*M. Majewski*
Getting Started with MuPAD

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# Getting Started with MuPAD

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# Preface

No book is born in a vacuum. There must always be somebody who needs the book, somebody who will read and use it, and somebody who will write it. I walked with the idea of this book for a long time. However, its final concept came into reality during my lectures, in February 2005, at the Universiti Malaysia Sabah in Borneo. I realized that my students needed a bit more than just my lectures. They needed a text that they could follow during lab sessions or after classes so they could learn at any time, at their own pace. Therefore, I decided to write a small book with just a few chapters covering the different areas of applying the Computer Algebra System called MuPAD in different areas of mathematics. I intended each chapter to be short enough to be covered in a reasonably short time, about 2 to 4 hours.

Another important objective was to have each chapter completely independent of the others, so that the readers could easily select and read the chapters that they needed the most, without being forced to read the whole book. There was one obstacle for such a concept—the large number of graphics I used to visualize mathematics. Therefore, I finally decided to write a separate chapter covering the major concepts of MuPAD graphics. The graphics chapter, together with the introductory chapter, forms the base for all the remaining chapters. Therefore, chapters 1 and 2 should be read first and foremost, but the remaining chapters can be read in any order.

Allow me to mention what you will find in this book. Chapter 1 is the introductory chapter, and it covers some very basic information about MuPAD. You should read this chapter if you do not know anything about MuPAD. The second chapter contains a reasonably brief description of the concept of MuPAD graphics, and introduces the Virtual Camera tool and the major types of graphical objects. Chapters 3 and 4 are devoted to calculus of one variable and several variables, respectively; they require a basic knowledge of calculus. This book isn't intended to teach mathematics; it teaches only how to apply MuPAD in mathematics. Chapter 5 is devoted to working with selected algebraic concepts with MuPAD. Chapter 6 shows how MuPAD can be used in elementary statistics and in the visualization of discrete data. Finally, chapter 7 introduces some basic programming concepts in MuPAD.

The chapters in this book aren't intended to cover specific mathematical disciplines in detail. However, the book gives the readers enough basic knowledge to start using MuPAD in several disciplines. Readers who are interested in a more detailed book on programming with MuPAD may read my *MuPAD Pro Computing Essentials*, which focuses on programming and using MuPAD programs in mathematics. I know that there are many other mathematical disciplines that are worth exploring with MuPAD. Perhaps one day I, or somebody else, will write the missing chapters on analytic geometry, number theory, abstract algebra, discrete mathematics, and so on. Finally, I know also that each chapter of this book could be the staring point for a new exciting textbook for a regular college or university course. For example, writing a textbook for a course on abstract algebra or discrete mathematics with MuPAD could be a very fascinating project. Therefore, this book is not only a collection of MuPAD workshops, but also can be considered as a starting point for many interesting new projects.

I am very grateful to Prof. Fred Szabo and Prof. Bernard Liengme as well as to my son Jakub for proofreading and commenting on the manuscript of this book. I thank them for their wonderful support and for the time they spent helping me in this project. The first two chapters of this book were used during my lectures at Mumbai University in April 2005, and they were received with great enthusiasm by my colleagues and the participants in my lectures.

The manuscript of this book, as well as all other my books, was written using Scientific Workplace, the best tool ever made for writing mathematical texts and experimenting with mathematics. I would like to thank Barry MacKichan and his excellent team for creating Scientific Workplace and for their never ending support of my work.

Some additions, corrections, and the MuPAD code for this book can be found on the web site at http://majewski.mupad.com.

*Miroslaw Majewski*
*Zayed University, June 2005*

# 1

# Introduction to MuPAD

## 1.1 A brief history of MuPAD

MuPAD is a *Computer Algebra System*. Such systems are very unusual computer programs. They contain procedures that can be used to manipulate mathematical objects and perform operations such as solving mathematical equations, integrating functions and plotting graphs. In addition, they usually contain a programming language that allows users to define their own commands and expand the functionality of the system. These features make computer algebra systems useful for solving mathematical problems such as problems encountered in high school algebra, as well as for solving very advanced mathematical or engineering problems. Another important feature of computer algebra systems is their ability to perform symbolic operations, that is to say, operations on expressions containing symbols, and producing results in symbolic form. They can also carry out operations on numbers with high precision. They can, for example, calculate the number $\pi$ to thousands of places.

The beginning of MuPAD goes back to the year 1989, when Professor Benno Fuchssteiner from Paderborn University and his student Waldemar Wiwianka started a research project on handling large data generated by algorithms, used to investigate the structure of nonlinear systems. The first students taking part in this project were Oliver Kluge and Karsten Morisse. Their joint Master's thesis was the first successful outcome of the project. At the same time, in the early stages of MuPAD, Gudrun Oevel, another student of Professor Fuchssteiner, developed the foundations of MuPAD graphics. Since then, MuPAD has become a major research project, carried out by researchers and students at Paderborn University.

