SOLUTIONS MANUAL

Logic and Discrete Mathematics

A Concise Introduction

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Logic and Discrete Mathematics

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A Concise Introduction, Solutions Manual

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Preface

This manual contains answers and solutions to roughly three quarters of the exercises in *Logic and Discrete Mathematics: A Concise Introduction* by Willem Conradie and Valentin Goranko. Most solutions are worked out in full detail. In deciding which solutions to include we were guided by two principles: fundamental exercises were given preference above the more esoteric ones intended mainly for enrichment; where a number of very similar exercises occur in succession, complete solutions were given for a few while the others were omitted or provided with answers only. We trust that these solutions will be a very valuable resource to students and instructors using *Logic and Discrete Mathematics*.

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/conradie/logic

The website includes:

- Lecture Slides
- Quizzes

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Preliminaries

1.1. Sets

(1a) $A = \{-1, 4\}$

(1b) $B = \{-5, -3, -\frac{3}{2}, 1\}$

(1c) $C = \{-3, -2, -1, 0, 1, 2\}$

(1d) $D = \{-1\}$

(1e) $E = \{-3, -2, 0, 1, 2\}$

(1f) $F = \{1, 3, 5, 7, 9\}$

(2a) $B = \{x \in \mathbb{R} \mid x^2 - 3 = 0\}$

(2b) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z} \text{ and } 1 \leq k \leq 4\}$

(2c) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z}^+\}$

(2d) $A = \{x \mid x = 3k + 2, \text{ where } k \in \mathbb{Z}\}$

(2e) $A = \{x \mid x = n^3, \text{ where } n \in \mathbb{N}\}$

(2f) $B = \{x \in \mathbb{Z} \mid -3 \leq x \leq 6\}$

(2g) $B = \{x \in \mathbb{Z} \mid x \leq -3 \vee x \geq 5\}$

(3) Only a, d e, g, h and j are true.

(4) Only a, g, j and l are true.

(5a) $A \cap B = \{2, 7\}$

(5b) $A \cup B = \{0, 1, 2, 4, 5, 6, 7, 9\}$

(5c) $A - B = \{1, 4\}$

(5d) $B - A = \{0, 5, 6, 9\}$

(5e) $A - (A - B) = \{2, 7\}$

(5f) $B - (A - B) = B$

(5g) $B' = \{1, 3, 4, 8\}$

(5h) $(A \cap B)' = \{0, 1, 3, 4, 5, 6, 8, 9\}$

(5i) $A \cap B' = \{1, 4\}$

(5j) $(A \cup B)' = \{3, 8\}$

(5k) $(A \cup B)' - (B - A)' = \varnothing$

(6a) $A \cap B = \{0\}$

(6b) $A \cup B = \{0, A\}$

(6c) $B - A = \{A\}$

(6d) $A \cap (B \cup C) = \{0\}$

(6e) $(B \cup C) - A = \{A, \{0\}, \{A\}\}$

(6f) $(A \cap B) \cup (A \cap C) = \{0\}$

(6g) $(C - B) - A = \{\{0\}, \{A\}\}$

(6h) $A \cap (C - A) = \varnothing$

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(7) $A \times B = \{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(4,0),(4,1)\}$
$B \times A = \{(0,1),(1,1),(0,2),(1,2),(0,3),(1,3),(0,4),(1,4)\}$

(8a) 30 (8c) 5 (8e) 42

(8b) 0 (8d) 0 (8f) 1

(9) Only a and e are graphs of functions.

(10) The right-to-left direction is obvious. For the converse direction, assume $(a_1, a_2) = (b_1, b_2)$. Then $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$. Hence,

$$\bigcap\{\{a_1\}, \{a_1, a_2\}\} = \bigcap\{\{b_1\}, \{b_1, b_2\}\},$$

and so $\{a_1\} = \{b_1\}$. This means that $a_1 = b_1$. Similarly,

$$\bigcup\{\{a_1\}, \{a_1, a_2\}\} = \bigcup\{\{b_1\}, \{b_1, b_2\}\},$$

which gives $\{a_1, a_2\} = \{b_1, b_2\}$. Thus, since $a_1 = b_1$, $a_2 = b_2$.

