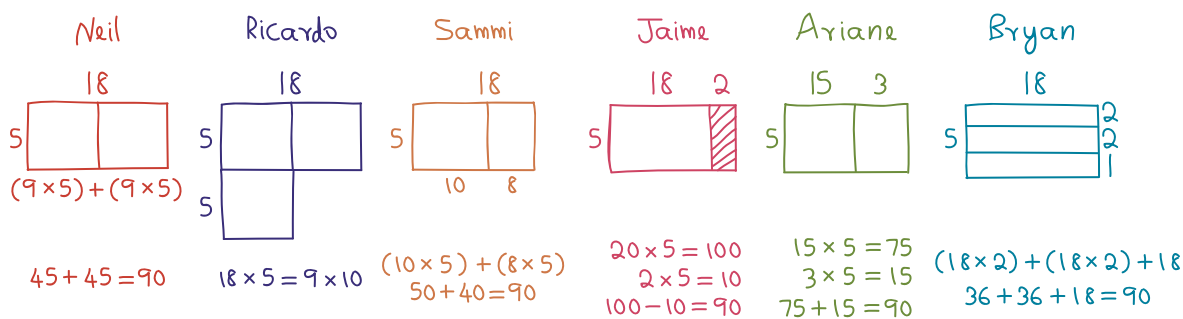


Figure I.6

that students are inspired by the creativity that is possible when mathematics is visual. When we were trialing the materials in a local middle school one day, a parent stopped me and asked what we had been doing. She said that her daughter had always said she hated and couldn't do math, but after working on our tasks, she came home saying she could see a future for herself in mathematics. We had been working on the number visuals that we use throughout these teaching materials, shown in Figure I.7.

The parent reported that when her daughter had seen the creativity possible in mathematics, everything had changed for her. I strongly believe that we can give these insights and inspirations to many more learners with the sort of creative, open mathematics tasks that fill this book.

We have also found that when we present visual activities to students, the status differences that often get in the way of good mathematics teaching disappear. I was visiting a first-grade classroom recently, and the teacher had set up four different stations around the room. In all of them, the students were working on arithmetic. In one, the teacher engaged students in a mini number talk; in another, a teaching assistant worked on an activity with coins; in the third, the students played a board game; and in the fourth, they worked on a number worksheet. In each of the first three stations, the students collaborated and worked really well, but as soon as students went to the worksheet station, conversations changed, and in every group I heard statements like "This is easy," "I've finished," "I can't do this," and "Haven't you finished yet?" These status comments are unfortunate and off-putting for many students. I now try to present mathematical tasks without numbers as often as possible, or I take out the calculation part of a task, as it is the numerical and calculational aspects that often cause students to feel less sure of themselves. This doesn't mean that students cannot have a wonderful and productive relationship with numbers, as we hope to promote in this book, but sometimes the key mathematical idea can be arrived at without any numbers at all.