In 1997, MuPAD became a commercial product. Its producer is *SciFace GmBH,* a computer company in Paderborn, with Dr. Oliver Kluge as manager. It is important to mention that at Paderborn University, a strong group of researchers, called *MuPAD Research Group*, is continually developing and implementing new algorithms for the system and works on expanding its functionality. It is important to mention that MuPAD has a number of features that distinguish it from other computer algebra systems. These include its open concept, where all users can expand the functionality of the program by developing their own libraries, the object-oriented concept, applied to all MuPAD elements, as well as its excellent graphics and animation tools.

Let us now find out how MuPAD works and what we can do with it. We will start by examining some of its features that make it resemble a super-calculator.

## 1.2 MuPAD as a calculator

After launching MuPAD, a new notebook opens and a large dot appears in the left margin on the screen. This is the area, the so-called input region, where we type our commands. Let us write something there, something that we usually type when testing a new calculator, such as

```
• 234 + 675
```

Now let us press [Enter]. MuPAD produces a result, similar to

```
909
```

The area where MuPAD displays its results is called the output region. In addition, MuPAD notebooks contain another type of region, the text region. We can insert text regions from the menu Insert>Input Above or Insert>Input Below. A text region is the space where we can type any text.

Technical comment *In order to improve readability of our text, all MuPAD statements, as well as all results obtained with MuPAD, will be printed using* `Lucida Sans Typewriter` *font. In cases where the output is a mathematical formula we will use* Times New Roman*. Wherever convenient, we will also omit the dot from the input region.*

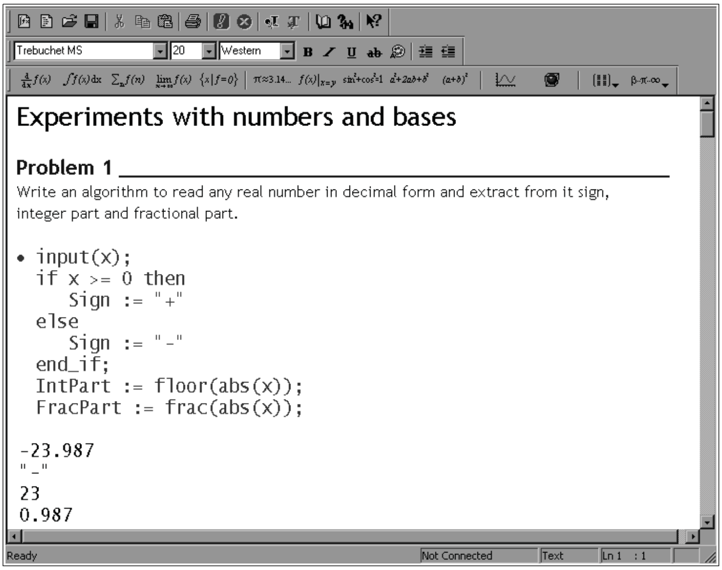



*Fig. 1.1 Text, input and output regions in MuPAD notebook*

MuPAD, as a calculator, can do much more than any calculator we ever used. Let us examine a few more examples.

```
• 98765467.89765/3456.987654
```

```
28569.80637
```

```
• 234!
```

226701500510243392088429967939458536596181051494323328919091\
627814531884961852861632986065166948869056073651901909595630\
646129453387036941368576843569614678171932751596833773143536\
624437808408963972397082885810366994838323956362463109712412\
249963159146761732364354657983880952611680981197119192191774\
651724175318705330712109638909510796006513526492598656929841\
724590466979051697750368473257330343936000000000000000000000\
00000000000000000000000000000000000

This was quite simple. In a similar way, MuPAD can perform all other arithmetical operations with any desired accuracy. For example, a square root of 2 calculated with an accuracy of 49 digits after the decimal point, can be calculated in the following way.

```
• DIGITS := 50:
• sqrt(2.0)
  1.4142135623730950488016887242096980785696718753769
```

The accuracy of the calculations can be very high. If needed, we can ask MuPAD to produce results with 1000 or 100,000 places of accuracy or more. This cannot of course be done with a normal calculator used for teaching or scientific calculations.

Technical comment *There are two important symbols, "`:`" and "`//`", useful for formatting MuPAD output. If we use the symbol "`:`" at the end of a MuPAD statement, then MuPAD produces the result, but it does not display it on the screen. If we use the symbol "`//`" in a MuPAD statement, then anything written to the right of it is treated as a comment and is not executed.*

The symbol "`:=`", used earlier, is what we will call an assignment operator. This operator assigns a value to a given variable or constant. In the above example, the word `DIGITS` denotes one of the MuPAD variables, defining the number of digits used to display decimal numbers on the screen. In our example, we requested 50 digits for displaying decimal numbers (the integer part plus a fractional part).

