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Robert Tubbs

Hilbert's Seventh Problem

Solutions and Extensions

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Preface

The twenty-three problems David Hilbert posed at the Second International Congress of Mathematicians in 1900 proved to inspire many of the mathematical breakthroughs of the twentieth, and nascent twenty-first, centuries. Hilbert's seventh problem, whose solutions are the topic of this monograph, concerned the transcendence of numbers either of a particular form or that are given as special values of transcendental functions. In this short report we will look first at the results in transcendence theory that preceded Hilbert's lecture. This brief introduction to the earliest results in transcendence theory is followed by a dissection of Hilbert's statement of his seventh problem. We will then look at three partial solutions that were given some thirty years later. These partial solutions were soon followed by two solutions to the most commonly cited portion of the seventh problem. We will then look at some early progress on another aspect of Hilbert's problem and finally look at a particularly interesting late-twentieth century advance.

This monograph grew out of some notes I prepared for students attending my short course *Hilbert's Seventh Problem: Its solutions and extensions* at the Institute for Mathematical Sciences in Chennai, India in December 2010. It is written for students and faculty who want to explore the progression of mathematical ideas that led to the partial solutions then complete solutions to one portion of Hilbert's problem. I thank the gifted students who attended my lectures for their many comments, questions, and corrections. This text owes a great deal to them. Of course I am solely responsible for any errors that remain.

I thank Professor Sanoli Gun of the IMSc for inviting me to participate in the institute's Number Theory Year dedicated to the institute's talented director Professor R. Balasubramanian. I also thank Professor Gun and Professor Purusottam Rath of the Chennai Mathematical Institute for the hospitality they extended to me during my stay in Chennai. I also want to acknowledge the referee who made several thoughtful comments. Lastly I thank Professor Michel Waldschmidt of Paris for suggesting that I participate in the institute's stimulating program.

About the Author

Robert Tubbs is Associate Professor of Mathematics at the University of Colorado Boulder, United States. His research interest lies in number theory, especially transcendental number theory, the intellectual history of mathematical ideas and mathematics and the humanities.

Chapter 1

Hilbert's seventh problem: Its statement and origins

At the second International Congress of Mathematicians in Paris, in 1900, the mathematician David Hilbert was invited to deliver a keynote address, just as Henri Poincaré had been invited to do at the first International Congress of Mathematicians in Zurich in 1896.

According to a published version of Hilbert's lecture [14], which appeared soon thereafter, he began his lecture with a bit of a motivation for offering a list of problems to inspire mathematical research:

...the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future. The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as every human undertaking pursues certain objects, so also mathematical research requires its problems.

In his lecture Hilbert posed ten problems. In the published versions of his lecture Hilbert offered twenty-three problems (only eighteen could really be considered to be problems rather than areas for further research). The distribution of his published problems is, roughly: two in logic, three in geometry, seven in number theory, ten in analysis/geometry, and one in physics (and its foundations). To date sixteen of these problems have been either solved or given counterexamples.

What will concern us in these notes is the seventh problem on Hilbert's list, which concerns the arithmetic nature of certain numbers–in particular, Hilbert proposed that certain, specific, numbers are transcendental, i.e., not algebraic and so not the solution of any integral polynomial equation $P(X) = 0$.

R. Tubbs, *Hilbert's Seventh Problem*, HBA Lecture Notes in Mathematics, DOI 10.1007/978-981-10-2645-4_1

Some relevant early developments

By the time Hilbert spoke in Paris, transcendental number theory had already had a short yet fairly glorious history. The beginning of this history, so far as we can tell from what was written down, began with either L. Euler or G. Leibniz. These two mathematicians must have felt a certain exasperation when they sensed that some numbers were just beyond their grasps; these *transcendental* numbers were not numbers that one would ordinarily encounter through algebraic methods. Rather than trace this history here we highlight research that influenced the early development of the study of transcendental numbers and then the nineteenth-century developments that inspired Hilbert's seventh problem.

A natural place to begin is with Euler. In 1748 Euler published *Introductio in analysin infinitorum* (Introduction to Analysis of the Infinite) [4] in which he did several things that are relevant to the development of transcendental number theory and to these notes:

• Euler summed the series $1/1^k + 1/2^k + 1/3^k + \ldots$ for all even $k, 2 \leq k \leq 26$, which we represent by $\zeta(k)$. (Euler showed that $\zeta(2) = \frac{\pi^2}{6}$, [1].)

• Euler derived the formula

$$e^{ix} = \cos(x) + i\sin(x).$$

• From the above equality, Euler obtained the relationship

$$e^{i\pi} = -1.$$

• Finally, Euler found continued fraction expansions for e and e^2 (which were non-terminating thus showing that each of these numbers is irrational).

In his book Euler also made an interesting conjecture concerning the nature of certain logarithms of rational numbers. As Euler's conjecture remarkably foreshadows one part of Hilbert's seventh problem, it is worth stating. Euler wrote:

... the logarithms of [rational] numbers which are not the powers of the base are neither rational nor irrational ...

When Euler used the terminology "irrational" he meant what we would call "irrational and algebraic." Euler went on to say that *... it is with justice that [the above logarithms] are called transcendental quantities.*

Euler's conjecture has the simple, more modern, formulation:

Conjecture (Euler). *For any two positive rational numbers $r \neq 1$ and s, the number*

$$log_r s \quad \left(= \frac{\log s}{\log r}\right)$$

is either rational or transcendental.