(11) Given the objects $a_1, a_2, \ldots, a_n$, the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is defined by

$$(a_1, (a_2, \ldots, (a_{n-1}, a_n))).$$

1.2. Basics of logical connectives and expressions

(1) a, b, e, f, and h are propositions.

(2a) $A \wedge (\neg B \vee C) = \mathrm{T} \wedge (\neg \mathrm{T} \vee \mathrm{F}) = \mathrm{T} \wedge (\mathrm{F} \vee \mathrm{F}) = \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$

(2e) $C \to (A \to D) = \mathrm{F} \to (\mathrm{T} \to \mathrm{F}) = \mathrm{F} \to \mathrm{F} = \mathrm{T}$

(3a) Denote "The sun is hot." by p, "The earth is larger than Jupiter." by q, and "There is life on Jupiter." by r. Then the composite proposition in symbolic form is $p \wedge (q \to r)$. It is true.

(3b) Denote "The sun rotates around the earth." by p, "The earth rotates around the moon." by q and "The sun rotates around the moon." by r. Then the composite proposition in symbolic form is $p \vee q \to r$. It is true.

(3c) Denote "The moon rotates around the earth." by p, "The sun rotates around the earth." by q and "The earth rotates around the moon." by r. Then the composite proposition in symbolic form is $\neg q \wedge \neg r \to \neg p$. It is false.

(3d) Denote "The earth rotates around itself." by p, "The sun rotates around the earth." by q and "The moon rotates around the earth." by r. Then the composite proposition in symbolic form is $p \to q \vee \neg r$. It is false.

(3e) Denote "The earth rotates around itself." by p, "The sun rotates around itself." by q and "The moon rotates around itself." by r. Then the composite proposition in symbolic form is $p \leftrightarrow (\neg q \vee \neg r)$. It is true.

(4a) Since B is true and $B \to A$ must be true, A must be true.

(4b) Since B is false and $A \to B$ must be true, A must be false.

(4c) B is false since $\neg B$ is true. Hence, since $A \vee B$ must be true, A is true.

(4d) $\neg C$ must be false, and since $\neg B \to \neg C$ is true, $\neg B$ must be false, i.e. B is true. Hence, since $B \to \neg A$ is true, $\neg A$ is true and so A is false.

(4e) Since $\neg C \wedge B$ is true, B and $\neg C$ are true. Hence, C is false. Since $\neg(A \vee C) \to C$ is true and C is false, $\neg(A \vee C)$ is false, i.e. $A \vee C$ is true. Therefore, since C is false, A must be true.

(5a) Not every number is different from 0. True in $\mathcal{R}$ and $\mathcal{N}$.

(5b) Every number is less than or equal to its cube. False in $\mathcal{R}$, take $x = -2$. True in $\mathcal{N}$.

(5c) Every number equal to its square is positive. False in $\mathcal{R}$ and $\mathcal{N}$, take $x = 0$.

(5d) There is a negative number equal to its square. False in both $\mathcal{R}$ and $\mathcal{N}$. The square of any number is always positive.

(5e) Any positive number is less than its square. False in both $\mathcal{R}$ and $\mathcal{N}$, take $x = 1$.

(5f) Every number is either zero or it is not equal to twice itself. True in $\mathcal{R}$ and $\mathcal{N}$.

(5g) For every pair of numbers x and y, one of them is less than the other. False in $\mathcal{R}$ and $\mathcal{N}$, take $x = y$.

(5h) Every number is greater than the square of some number. False in $\mathcal{R}$ and $\mathcal{N}$, as 0 is not greater than any square.

(5i) For every number x, there is a number y that is either positive or whose square is less than x. True in $\mathcal{R}$ and $\mathcal{N}$.

(5j) Every non-negative number is the square of a positive number. False in $\mathcal{R}$ and $\mathcal{N}$, as 0 is not equal to the square of any positive number.

(5k) For every number x, there is a number y such that if x is greater than y, then it is also greater than the square of y. True in $\mathcal{R}$ and $\mathcal{N}$. Given x, take $y = x$, which makes the antecedent false and hence the implication true.

(5l) For every number x, there is a number y such that, if it is different from x, then its square is less than x. True in $\mathcal{R}$ and $\mathcal{N}$. Given x, take $y = x$, which makes the antecedent false and hence the implication true.