As names for variables and constants we can use any expressions, not starting with a digit, as well as letters, digits and the underscore character. Examples of correctly formed names of variables are `MySet`, `NumberOfData`, `A234`, `John_Brown_on_a_red_bike`, and so on. Names of constants usually contain capital letters only, for example `E`, `PI`, `DIGITS`, `CATALAN`, `EULER`.

Finally, let us note that MuPAD treats the mathematical equality symbol "=" as a relation, used to compare two objects. Therefore `A:=123` means *let A be equal* 123, but `A=123` denotes a true or false equality.

Technical comment *While writing MuPAD statements we should avoid using characters that do not exist in the English alphabet such as special symbols in Polish, Czech, Portuguese, etc. Usually MuPAD produces an error message if we do. However, we can use these characters in comments to the right of "//".*

It is worth noting that when doing arithmetic with MuPAD, the program always tries to produce exact or symbolic values, whereas decimal values, often approximate, are produced only when explicitly asked for. Therefore, if we wish to obtain results in decimal form, we must use a special `float` procedure, or stipulate that decimal numbers are required. This is how it might look.

```
• 123/345  // Here we produce an exact result.
```

$$\frac{41}{115}$$

```
• 123.0/345 // Here we produce a decimal result.
```

0.3565217391304347826086956521739130434782608695652 2

```
• float(123/345) // Here is another way of getting
                 // decimal result.
```

0.35652173913043478260869565217391304347826086956522

Exercise 1.1 *Before we continue exploring MuPAD, it might be wise to practice a bit. Here are some exercises for you to try.*

1. *Calculate the decimal value of the fraction* 123/456 *with an accuracy of* 10, 100, *and* 1000 *places.*
2. *In MuPAD, the symbols* `PI`, `E` *denote the constants* $\pi$ *and* $e$ *respectively. Calculate* $\pi$ *and* $e$ *with an accuracy of* 150 *decimal digits.*

---

For MuPAD beginners, it is important to learn how formulas are typed in a MuPAD notebook. As you would expect, we can use the arithmetical operations +, –, $*$, / (division). Products, for example $2xy$, must be written as `2*x*y`. Powers, $3^2$, $5^3$, $a^n$ are written as `3^2`, `5^3`, `a^n`. Square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{a}$ are written as `sqrt(2)`, `sqrt(3)` and `sqrt(a)`. We can use most of the well-known mathematical functions such as $\sin x$, $\cos x$,

$\tan x$, $\cot x$. However, we must enclose the functional arguments in parenthesis, as in `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`. In order to write $e^x$, for example, we must write `exp(x)` or `E^x`.

Finally, it should be noted that several functions are used in infix form, that is, as operations between two arguments. Such functions are, for example, the two well-known integer operations mod and div. The operation mod produces the remainder of a division of one integer by another. The operation div, meanwhile, produces the result of one integer divided by another. Such operations we write in the traditional way, using parenthesis where necessary, as in `(34 mod 3)+(34 div 3)`. Note that using parenthesis may be important. For example, the two expressions `(34 mod 3)+(34 div 3)` and `34 mod 3 + 34 div 3` will produce different results. This suggests that in the second formula, `3+34` was produced first and then the remaining operations were carried out. This may not be what we intend.

Exercise 1.2 *In a MuPAD notebook, type statements to declare the following variables:*

1. $MyRoot = \sqrt{\sqrt{x} + 1}$
2. $MyFraction = \dfrac{2}{\sqrt{1 + x - \frac{1}{x}}}$
3. $Poly1 = 3x^3 + 2x^2 + x + 1$
4. $Fractions = b + \dfrac{1}{a + \frac{1}{x}} + \dfrac{1}{b + \frac{1}{a+\frac{1}{x}}} + \dfrac{1}{c + \dfrac{1}{b + \frac{1}{a+\frac{1}{x}}}}$
5. $ManyRad = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{x}}}}}$

## 1.3 Solving equations

Solving equations and systems of equations is one of the most important activities in mathematics. Let us examine how MuPAD solves equations.

```
• solve(x^4 - 4*x^2 - 1 = 0, x)
```

$$\left\{-\sqrt{\sqrt{5}+2}, \sqrt{2-\sqrt{5}}, -\sqrt{2-\sqrt{5}}, \sqrt{\sqrt{5}+1}\right\}$$

In this example, we obtained a very elegant result. However, it may happen that MuPAD also produces strange looking results such as the next one.

```
• solve(x^3 - 4*x^2 - 1 = 0, x)
```

$$RootOf(X1^3 - 4 \cdot X1^2 - 1, X1)$$

Such a result does not mean that MuPAD does not know how to solve the given equation. It often happens that the results obtained by MuPAD are too complex or too large to be displayed on a computer screen. In such cases, MuPAD uses built-in restrictions. If we do not care about exact results, we can try to obtain approximate values for our solutions. Here is how we would do this in the previous example.