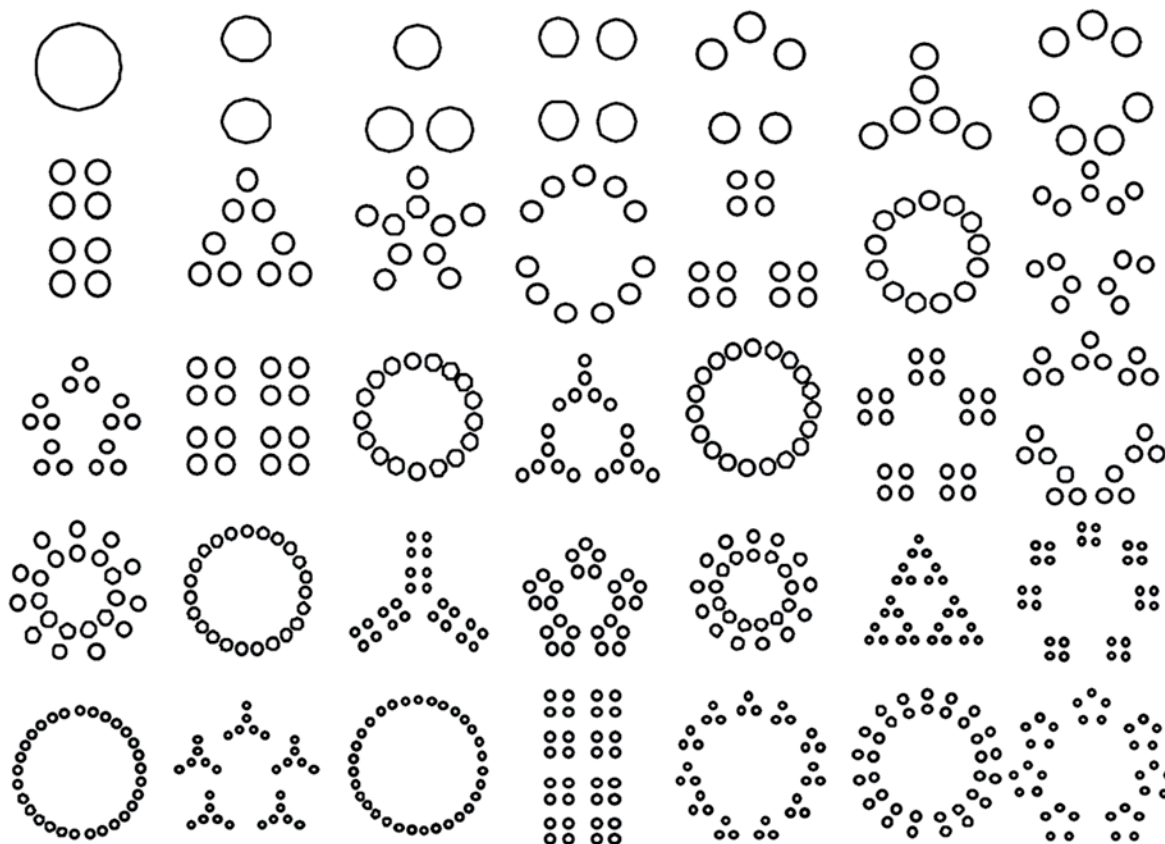


Figure I.7

Almost all the tasks in our book invite students to think visually about mathematics and to connect visual and numerical representations. This encourages important brain connections as well as deep student engagement.



The key to reducing status differences in mathematics classrooms, in my view, comes from *opening* mathematics. When we teach students that we can see or approach any mathematical idea in different ways, they start to respect the different thinking of all students. Opening mathematics involves inviting students to see ideas differently, explore with ideas, and ask their own questions. Students can gain access to the same mathematical ideas and methods through creativity and exploration that they can by being taught methods that they practice. As well as reducing or removing status differences, open mathematics is more engaging for students. This is why we are inviting students, through these mathematics materials, to play with mathematics. Albert Einstein famously once said that “play is the highest form of research.” This

is because play is an opportunity for ideas to be used and developed in the service of something enjoyable. In the Play activities of our materials, students are invited to work with an important idea in a free space where they can enjoy the freedom of mathematical play. This does not mean that the activities do not teach essential mathematical content and practices—they do, as they invite students to work with the ideas. We have designed the Play activities to downplay competition and instead invite students to work with each other, building understanding together.

Investigate

Our Investigate activities add something very important: they give students opportunities to take ideas to the sky. They also have a playful element, but the difference is that they pose questions that students can explore and take to very high levels. As I mentioned earlier, all of our tasks are designed to be as low floor and high ceiling as possible, as these provide the best conditions for engaging all students, whatever their prior knowledge. Any student can access them, and students can take the ideas to high levels. We should always be open to being surprised by what our learners can do, and always provide all students with opportunities to take work to high levels and to be challenged.

A crucial finding from neuroscience is the importance of students struggling and making mistakes—these are the times when brains grow the most. In one of my meetings with a leading neuroscientist, he stated it very clearly: if students are not struggling, they are not learning. We want to put students into situations where they feel that work is hard, but within their reach. Do not worry if students ask questions that you don't know the answer to; that is a good thing. One of the damaging ideas that teachers and students share in education is that teachers of mathematics know everything. This gives students the idea that mathematics people are those who know a lot and never make mistakes, which is an incorrect and harmful message. It is good to say to your students, “That is a great question that we can all think about” or “I have never thought about that idea; let's investigate it together.” It is even good to make mistakes in front of students, as it shows them that mistakes are an important part of mathematical work. As they investigate, they should be going to places you have never thought about—taking ideas in new directions and exploring uncharted territory. Model for students what it means to be a curious mathematics learner, always open to learning new ideas and being challenged yourself.

* * *

We have designed activities to take at least a class period, but some of them could go longer, especially if students ask deep questions or start an investigation into a cool idea. If you can be flexible about students' time on activities, that is ideal, or you may wish to suggest that students continue activities at home. In our teaching of these activities, we have found that students are so excited by the ideas that they take them home to their families and continue working on them, which is wonderful. At all times, celebrate deep thinking over speed, as that is the nature of real mathematical thought. Ask students to come up with creative representations of their ideas; celebrate their drawing, modeling, and any form of creativity. Invite your students into a journey of mathematical curiosity and take that journey with them, walking by their side as they experience the wonder of open, mindset mathematics.