(5m) There is number greater than all numbers. False in $\mathcal{R}$ and $\mathcal{N}$.

(5n) There is a number x such that adding any number to it again yields x. False in $\mathcal{R}$ and $\mathcal{N}$.

(5o) There is a number x that can be added to any number y to obtain y again. True in $\mathcal{R}$ and $\mathcal{N}$, take $x = 0$.

(5p) There is a number such that every number is either less than it or less than its additive inverse. False in $\mathcal{R}$. It is not a formula in the language $\mathcal{L}_{\mathcal{N}}$.

(5q) There is a number x that is greater than or not greater than any given number y. True in $\mathcal{R}$ and $\mathcal{N}$.

(5r) There is a number such that the square of every number greater than it is also greater than it. True in $\mathcal{R}$ and $\mathcal{N}$, take, for example, $x = 0$.

(5s) There is a number such that the square of every number less than it is also less than it. False in $\mathcal{R}$, while it is true in $\mathcal{N}$, take $x = 0$.

(5t) There exist two numbers whose sum is equal to their product. True in $\mathcal{R}$ and $\mathcal{N}$, take $x = y = 0$.

(5u) Between any two distinct numbers, there is another number. True in $\mathcal{R}$, while it is false in $\mathcal{N}$.

1.3. Mathematical induction

(1) For $n = 0$, $0^2 + 0 = 0$, which is clearly even. Our inductive hypothesis is that $k^2 + k$ is even. Now, for $n = k + 1$,

$$(k + 1)^2 + k + 1 = k^2 + 2k + 1 + k + 1 = k^2 + k + 2(k + 1).$$

However, we know that $k^2 + k$ is even from the inductive hypothesis and, furthermore, $2(k + 1)$ is also clearly even. Therefore, since the sum of two even numbers is even, $k^2 + k$ is even.

(2) For $n = 4$, $2^4 = 16$ and $4! = 24$, so, clearly, $2^4 < 4!$. Our inductive hypothesis is that $2^k < k!$ for some $k \geq 4$. Now, for $n = k + 1$,

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < 5 \cdot k! \leq (k + 1) \cdot k! = (k + 1)!$$

(3) First, the set $\{1\}$ has two subsets, namely $\{1\}$ and $\emptyset$, so, clearly, the power set of $\{1\}$ has 2^1 elements. Our inductive hypothesis is that the power set of $\{1, 2, 3, \ldots, k\}$ has 2^k elements for some $k \geq 1$. Now, let $A = \{1, 2, \ldots, k, k+1\}$. Choose an element $a \in A$ and set $A' = A - \{a\}$. Note that $\mathcal{P}(A) = \{X \subseteq A \mid a \notin X\} \cup \{X \subseteq A \mid a \in X\}$. It is clear that these sets are disjoint, so to find the number of elements in $\mathcal{P}(A)$, we need only find the number of the elements in each of these sets and add them together. First, clearly, $\mathcal{P}(A') = \{X \subseteq A \mid a \notin X\}$, and so, since A' has k elements, $\{X \subseteq A \mid a \notin X\}$ has 2^k elements by the inductive hypothesis. Next, note that $X = Y \cup \{a\}$ for all $X \in \{X \subseteq A \mid a \in X\}$, where $Y \in \mathcal{P}(A')$. Since there are k elements in the set A', there are 2^k such Y by the inductive hypothesis. Hence, the set $\{X \subseteq A \mid a \in X\}$ has 2^k elements. Thus, in total, $\mathcal{P}(A)$ has $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ elements.

2

Sets, Relations, Orders

2.1. Set inclusions and equalities

To keep the solutions as clear and concise as possible, we will make use of the logical notation introduced in Section 1.2. We will also be making use of the logical constant $\top$, which always takes the truth value "true", and the logical constant $\bot$, which always takes the truth value "false".