To produce the decimal values of our roots, first we declare the set of roots of the equation and then use the `float` procedure to produce roots in decimal form.

```
• A := solve(x^3 - 4*x^2 - 1 = 0, x)
```

$$RootOf(X2^3 - 4 \cdot X2^2 - 1, X2)$$

```
• float(A)
```

```
{4.060647028, - 0.03032351378 + 0.4953247992*i,
-0.03032351378 - 0.4953247992*i}
```

**Technical comment** *For polynomial equations of order higher than* 2*, MuPAD often produces results as RootOf if their solutions contain radicals. This is because the resulting formula may be very complex and may not fit on the screen. However, we can force the program to display results in exact form. This can be done by using the solve statement, with parameters* `MaxDegree=3` *or* `MaxDegree=4`*. For example, a statement like* `solve(x^3-4*x^2-1=0,x,MaxDegree=3)` *can produce the desired formula.*

**Technical comment** *MuPAD statements use specific words such as* `solve`*,* `float`*, etc. We will call such words the names of procedures or, for short, procedures. Procedures are programs that have been* *written by MuPAD developers or sometimes by users of the program. Usually MuPAD users are not interested in the content of a procedure. Its name and how to use it is enough. In this book we will distinguish the names of procedures from statements using them. It is important to notice that usually there are many ways to formulate such statement. It also happens that a given procedure can be used in several contexts. For example, the procedure* `solve` *can be used to solve linear equations, quadratic equations, systems of equations and even differential equations.*

Solving systems of equations with MuPAD is as simple as solving single equations. We declare a set of equations and then specify the set of variables for which we wish to obtain the solutions. Here is such an example.

```
• solve({3*x+2*y-1 = 0, -6*x+4*y-6 = 0}, {x,y})
```

$$\left\{\left[x = -\frac{1}{3}, y = 1\right]\right\}$$

Here is another, slightly more complex, example.

```
• solve({x+2*y-1 = 0, -2*x+4*y^2-2 = 0}, {x,y})
```

$$\left\{\left[x = 2 - \sqrt{5}, y = \frac{\sqrt{5}}{2} - \frac{1}{2}\right], \left[x = 2 + \sqrt{5}, y = -\frac{\sqrt{5}}{2} - \frac{1}{2}\right]\right\}$$

In these examples we solved systems of two equations, considered as a set, with respect to the set of two variables $\{x,y\}$. Observe that exactly as in a high school textbook, we denoted the sets using curly brackets.

Often we simplify the code by declaring the equations and variables at the beginning, as in the next example.

```
• EQ1 := {x + 2*y - 1 = 0, -2*x + 4*y^2 - 2 = 0}
```

$$\{-2 \cdot x + 4 \cdot y^2 - 2 = 0, x + 2 \cdot y - 1 = 0\}$$

```
• Var1 := {x,y}
```

$$\{x,y\}$$

```
• solve(EQ1, Var1)
```

$$\left\{\left[x = 2 - \sqrt{5}, y = \frac{\sqrt{5}}{2} - \frac{1}{2}\right], \left[x = 2 + \sqrt{5}, y = -\frac{\sqrt{5}}{2} - \frac{1}{2}\right]\right\}$$

We will talk about sets later at the end of this chapter. Before then, let us do a few exercises.

Exercise 1.3 *Declare the given equations or sets of equations and solve them with MuPAD.*

1. $x^2 - 49 = 0$
2. $x^2 + x - 6 = 0$
3. $20x^2 - 31x - 9 = 0$
4. $x^2 - 3x - 1 = 0$
5. $x^2 - 6x + 1 = 0$
6. $x^2 + 7x + 9 = 0$
7. $x^2 + y^2 + 4xy - 1 = 0$, *solve in respect to the variable* $x$, *and then do the same for variable* $y$.
8. $x^3 - x^2 + x + 1 = 0$, *obtain the results in decimal form. Find out how to produce results in an exact form.*
9. $x - 2y = 7$ *and* $x^2 + 4y^2 = 37$
10. $x + y = 1$ *and* $16x^2 + y^2 = 65$
11. $y - x = a$ *and* $2x^2 + xy + y^2 = 8$

## 1.4 Transforming mathematical expressions

Transforming mathematical expressions is one of the most frequent activities starting in primary school, all the way up to advanced research. MuPAD contains a number of procedures that can help. Starting with version 3.0, we MuPAD provides powerful tools for simplifying mathematical expressions, in particular, trigonometric ones. Let us analyze a few simple examples.

