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Carlos Matheus Silva Santos

Dynamical Aspects of Teichmüller Theory

$SL(2, \mathbb{R})$-Action on Moduli Spaces of Flat Surfaces

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To Jean-Christophe Yoccoz (in memoriam)

Preface

This memoir is based on some of my works on *Teichmüller dynamics*, or, more precisely, the dynamics of the $SL(2,\mathbb{R})$-action on moduli spaces of flat surfaces.

Chapter 1 serves to introduce several basic aspects of Teichmüller dynamics. Its content is based on the survey texts written by Zorich [71] and Yoccoz [69], and the lecture notes [31] from a minicourse delivered by Forni and the author in 2011 at Banach Center (Bedlewo, Poland). In particular, Chap. 1 is a general-purpose introduction to all subsequent chapters, so that there is a foundation from which to build later discussions.

After reading Chap. 1, the reader can freely choose in which order to read the remaining chapters. In fact, the discussions in Chaps. 2–6 are completely independent from each other.

In Chap. 2, we investigate the regularity of $SL(2,\mathbb{R})$-invariant measures on moduli spaces of flat surfaces. In general, such a measure is called regular if the majority of the flat surfaces in its support have all of its shortest saddle connections parallel between themselves. The regularity property was used by Eskin–Kontsevich–Zorich [19] to justify a sophisticated integration by parts argument in the proof of their famous formula for the sum of nonnegative Lyapunov exponents of the Kontsevich–Zorich cocycle (the interesting part of the derivative of the $SL(2,\mathbb{R})$-action on flat surfaces). In the same article, Eskin–Kontsevich–Zorich [19] conjectured that the regularity property is always valid, so that their formula for the sum of nonnegative Lyapunov exponents could be applied without restrictions to all ergodic $SL(2,\mathbb{R})$-invariant probability measures on moduli spaces of flat surfaces. The goal of Chap. 2 is to explain my joint work [5] with Avila and Yoccoz where the regularity conjecture of Eskin–Kontsevich–Zorich is solved affirmatively.

Chapter 3 is dedicated to the study of the rate of mixing of the Teichmüller flow. The question of determining how fast is the decay of correlations of the so-called Masur–Veech measures was solved in a celebrated article of Avila, Gouëzel and Yoccoz [4]: the rate of mixing of the Teichmüller flow with respect to such measures is always exponential. After that, Avila and Gouëzel [3] extended this result to all ergodic $SL(2,\mathbb{R})$-invariant probability measures on moduli spaces of flat surfaces. A natural problem motivated by the results of Avila, Gouëzel and Yoccoz is to decide if the rates of mixing of these measures have some sort of uniformity. The main result of Chap. 3 is a theorem obtained in collaboration with Schmithüsen [49] saying that there is no uniformity in the rate of mixing of such measures when one looks into moduli spaces of flat surfaces of arbitrarily high genus.

Chapter 4 is devoted to the problem of classifying closures of $SL(2,\mathbb{R})$-orbits on moduli spaces of flat surfaces. Many applications of Teichmüller dynamics to mathematical billiards depend on the precise knowledge of the closures of certain $SL(2,\mathbb{R})$-orbits of flat surfaces, and this partly explains the interest in classifying such objects. The groundbreaking works of Eskin–Mirzakhani [22], Eskin–Mirzakhani–Mohammadi [23] and Filip [24] say that the closures of $SL(2,\mathbb{R})$-orbits of translation surfaces are extremely well-behaved objects: they are affine in period coordinates, quasi-projective in the coordinates induced by the moduli spaces of curves, and the totality of such objects is a countable collection. In particular, it is reasonable to try to classify these objects. The works of Calta [11] and McMullen [54, 56] provide a quite satisfactory classification of closures of $SL(2,\mathbb{R})$-orbits of translation surfaces of genus two. On the other hand, the situation in higher genus is still not completely understood despite many recent partial results. Nevertheless, this situation improves a little when we concentrate on the so-called Teichmüller curves, i.e. closed $SL(2,\mathbb{R})$-orbits; for example, Bainbridge, Habegger and Möller [9] showed the finiteness of algebraically primitive Teichmüller curves generated by translation surfaces of genus three. In Chap. 4, we outline the proof of a result obtained together with Wright [51] ensuring the finiteness of algebraically primitive Teichmüller curves generated by translation surfaces of genus $g > 2$ prime possessing a single conical singularity.