(1a) $x \in A \cap A$
$\iff x \in A \wedge x \in A$
$\iff x \in A$

(1b) $x \in A \cap B$
$\iff x \in A \wedge x \in B$
$\iff x \in B \wedge x \in A$
$\iff x \in B \cap A$

(1c) $x \in A \cap (B \cap C)$
$\iff x \in A \wedge x \in B \cap C$
$\iff x \in A \wedge (x \in B \wedge x \in C)$
$\iff (x \in A \wedge x \in B) \wedge x \in C$
$\iff x \in A \cap B \wedge x \in C$
$\iff x \in (A \cap B) \cap C$

(1d) Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, which means that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(1e) Assume $A \subseteq B$. First, the left-to-right inclusion follows from 1(d). For the other inclusion, let $x \in A$. However, since $A \subseteq B$, $x \in B$. Hence, $x \in A \cap B$. For the converse direction, assume $A \nsubseteq B$. Then there is some x such that $x \in A$ but $x \notin B$. Hence, $x \in A$ but $x \notin A \cap B$, which means that $A \neq A \cap B$.

(1f) $x \in A \cup A$
$\iff x \in A \vee x \in A$
$\iff x \in A$

(1g) $x \in A \cup B$
$\iff x \in A \vee x \in B$
$\iff x \in B \vee x \in A$
$\iff x \in B \cup A$

(1h) $x \in A \cup (B \cup C)$
$\iff x \in A \vee x \in B \cup C$
$\iff x \in A \vee (x \in B \vee x \in C)$
$\iff (x \in A \vee x \in B) \vee x \in C$
$\iff x \in A \cup B \vee x \in C$
$\iff x \in (A \cup B) \cup C$

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(1i) For the sake of a contradiction, assume $A \nsubseteq A \cup B$ or $B \nsubseteq A \cup B$. If $A \nsubseteq A \cup B$, then there is some x such that $x \in A$ but $x \notin A \cup B$; $x \notin A \cup B$ implies that $x \notin A$ and $x \notin B$, which contradicts the fact that $x \in A$. On the other hand, if $B \nsubseteq A \cup B$, then there is some x such that $x \in B$ but $x \notin A \cup B$. Hence, $x \notin A$ and $x \notin B$, which again is a contradiction.

(1j) Assume $A \subseteq B$ and let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$, we are done. On the other hand, if $x \in A$, then $x \in B$ since $A \subseteq B$. Hence, $A \cup B \subseteq B$. For the other inclusion, let $x \in B$; then, clearly, $x \in A \cup B$. For the converse, assume $A \cup B = B$ and let $x \in A$. We have to show that $x \in B$. However, $x \in A$ implies that $x \in A \cup B$. Hence, since $A \cup B = B$, $x \in B$.

(1k) Assume $A \subseteq X$ and $B \subseteq X$, and let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in X$ since $A \subseteq X$ and, similiarly, if $x \in B$, then $x \in X$ since $B \subseteq X$. Hence, in both cases, $x \in X$, which means that $A \cup B \subseteq X$.

(2a) For the sake of a contradiction, assume $\varnothing - A \neq \varnothing$. Then $\varnothing - A \nsubseteq \varnothing$ or $\varnothing \nsubseteq \varnothing - A$. However, since $\varnothing$ is a subset of any set, $\varnothing - A \nsubseteq \varnothing$. This means that there is some x such that $x \in \varnothing - A$ but $x \notin \varnothing$. Hence, $x \in \varnothing$ and $x \notin A$. This is a contradiction since $\varnothing$ has no elements.

(2b) For the right-to-left inclusion, let $x \in A$. However, $\varnothing$ has no elements, so $x \notin \varnothing$. Hence, $x \in A - \varnothing$. For the left-to-right inclusion, suppose $A - \varnothing \nsubseteq A$. Then there is some x such that $x \in A - \varnothing$ but $x \notin A$; $x \in A - \varnothing$ implies that $x \in A$ and $x \notin \varnothing$, which contradicts the fact that $x \notin A$.

(2c) For the sake of a contradiction, assume $A - B \nsubseteq A$. Then there is some x such that $x \in A - B$ and $x \notin A$. However, $x \in A - B$ implies that $x \in A$ and $x \notin B$, which contradicts the fact that $x \notin A$.

(2d) For the left-to-right direction, assume $A \nsubseteq B$. Then there is some x such that $x \in A$ but $x \notin B$. However, this means that $x \in A - B$. Hence, $A - B \neq \varnothing$. For the converse, assume $A - B \neq \varnothing$. Then there is some x in $A - B$. Hence, $x \in A$ but $x \notin B$, which means that $A \nsubseteq B$.