We declare a polynomial and see what we can do with it.

```
• A := (x - 3)^2*(x - 1)^3*(x - 5)
```

$$(x-1)^3 \cdot (x-3)^2 \cdot (x-5)$$

Next let us expand the polynomial.

```
• B := expand(A)
```

$$x^6 - 14 \cdot x^5 + 75 \cdot x^4 - 196 \cdot x^3 + 263 \cdot x^2 - 174 \cdot x + 45$$

We can get back to the beginning by factorizing the result.

```
• factor(B)
```

$$(x - 5) \cdot (x - 3)^2 \cdot (x - 1)^3$$

Simplification of formulas is slightly trickier. This is because the word *simplify* may have a different meaning for all of us and depends heavily on the given context. Let us see how MuPAD simplifies mathematical formulas.

```
• simplify(sin(2*x)^2 + cos(2*x)^2)
```

1

```
• Simplify(sin(3*x) * cos(3*x)) //Note, capital S
```

$$\frac{\sin(6 \cdot x)}{2}$$

```
• simplify(exp(x)*exp(3*x)-exp(x)*exp(2*x))
```

$$(e^x)^3 \cdot (e^x - 1)$$

Technical comment *Starting from version 3.0, MuPAD contains two procedures:* `simplify` *and* `Simplify`. *Their names begin with a lowercase and an uppercase "s". The procedure* `simplify` *is sufficient for most elementary examples. The procedure* `Simplify` *is more "intelligent" and can handle more complex expressions. However, in many situations this procedure is slower.*

Procedures to simplify formulae may involve several parameters. For example, as in the case below, we can ask to simplify an expression with respect to a specific function. Here it is the function `sqrt`.

```
• simplify(sqrt(4 + 2*sqrt(3)), sqrt)
```

$$\sqrt{3} + 1$$

MuPAD provides several procedures for manipulating polynomials. Let us define a polynomial in two variables $P(x,y)$ and illustrate this point.

```
• P := x*y + a*x*y + x^2*y - a*x*y^2 + x + a*x
```

$$x + x^2 \cdot y + a \cdot x + x \cdot y - a \cdot x \cdot y^2 + a \cdot x \cdot y$$

```
• collect(P, x) // collecting in respect to x
```

$$y \cdot x^2 + (a + y - a \cdot y^2 + a \cdot y + 1) \cdot x$$

```
• collect(P, y) // collecting in respect to y
```

$$-a \cdot x \cdot y^2 + (x + a \cdot x + x^2) \cdot y + x + a \cdot x$$

```
• collect(P,[x,y]) // collecting for x and y
```

$$x^2 \cdot y - a \cdot x \cdot y^2 + (a + 1) \cdot x \cdot y + (a + 1) \cdot x$$

In calculus, we often decompose rational expressions into fractions, usually called partial fractions, where the denominator contains a single irreducible polynomial of the first or second degree and the numerator is a polynomial of the degree zero or one. In MuPAD this can be done using the procedure `partfrac`.

```
• w := x^2/(x^3 - 3*x + 2)
```

$$\frac{x^2}{x^3 - 3 \cdot x + 2}$$

```
• partfrac(w)
```

$$\frac{5}{9 \cdot (x - 1)} + \frac{1}{3 \cdot (x - 1)^2} + \frac{4}{9 \cdot (x + 2)}$$

Exercise 1.4 *Use MuPAD to perform the required transformations.*

1. *Expand the expressions:*
   a. $(2x + 3)^2$,
   b. $(4x^2 - 1)^2$,
   c. $(x - 1)^2(x - 3)^4$,
   d. $(x - 2)(x - 3)^5(x - 4)^2$,
   e. $(1 + x)^n$, *where* $n = 2, 3, 4, 5$.
2. *Expand these trigonometric expressions:*
   a. $(\sin x + \cos x)^3$,
   b. $\left(\frac{1 + \sin x}{\cos x}\right)^5$,
   c. $\tan(x + 1)$,
   d. $\cot(x^2 + 1)$

3. *Factorize the following expressions:*

   a. $2x^2 - 3x^3$,

   b. $4x^4 - 5x^2 + 1$,

   c. $9x^2 + 21x + 10$,

   d. $6x^2 - 4x - 4x^3 + x^4 + 1$,

   e. $\frac{1}{y+1}\left(\frac{1}{2}x + \frac{1}{2}\right) + \frac{1}{1-y}\left(\frac{1}{2}x + \frac{1}{2}\right)$

4. *Simplify the following expressions:*

   a. $\cos x \sin y + \cos y \sin x$

   b. $5x + 10x^2 + 10x^3 + 5x^4 + x^5 + 1$

   c. $\frac{1}{y+1}\left(\frac{1}{2}x + \frac{1}{2}\right) + \frac{1}{1-y}\left(\frac{1}{2}x + \frac{1}{2}\right)$

   d. $\dfrac{(1+3x)^5}{(1+3x)(3+9x)}$

   e. $\dfrac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}}$,

5. *Decompose the following expressions into partial fractions:*

   a. $\dfrac{x+7}{x^2 - x - 6}$,

   b. $\dfrac{50x^2 + 20x + 6}{x^3 + 2x^2 + x}$.