Chapter 5 discusses the Lyapunov exponents of the so-called Kontsevich–Zorich cocycle, i.e. the interesting part of the derivative of the $SL(2,\mathbb{R})$-action on moduli spaces of translation surfaces. The qualitative and/or quantitative properties of the Lyapunov exponents of the KZ cocycle are usually important in many applications of Teichmüller dynamics; for example, Avila and Forni [2] exploited a result of Forni [28] on the non-uniform hyperbolicity of the KZ cocycle with respect to Masur–Veech measures to show that typical, non-rotational interval exchange transformations are weak mixing. From the qualitative point of view, the Lyapunov exponents of the KZ cocycle with respect to Masur–Veech measures are well understood thanks to a celebrated work of Avila and Viana [6] asserting the simplicity, i.e. multiplicity one, of such exponents (thus confirming a conjecture of Kontsevich and Zorich). On the other hand, this is not true for other measures; Forni and I (see [31], for example) constructed some examples of $SL(2,\mathbb{R})$-invariant measures on moduli spaces of translation surfaces such that the Lyapunov exponents of the KZ cocycle with respect to these measures are far from being simple. The starting point of Chap. 5 is a result in collaboration with Eskin [21] guaranteeing that the Lyapunov exponents of the KZ cocycles over Teichmüller curves (closed $SL(2,\mathbb{R})$-orbits) can be computed from random products of matrices. Next, we exploit this result and the techniques of Avila and Viana [6] to show an effective criterion (based on Galois theory) obtained with Möller and Yoccoz [50] for the simplicity of Lyapunov exponents of the KZ cocycle over arithmetic

Teichmüller curves. Finally, we employ this Galois-theoretical simplicity criterion to discuss a counterexample by Delecroix and me [17] to a conjecture of Forni.

Chapter 6 is dedicated to the structure of the group of matrices associated with the KZ cocycle, i.e. the interesting part of the derivative of the $SL(2,\mathbb{R})$-action on moduli spaces of translation surfaces. The groups of matrices generated by the KZ cocycle deserve a special attention because they play a fundamental role in the study of the $SL(2,\mathbb{R})$-action on translation surfaces; for example, the famous work of Eskin–Mirzakhani [22] on the classification of $SL(2,\mathbb{R})$-invariant probability measures on moduli spaces of translation surfaces is based in a fine analysis of this cocycle. A recent work of Filip [26] gives a list of all possible Zariski closures of groups of matrices associated with the KZ cocycle (modulo compact factors and finite index); in particular, Filip confirmed a conjecture by Forni, Zorich and me [33] about the mechanisms behind zero Lyapunov exponents for the KZ cocycle. However, Filip's list is produced from considerations on variations of Hodge structures of weight one over quasi-projective varieties, and for this reason, a natural question is to know what items in this list actually occur in the context of the KZ cocycle. In this direction, the main result in Chap. 6 is a theorem by Filip, Forni and me [27] exhibiting an example where one of the groups of quaternionic matrices in Filip’s list is realized as part of the KZ cocycle.

Villetaneuse, France Carlos Matheus Silva Santos

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Contents

Chapter 1
Introduction

This section serves as a general-purpose introduction to all other sections of this memoir. In particular, we'll always assume familiarity with the content of this section in subsequent discussions.

The basic references for this section are the survey texts of Zorich [71], Yoccoz [69], and Forni and the author [31].

1.1 Abelian Differentials and Their Moduli Spaces

Let $\mathcal{L}_g$ be the set of Abelian differentials on a Riemann surface of genus $g \geq 1$, that is, the set of pairs (Riemann surface structure on M, ω) where M is a compact topological surface of genus g and $\omega \not\equiv 0$ is a non-trivial 1-form which is holomorphic with respect to the underlying Riemann surface structure.