(2e) For the sake of a contradiction, assume $(A - B) \cap (B - A) \neq \varnothing$. Then there is some x such that $x \in (A - B) \cap (B - A)$. Hence, $x \in A - B$ and $x \in B - A$. This means that $x \in A$ while $x \notin B$ and $x \in B$ while $x \notin A$, which is a contradiction.

(2f) $x \in (A - B) \cup B$
$\iff x \in A - B \vee x \in B$
$\iff (x \in A \wedge \neg x \in B) \vee x \in B)$
$\iff (x \in A \vee x \in B) \wedge (\neg x \in B \vee x \in B)$
$\iff (x \in A \vee x \in B) \wedge \top$
$\iff x \in A \vee x \in B$
$\iff x \in A \cup B$

(2g) $x \in A - (A \cap B)$
$\iff x \in A \wedge \neg x \in A \cap B$
$\iff x \in A \wedge \neg(x \in A \wedge x \in B)$
$\iff x \in A \wedge (\neg x \in A \vee \neg x \in B)$
$\iff (x \in A \wedge \neg x \in A) \vee (x \in A \wedge \neg x \in B)$
$\iff \bot \vee x \in A - B$
$\iff x \in A - B$

(2h) $x \in A - (B \cap C)$
$\iff x \in A \wedge \neg x \in B \cap C$
$\iff x \in A \wedge \neg(x \in B \wedge x \in C)$
$\iff x \in A \wedge (\neg x \in B \vee \neg x \in C)$
$\iff (x \in A \wedge \neg x \in B) \vee (x \in A \wedge \neg x \in C)$
$\iff x \in A - B \vee x \in A - C$
$\iff x \in (A - B) \cup (A - C)$

(2i) $x \in (A - B) \cup (B - A) \cup (A \cap B)$
$\iff x \in A - B \vee x \in B - A \vee x \in A \cap B$
$\iff (x \in A \wedge \neg x \in B) \vee (x \in B \wedge \neg x \in A) \vee (x \in A \wedge x \in B)$
$\iff ((x \in A \wedge \neg x \in B) \vee (x \in A \wedge x \in B)) \vee (x \in B \wedge \neg x \in A)$
$\iff (x \in A \wedge (\neg x \in B \vee x \in B)) \vee (x \in B \wedge \neg x \in A)$
$\iff (x \in A \wedge \top) \vee (x \in B \wedge \neg x \in A)$
$\iff x \in A \vee (x \in B \wedge \neg x \in A)$
$\iff (x \in A \vee x \in B) \wedge (x \in A \vee \neg x \in A)$
$\iff (x \in A \vee x \in B) \wedge \top$
$\iff (x \in A \vee x \in B)$
$\iff x \in A \cup B$

(2j) $x \in A \cup (B \cap C)$
$\iff x \in A \vee x \in B \cap C$
$\iff x \in A \vee (x \in B \wedge x \in C)$
$\iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
$\iff x \in A \cup B \wedge x \in A \cup C$
$\iff x \in (A \cup B) \cap (A \cup C)$

(2k) $A \subseteq A \cup B$, so, by 1(d), $A \cap (A \cup B) = A$.

(2l) $A \cap B \subseteq A$, so, by 1(i), $A \cup (A \cap B) = A$.

(3a) Assume $A \subseteq B$ and let $X \in \mathcal{P}(A)$. Then $X \subseteq A$. However, $A \subseteq B$, so $X \subseteq B$, which means that $X \in \mathcal{P}(B)$.

(3b) Assume $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(X)$. Then $A \subseteq X$ and $B \subseteq X$. To show that $A \cap B \in \mathcal{P}(X)$, we have to show that $A \cap B \subseteq X$. To see this, let $x \in A \cap B$. Then $x \in A$ and $x \in B$. However, $A \subseteq X$ and $B \subseteq X$, so $x \in X$, which proves that $A \cap B \subseteq X$. The fact that $A \cup B \in \mathcal{P}(X)$ follows from 1(k). To show that $A - B \in \mathcal{P}(X)$, let $x \in A - B$. Then $x \in A$ and $x \notin B$. However, since $A \subseteq X$, $x \in X$, and so $A - B \subseteq X$, which proves that $A - B \in \mathcal{P}(X)$. Finally, by 2(c), $X - A \subseteq X$, so $X - A \in \mathcal{P}(X)$.