---

## 1.5 Differentiation and integration

Differentiation and integration are the basic operations of calculus. Of course calculus doesn't end there. It includes a wide range of interesting topics some of which we will come to it in a chapter later on. In this chapter, we concentrate on the basics of differentiation and integration.

Let us begin by defining an expression in a single variable $x$.

```
• w := x^2/(x^3 - x + 1)
```

$$\frac{x^2}{x^3 - x + 1}$$

We can calculate its first derivative.

```
• diff(w,x)
```

$$\frac{2x}{x^3-x+1} - \frac{x^2(3x^2-1)}{\left(x^3-x+1\right)^2}$$

In the same way, we can obtain the second, third, and higher derivatives.

```
• diff(w,x,x)
```

$$\frac{2}{x^3-x+1} - \frac{6x^3}{(x^3-x+1)^2} + \frac{2x^2(3x^2-1)^2}{(x^3-x+1)^3} - \frac{4x(3x^2-1)}{(x^3-x+1)^2}$$

Here is another example, where we calculate the derivatives of a trigonometric function.

```
• h := sin(x)/cos(x)
```

$$\frac{\sin(x)}{\cos(x)}$$

```
• diff(h,x,x,x)
```

$$\frac{8\sin(x)^2}{\cos(x)^2} + \frac{6\sin(x)^4}{\cos(x)^4} + 2$$

In exactly the same way we can obtain the partial derivatives of an expression in two or more variables.

```
• u := (sin(x) + cos(y))*(cos(x) - sin(y))
```

$$(\sin(x) + \cos(y)) \cdot (\cos(x) - \sin(y))$$

We can produce the second partial derivative in respect to $x$ and then $y$.

```
• diff(u,x,y)
```

$$\sin(x) \cdot \sin(y) - \cos(x) \cdot \cos(y)$$

MuPAD calculates definite as well as indefinite integrals.

```
• int(x^2 + 3*x + 1, x)
```

$$\frac{x^3}{3} + \frac{3x^2}{2} + x$$

```
• int(x^2 + 3*x + 1, x=0..10)
```

$$\frac{1480}{3}$$

Exercise 1.5 *Obtain the derivaties and integrals of the given expressions.*

1. $y = x^3 + x^2 + x + 1$
2. $y = \sin x + \cos x$
3. $y = \tan x + \cot x$
4. $y = e^{1+x+x^2}$
5. $y = \frac{1+x}{1+x+x^2}$

---

Until now, we have performed most of our operations on expressions with or without variables. However, mathematicians usually prefer to deal with functions. In the next section we will learn how to declare basic functions in MuPAD, and what the difference is between an expression in a variable and a function.

## 1.6 Declaring functions

The declaration of a function in MuPAD uses a notation similar to that encountered in school mathematics. For example, $f : x \to 1/(1 + x^3)$, where $f$ is the name of the function, $x$ is its argument, and the expression on the right side is a rule showing how to calculate values of the function $f$. In MuPAD, we have almost the same syntax.

```
• f := x->1/(1 + x^3)
```

$$x \longrightarrow \frac{1}{1+x^3}$$

Having declared the function $f$, we can use it in subsequent calculations, such as

```
• diff(f(x),x)
```

$$-\frac{3 \cdot x^2}{(x^3+1)^2}$$

or

```
• int(f(x),x)
```

$$\frac{\ln(x+1)}{3} - \frac{\ln\left((x-\frac{1}{2})^2+\frac{3}{4}\right)}{6} + \frac{\sqrt{3}\left(2\arctan\left(\frac{2\sqrt{3}\left(x-\frac{1}{2}\right)}{3}\right)-\pi\right)}{6}$$

We can obtain the definite integral of this function on the finite interval $0 \le x \le 5$ by executing the statement

```
• int(f(x), x = 0..5)
```

$$\frac{\ln(6)}{3} - \frac{\ln(21)}{6} + \frac{\pi\sqrt{3}}{18} + \frac{\arctan(3\sqrt{3})\sqrt{3}}{3}$$

and produce the integral of $f$ over the infinite interval $(x, \infty)$ by executing the statement

```
• int(f(x), x = 0..+infinity)
```

$$\frac{2 \cdot \pi \cdot \sqrt{3}}{9}$$

Another important operation on functions is to find some of their limits. Let us check the limits of the function $f$ above for the points $x = 0$ and $x = -1$.

```
• f := x->1/(1+x^3)
```

$$x \rightarrow \frac{1}{1+x^3}$$

```
• limit(f(x), x=0)
```

1

```
• limit(f(x),x = -1)
```

undefined

Although the limit of $f$ at $x = -1$ is undefined, we can check the one-sided limits from the left and right at this point.

```
• limit(f(x), x = -1, Left)
```