The *Teichmüller space* of Abelian differentials of genus $g \geq 1$ is the quotient $\mathcal{TH}_g := \mathcal{L}_g/\text{Diff}_0^+(M)$ and the *moduli space* of Abelian differentials of genus $g \geq 1$ is the quotient $\mathcal{H}_g := \mathcal{L}_g/\Gamma_g$. Here $\text{Diff}_0^+(M)$ is the set of diffeomorphisms isotopic to the identity and $\Gamma_g := \text{Diff}^+(M)/\text{Diff}_0^+(M)$ is the mapping class group (i.e., the set of isotopy classes of orientation-preserving diffeomorphisms), and both $\text{Diff}_0^+(M)$ and Γ_g act on the set of Riemann surface structure in the usual manner,[1] while they act on Abelian differentials by pull-back.

Before equipping $\mathcal{TH}_g$ and $\mathcal{H}_g$ with nice structures, let us give a *concrete* description of Abelian differentials in terms of *translation structures*.

[1]By precomposition with coordinate charts.

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1.2 Translation Structures

Let $(M, \omega) \in \mathcal{L}_g$ and denote by $\Sigma \subset M$ the set of singularities of ω, or, equivalently, the *divisor* of ω, i.e., the finite set

$$\Sigma := \mathrm{div}(\omega) := \{p \in M : \omega(p) = 0\}$$

For each $p \in M - \Sigma$, let us select a small simply-connected neighborhood U_p of p such that $U_p \cap \Sigma = \emptyset$. In this context, the "period" map $\phi_p : U_p \to \mathbb{C}$, $\phi_p(x) := \int_p^x \omega$ given by integration along *any* path inside U_p joining p and x is well-defined: in fact, any holomorphic 1-form ω is closed and, thus, the integral $\int_p^x \omega$ does not depend on the choice of the path inside U_p connecting p and x. Furthermore, since $p \notin \Sigma$ (i.e., $\omega(p) \neq 0$), we have that, after reducing U_p if necessary, this "period" map ϕ_p is a biholomorphism.

In other words, the collection $\{(U_p, \phi_p)\}_{p \in M - \Sigma}$ of all such "period" maps is an atlas of $M - \Sigma$ which is compatible with the Riemann surface structure. By definition, the local expression of Abelian differential ω in these coordinates is $(\phi_p)_*(\omega) = dz$ (on $\mathbb{C}$). Also, the local equality $\int_p^x \omega = \int_p^q \omega + \int_q^x \omega$ implies that all coordinate changes are $\phi_q \circ \phi_p^{-1}(z) = z + c$ where $c = \int_q^p \omega \in \mathbb{C}$ is a constant independent of z. Moreover, since $\mathrm{div}(\omega)$ is finite, Riemann's theorem on removable singularities implies that this atlas of "period" charts on $M - \Sigma$ can be extended to M in such a way that the local expression of ω in a chart around a zero $p \in \Sigma$ of ω of order k is the holomorphic 1-form $z^k dz$.

In the literature, a maximal atlas of compatible charts on the complement $M - \Sigma$ of a finite subset Σ of a surface M whose changes of coordinates are translations $z \mapsto z + c$ of the complex plane is called a *translation structure* on M. In this language, the discussion in the previous paragraph says that (M, ω) determines a translation structure on M. On the other hand, it is clear that a translation structure on M determines a Riemann surface structure[2] and an Abelian[3] differential ω on M.

In summary, we proved the following proposition.

Proposition 1 *The set $\mathcal{L}_g$ of all non-trivial Abelian differentials on compact Riemann surfaces of genus $g \geq 1$ is canonically identified to the set of all translation structures on the compact surfaces of genus $g \geq 1$.*

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[2]Since translations are particular cases of biholomorphisms.

[3]We define ω by locally pulling-back dz via the charts: this gives a globally defined Abelian differential because the changes of coordinates are translations and, hence, dz is invariant under changes of coordinates.