(4a) $(x, y) \in (B \cup C) \times A$
$\iff x \in B \cup C \wedge y \in A$
$\iff (x \in B \vee x \in C) \wedge y \in A$
$\iff (x \in B \wedge y \in A) \vee (x \in C \wedge y \in A)$
$\iff (x, y) \in B \times A \vee (x, y) \in C \times A$
$\iff (x, y) \in (B \times A) \cup (C \times A)$

(4b) $(x, y) \in A \times (B \cap C)$
$\iff x \in A \wedge y \in B \cap C$
$\iff x \in A \wedge (y \in B \wedge y \in C)$
$\iff (x \in A \wedge x \in A) \wedge (y \in B \wedge y \in C)$
$\iff x \in A \wedge (x \in A \wedge y \in B) \wedge y \in C$
$\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)$
$\iff (x, y) \in (A \times B) \wedge (x, y) \in (A \times C)$
$\iff (x, y) \in (A \times B) \cap (A \times C)$

(4c) $(x, y) \in (B \cap C) \times A$
$\iff x \in B \cap C \wedge y \in A$
$\iff (x \in B \wedge x \in C) \wedge y \in A$
$\iff (x \in B \wedge x \in C) \wedge (y \in A \wedge y \in A)$
$\iff x \in B \wedge (x \in C \wedge y \in A) \wedge y \in A$
$\iff (x \in B \wedge y \in A) \wedge (x \in C \wedge y \in A)$
$\iff (x, y) \in B \times A \wedge (x, y) \in C \times A$
$\iff (x, y) \in (B \times A) \cap (C \times A)$

(4d) $(x, y) \in (A \times B) - (A \times C)$
$\iff (x, y) \in (A \times B) \wedge \neg(x, y) \in A \times C$
$\iff (x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)$
$\iff (x \in A \wedge y \in B) \wedge (\neg x \in A \vee \neg y \in C)$
$\iff (x \in A \wedge y \in B \wedge \neg x \in A) \vee (x \in A \wedge y \in B \wedge \neg y \in C)$
$\iff \bot \vee (x \in A \wedge y \in B \wedge \neg y \in C)$
$\iff x \in A \wedge y \in B \wedge \neg y \in C$
$\iff x \in A \wedge y \in (B - C)$
$\iff (x, y) \in A \times (B - C)$

2.2. Functions

(1a) gf is the function from $\mathbb{Z}$ to $\mathbb{R}$ given by $x \mapsto x^3/27$, and, furthermore, $\text{dom}(gf) = \mathbb{Z}$ $\text{rng}(gf) = \left\{ \frac{m^3}{27} \mid m \in \mathbb{Z} \right\}$

(1b) gh is the function from $\mathbb{R}^2$ to $\mathbb{R}$ given by $(x, y) \mapsto (x \times y)^2$, and, furthermore, $\text{dom}(gh) = \mathbb{R}^2$ $\text{rng}(gh)$ is the set of non-negative real numbers.

(1c) No, fg is not defined.

(1d) No, h is binary, not unary.

(1e) No, g has range $\mathbb{R}$, while f has domain $\mathbb{Z}$.

(2a) $f_1(x) = x^2$

(2b) $f_2(x) = x^3 - x$

(2c) $f_3(x) = 2^x$

(2d) $f_4(x) = x + 2$

(3) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt{x}$ and $g(x) = x^2$.

(4) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}^+$, where $f(x) = \sqrt{x}$ and $g(x) = x^2$.

(5a) $\text{cod}(h) = \mathbb{R}$

(5b) $\text{rng}(f) = \mathbb{R}$
$\text{rng}(g) = \mathbb{R}^+$
$\text{rng}(h) = \mathbb{R}$

(5c) f and g are injective.

(5d) f and h are surjective.

(5e) f is the only bijective function.

(6a) The fixed points of sq are 0 and 1.

(6b) $f(x) = x + 1$

(6c) abs has infinitely many fixed points.