References

- Lockhart, P. (2012). *Measurement*. Cambridge, MA: Harvard University Press.
- Siegler, R. S., & Ramani, G. B. (2008). Playing linear numerical board games promotes low income children's numerical development. *Developmental Science*, 11(5), 655–661. doi:10.1111/j.1467-7687.2008.00714.x
- Thurston, W. (1990). Mathematical education. *Notices of the American Mathematical Society*, 37(7), 844–850.

Note on Materials

In middle schools, we often find that there is little use of manipulatives and that few may be available in the building for teachers to choose from. But we believe, and extensive research supports, that all math learners benefit from mathematics that is visual, concrete, and modeled in multiple representations. Students need to physically create, draw, and construct mathematics to build deep understanding of what concepts represent and mean. Students need to interact with mathematics, manipulating representations to pose and investigate questions. Apps and digital games are another option, and we have found them to be valuable because they can be organized and manipulated with an unending supply. However, we want to emphasize that they should not be a replacement for the tactile experience of working with physical manipulatives. We support making different tools available for students to use as they see fit for the representation, and, following the activity, we encourage you to ask students to reflect on what the tools allowed them to see mathematically.

In our books for middle grades, you will find the same emphasis on visual mathematics and using manipulatives as in our elementary books, because these representations of mathematics are critical for all learners. If manipulatives are in short supply in your building, we encourage you to advocate for their purchase for the long-term benefit of your students. In the near term, you may be able to borrow the manipulatives we use in this book from your district's elementary schools.

Manipulatives and Materials Used in This Book

- **Snap cubes.** Snap or multilink cubes can be used for patterning, representing mathematical situations, and measurement. In eighth grade, we use them as tools for exploring and representing slope.
- **Square tiles.** Square tiles are flexible manipulatives that can be used to represent patterns visually that are too often only represented symbolically, making them a useful algebraic manipulative.
- **Cuisenaire rods.** Cuisenaire rods are mathematical manipulatives more common in elementary schools, where they are often used to represent part-whole relationships in operations and fractions. They can also be used like square tiles to represent some kinds of pile patterns or explore the Pythagorean theorem, which we offer as an option in eighth grade.
- **Fillable, relational geometric solids.** Geometric solids enable students to touch, rotate, and compare three-dimensional figures of different types. Some solids can be filled to explore volume. For eighth grade, we recommend a set of fillable relational solids that all have one dimension that is the same (height or diameter), so that students can fill and compare volumes.
- **Rulers.** We use rulers as linear measurement tools and as straightedges in the construction of graphs and geometric figures.
- **Meter sticks or yardsticks and balls.** As students investigate proportional relationships, we use meter sticks or yardsticks to create ramps that students can roll balls down to explore the relationships between how far the balls roll, the length of the ramp, and the height of the ramp.
- **Angle rulers or protractors.** Students need tools for measuring angles. Protractors are the most common tool, but are notoriously hard for students to use accurately. We prefer angle rulers, which have two arms that can be adjusted to fit an angle and are easier to read. They avoid the issue of needing to extend the sides of the angle being measured, which students encounter with protractors.
- **Compasses.** Compasses are tools both for drawing and for pivoting from a fixed point. In eighth grade, we use compasses to draw arcs and circles in geometric explorations. We use compasses to pivot when moving through an angle to explore rotations.

- **Colors.** Color-coding work is a powerful tool to support decomposition, patterning, and connecting representations. We often ask that students have access to colors; whether they are markers, colored pencils, or colored pens, we leave up to you.
- **Patty paper.** Patty paper is flexible and inexpensive, and it supports mistakes. Patty paper is a thin, transparent paper used in the food industry to separate food items. Its advantages in mathematics are that it is square paper that can be folded, cut, written on, and seen through for tracing or comparing. In eighth grade, we use patty paper to support explorations with transformations, enabling students to slide, flip, and turn figures and overlay them to look for congruence or similarity.
- **Calculators.** Calculators are used as a resource in the book to allow students to focus on the bigger mathematical ideas, rather than spending their time performing calculations. Calculators are a tool that students need to know how to use, and they should increase students' access to big mathematical ideas.
- **Office supplies, such as card stock, tape, glue sticks, and masking tape.** We use these across the book to construct charts, mark spaces, display thinking, or piece together work.