$-\infty$

```
• limit(f(x), x = -1, Right)
```

$\infty$

MuPAD also has tools for analyzing sequences. We treat sequences as functions with integer arguments. For example, the limit of the sequence $e_n = (1 + 1/n)^n$ for $n \rightarrow \infty$ can be produced as follows.

```
• en := n->(1+1/n)^n
```

$$n \to \left(1+\frac{1}{n}\right)^n$$

```
• limit(en(n), n = +infinity)
```

e

After having declared a function, we can use MuPAD to calculate the values of the function. Here is an example.

Let $f$ be the function used in one of our earlier examples.

```
• f := x->1/(1+x^3)
```

$$\frac{1}{1+x^3}$$

To calculate the value $f(1.234)$, we need to execute this simple statement.

```
• f(1.234)
```

0.3473330668

At this stage it is important to understand the difference between a variable, say F, declared as a formula and F declared as a function. Let us consider the example of two very similar declarations `F1:=x^3+x^2+x` and `F2:=x->x^3+x^2+x`. In the first case, `F1` is just a label for the formula `x^3+x^2+x`. We can put it anywhere where we would put the formula `x^3+x^2+x`, for example in the statement `solve(F1-1=0,x)`. The symbol `F2`, on the other hand, denotes a function and we should always use it with an argument, for example `solve(F2(x)-1=0,x)` or `solve(F2(y)-1=0,y)`. Concluding, we cannot use `F1(x)` but must use `F1` only, while at the same time we cannot use `F2` only, we have to use it with an argument, `F2(x)`.

Exercise 1.6 *Declare the following functions in MuPAD and calculate their values for* $x = 1$ *and* $x = 0$.

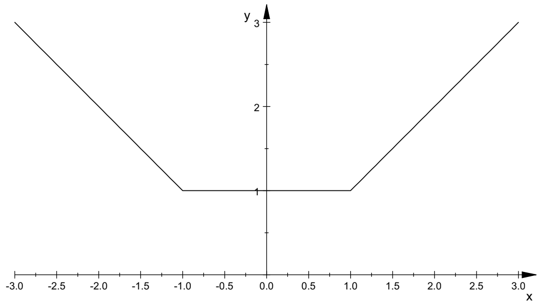
1. $f(x) = \sqrt{1+\sqrt{x}}$
2. $g(x) = \dfrac{1}{\sqrt{1+2x+3x^2}}$
3. $h(x) = x^3 + 2x^2 + 3x + 4$

4. $Fr(x) = \left(1 + \frac{1}{1 + \frac{1}{x}}\right) \cdot \frac{1}{1 + \frac{1}{1+\frac{1}{x}}}$

5. $Rts(x) = \sqrt{x + \sqrt{2x + \sqrt{3x + \sqrt{4x + \sqrt{x}}}}}$

---

From time to time it happens that we must deal with functions build from pieces of two or more functions. For example, the function presented in the picture below can be treated as a function built from three pieces, with each piece determined by a different formula.




Such functions can be declared as follows:

```
• f := x -> piecewise([x<-1,-x],[x<1,1],[x>1,x])
```

Declaring piecewise functions requires some thought on how to declare each piece and how the pieces connect. The order of the pieces in such declarations is crucial. We will discuss piecewise functions in detail in the chapter about calculus. MuPAD can be used to plot the graphs of declared functions. Here is an example.

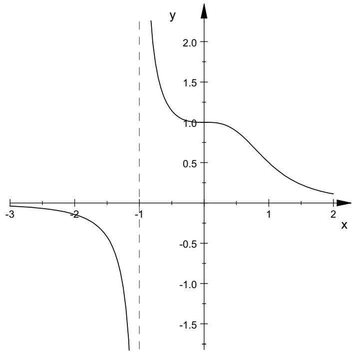
Let *f* be the function used in one of our earlier examples.

```
• f := x->1/(1+x^3)
```

$$\frac{1}{1 + x^3}$$

Producing the most basic plot of this function involves a simple statement.

```
• plotfunc2d(f(x), x=-3..2)
```

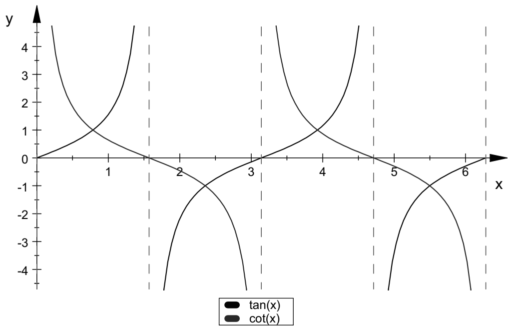



The consideration of this example has lead us to MuPAD graphics. In the next chapter we will discuss graphics in MuPAD in detail. Here we only examine two basic examples.

## 1.7 Graphs of functions in one variable

As we noticed in the previous section, obtaining the graph of a function can be quite easy and even with a simple tool like `plotfunc2d` we can produce sophisticated graphs. Here is another interesting example. We will produce graphs of two functions in the same coordinate system.

```
• plotfunc2d(tan(x), cot(x), x = 0..2*PI):
```




Exercise 1.7 *Produce graphs for the given expressions or groups of expressions.*

1. $\sin x,\ \sin 2x,\ \sin 3x,\ \sin 4x$
2. $\sin x,\ \sin x + 1,\ \sin x + 2,\ \sin x + 3$

3. $\sin x,\ \sin(x+1),\ \sin(x+2),\ \sin(x+3)$

4. $\sin 3x + \cos 2x$

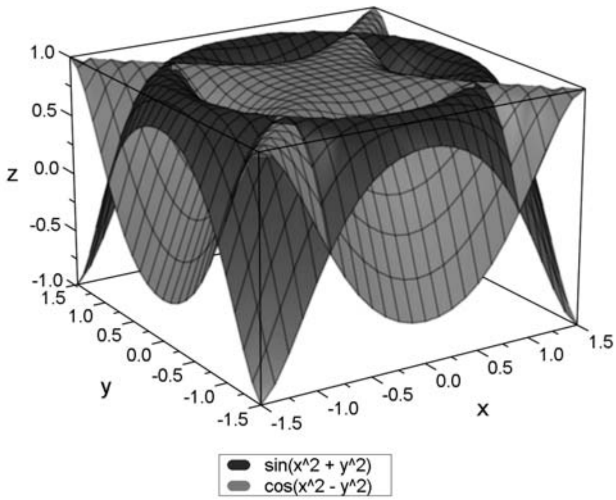
5. $\dfrac{e^x + e^{-x}}{2}$

---

## 1.8 Graphs of functions of two variables

In almost the analogous way, we can produce graphs of functions in two variables. In the next example, we produce a graph of the two expressions $\sin(x^2 + y^2)$ and $\cos(x^2 - y^2)$.

```
• plotfunc3d(
      sin(x^2 + y^2), cos(x^2 - y^2),
      x=-1.5..1.5, y=-1.5..1.5
  ):
```




Technical comment *As we already know, pressing the* [Enter] *key inside an input region forces MuPAD to execute a statement. If for some reason we want to split our statement into two or more lines, we can create line breaks by pressing the keys* [Shift] *and* [Enter] *simultaneously.*

Technical comment *Any two consecutive statements in the same input region must always be separated by "*;*" or "*:*" symbols.*

Exercise 1.8 *Produce graphs for the given expressions or groups of expressions.*

1. $1,\ \sin x,\ \sin 2x,\ \sin 3x,\ \sin 4x$
2. $\sin xy,\ \sin xy+1,\ \sin xy+2,\ \sin xy+3$
3. $\sin xy,\ \sin(xy+1),\ \sin(xy+2),\ \sin(xy+3)$
4. $xy,\ x(y+1),\ (x+1)y,\ (x+1)(y+1)$
5. $x^3+y^2+x+y$
6. $\sin(x+y)$

## 1.9 Selected mathematical structures in MuPAD

In mathematics, we frequently use mathematical objects that have some internal structure, for example sequences, vectors, matrices, or sets. In order to distinguish such objects from individual numbers or symbols, we call them mathematical structures. A sequence is an ordered collection of elements. In mathematics, we use two methods to define sequences: by listing their elements or by using a formula such as $a_n = n/(n+1)$. In the latter case, a sequence is treated as a function of one variable with integer or natural arguments.

Here is an example of a squence A declared by listing its elements.

```
• A := 1, -2, 3, -4, 5, -5, 6, -8, 8, -3, 4
  1, -2, 3, -4, 5, -5, 6, -8, 8, -3, 4
```

A sequence declared this way differs significantly from a sequence declared as a function. In the first case, we deal with a collection of almost physical objects, whereas in the second case, we do not have any objects as such, but rather a recipe showing how to calculate the elements of the sequence.

Sets are another type mathematical structure in MuPAD. We declare them by listing their elements and surrounding them with curly brackets.

```
• B := {1, -2, 3, -4, 5, -5, 6, -8, 8, -3, 4}:
• C := {-1, 2, 7, 4, -5, 5, 9, -8, 10, 3, -4}:
```

Once we have declared two sets, we can perform set-theoretical