(7) Let $f\colon A \to B$ and assume f is bijective. Then f has an inverse f^{-1} which is also bijective by Proposition 4.2.11. Hence, f^{-1} has an inverse $(f^{-1})^{-1}$. Now, let $a \in A$. Then $(f^{-1})^{-1}(a) = (f^{-1})^{-1}(f^{-1}(f(a))) = f(a)$, where the second and last steps follow from Definition 4.2.10.

(8a) Let $f\colon A \to B$ and $g\colon B \to C$ be surjective mappings and let $c \in C$. To show that gf is surjective, we have to find some $a \in A$ such that $gf(a) = c$. However, since g is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since f is surjective, there is some $a \in A$ such that $f(a) = b$. Hence, $g(f(a)) = c$, as desired.

(8b) Let $f\colon A \to B$ and $g\colon B \to C$, and assume that gf is injective. Now, suppose $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Hence, by definition, $gf(a_1) = gf(a_2)$. However, gf is injective, so $a_1 = a_2$, and we are done.

(9) Let $h\colon A \to B$, $g\colon B \to C$ and $f\colon C \to D$. Then $f(gh)(a) = f(gh(a)) = f(g(h(a))) = fg(h(a)) = (fg)h(a)$.

(10) Let $f\colon A \to B$. To prove the left-to-right direction, assume f is surjective, and consider the mappings $g_1\colon B \to C$ and $g_2\colon B \to C$. Furthermore, suppose $g_1 f = g_2 f$. We have to show that $g_1 = g_2$. Now, let $b \in B$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. Hence,

$$g_1(b) = g_1(f(a)) = g_1 f(a) = g_2 f(a) = g_2(f(a)) = g_2(b).$$

For the converse, suppose f is not surjective. Then there is some $b \in B$ such that, for all $a \in A$, $f(a) \neq b$. We have to find two mappings g_1 and g_2 such that $g_1 f = g_2 f$ but $g_1 \neq g_2$. Let $g_1\colon B \to B$ and $g_2\colon B \to B$

such that $g_1(b') = b'$ for all $b' \in B$, and $g_2(b') = b'$ for all $b' \in B - \{b\}$ but $g_2(b) = f(a_0)$ for an arbitrary fixed element $a_0 \in A$. Then

$$g_1 f(a) = g_1(f(a)) = f(a) = g_2(f(a)) = g_2 f(a),$$

while $g_1(b) = b \neq f(a_0) = g_2(b)$, and we are done.

2.3. Binary relations and operations on them

(1) dom(ParentOf) is the set of all humans who have children, while rng(ParentOf) is the set of all humans.
dom(IsMArriedTo) and rng(IsMarriedTo) are the set of all married humans.
dom(OwnsDog) is the set of all humans owning a dog, while rng(OwnsDog) is the set of all dogs owned by some human.
dom(IsCitizenOf) is the set of all humans who are citizens of some country, while rng(IsCitizenOf) are the set all countries.
dom(IsMemberOf) is the set of all humans who are member of some club, while rng(IsMemberOf) is the set of all clubs that have at least one member.

(2a) $\{4, 6, 8, 10, \ldots\} \cup \{10, 15, 20, 25, \ldots\}$

(2b) $\{2, 3, 4, 5, 6\}$

(2c) $\{x \in \mathbb{Z} \mid x \geq 13\}$

(2d) $\{x \in \mathbb{Z} \mid x \leq 302\}$

(3a) The set of all Dachshunds owned by citizens of European countries.

(3b) The set of all Scotish Terriers owned by the children of members of the Rotary Club.

(3c) The set of all Scotish Terriers owned by parents of members of the Rotary Club.

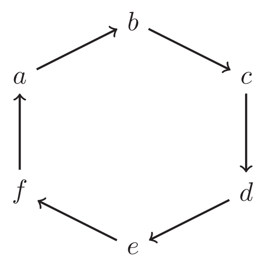
(3d) The set of all those humans who have a parent who is a citizen of a European country and a parent who is a citizen of an African country.

(3e) The set of parents whose children are citizens of both African and European countries.

(3f) The set of all sons of members of the Rotary Club.

(3g) The set of all mothers of members of the Rotary Club.

(4a) $R_1 \cap R_2$:



(4b) $R_1 \cup R_2$: