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Hanspeter Schmidli

Risk Theory

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Springer Actuarial
Lecture Notes
ISBN 978-3-319-72004-3 ISBN 978-3-319-72005-0 (eBook)
https://doi.org/10.1007/978-3-319-72005-0

Library of Congress Control Number: 2017962068

Mathematics Subject Classification (2010): 91B30, 60F10, 60G42, 60G44, 60G51, 60G55, 60J10, 60J25, 60K05, 60K20, 91B16

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book aims to give an introduction to the methods used in non-life insurance. In addition to providing an overview of classical actuarial methods, the main part deals with ruin models, which is particularly interesting from a mathematical point of view. However, ruin theory also gives a deeper insight into and understanding how losses or an insolvency happens, if it happens at all, and what precautions may be taken to avoid an undesirable situation. Even though "ruin in infinite time" is not considered by the solvency rules, the theory gives an understanding of the risks taken.

I started writing this book back in 1994, when I gave a lecture on risk theory at Heriot–Watt University in Edinburgh. The lecture was based on notes by my colleagues, in particular by Howard Waters. The main part of the book has its origins in a two-semester course I gave at the University of Aarhus from 1994 to 2000. Parts of the notes to this course were also used in the book project [110]. In the last year, I added the chapter on claims reserving and the discussions on some aspects of solvency. I hope that all the theoretical background that an actuary may need can now be found in this book.

Many colleagues have directly or indirectly contributed to this book. I want to thank my Ph.D. supervisor Paul Embrechts, who started my interest in this topic; my former colleagues in Edinburgh, in particular Howard Waters, who supported me during my stay in Scotland's capital; Jan Grandell and Søren Asmussen, whose experience has increased my research skills; Hansjörg Albrecher, who drew my attention to several interesting references; and Mario Wüthrich, whose fruitful discussions with me on claims reserving has improved the corresponding chapter. Last but not least, I thank my wife, who has always supported my scientific career, even though it was often quite hard for her, in particular, when the children were small.

Cologne, Germany — Hanspeter Schmidli
September 2017

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Principal Notation

$\mathbb{I}_A$	Indicator function
$\mathbb{E}$	Expected value
$\mathbb{N}$	The natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{P}$	The basic probability measure
$\mathbb{R}$	The real numbers
$\mathbb{R}_+$	The positive real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{Z}$	The integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathcal{F}$	σ-algebra of the probability space
$\{\mathcal{F}_t\}$	Filtration
$\{\mathcal{F}_t^X\}$	Natural filtration of the process X
$\mathcal{L}^\infty$	Space of bounded random variables
$\{C_t\}$	Surplus process
c	Premium rate
F	Distribution function or interclaim arrival distribution
G	Claim size distribution
$M_Y(r)$	Moment-generating function of the random variable Y
$N(= \{N_t\})$	Claim number or claim number process
R	Lundberg coefficient
T_i	ith occurrence time
$x^+ = x \vee 0$	Positive part of x
$x^- = (-x) \vee 0$	Negative part of x
$\lfloor x \rfloor$	Integer part of x, $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$
Y_k	Claim sizes
Ω	Event space on which probabilities are defined
$\delta(x)$	Survival probability
Φ	Standard normal distribution function
λ	Claim arrival intensity
μ, μ_n	nth moment of the claim sizes

$\psi(x)$	Ruin probability
τ	Time of ruin
$\boldsymbol{\Theta}_i$	Risk characteristics
$\theta(r)$	Cumulant-generating function of the surplus

Chapter 1
Risk Models

In this chapter we will consider a risk in a single time period. We will see how to approximate the distribution of a compound sum, how to calculate premia, and we will introduce risk measures.

1.1 Introduction

Let us consider a (collective) insurance contract in some fixed time period $(0, T]$, for instance $T = 1$ year. Let N denote the number of claims in $(0, T]$ and $Y_1, Y_2, \dots, Y_N$ the corresponding claims. Then

$$S = \sum_{i=1}^{N} Y_i$$

is the accumulated sum of claims. We assume

(i) N and $\{Y_1, Y_2, \dots\}$ are independent.
(ii) $Y_1, Y_2, \dots$ are independent.
(iii) $Y_1, Y_2, \dots$ have the same distribution function, G say.

We further assume that $G(0) = 0$, i.e. the claim amounts are positive. Let $M_Y(r) = \mathbb{E}[e^{rY_i}]$, $\mu_n = \mathbb{E}[Y_1^n]$ if the expressions exist and $\mu = \mu_1$. The distribution of S can be written as

$$\begin{aligned}\mathbb{P}[S \le x] &= \mathbb{E}[\mathbb{P}[S \le x \mid N]] = \sum_{n=0}^{\infty} \mathbb{P}[S \le x \mid N = n]\mathbb{P}[N = n] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[N = n] G^{*n}(x) \,.\end{aligned}$$

H. Schmidli, *Risk Theory*, Springer Actuarial,
DOI:10.1007/978-3-319-72005-0_1

In general, this is not easy to compute, but it is often enough to know only some characteristics of a distribution.

$$\mathbb{E}[S] = \mathbb{E}\Big[\sum_{i=1}^{N} Y_i\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N} Y_i \;\Big|\; N\Big]\Big] = \mathbb{E}\Big[\sum_{i=1}^{N} \mu\Big] = \mathbb{E}[N\mu] = \mathbb{E}[N]\mu$$

is called the first Wald formula, and

$$\begin{aligned}\mathbb{E}[S^2] &= \mathbb{E}\Big[\mathbb{E}\Big[\Big(\sum_{i=1}^{N} Y_i\Big)^2 \;\Big|\; N\Big]\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N} Y_i Y_j \;\Big|\; N\Big]\Big] \\ &= \mathbb{E}[N\mu_2 + N(N-1)\mu^2] = \mathbb{E}[N^2]\mu^2 + \mathbb{E}[N](\mu_2 - \mu^2)\end{aligned}$$

gives the second Wald formula

$$\mathrm{Var}[S] = \mathrm{Var}[N]\mu^2 + \mathbb{E}[N]\mathrm{Var}[Y_1]\,.$$

The moment generating function of S becomes

$$\begin{aligned}M_S(r) &= \mathbb{E}[\mathrm{e}^{rS}] = \mathbb{E}\Big[\exp\Big\{r\sum_{i=1}^{N} Y_i\Big\}\Big] = \mathbb{E}\Big[\prod_{i=1}^{N} \mathrm{e}^{rY_i}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\prod_{i=1}^{N} \mathrm{e}^{rY_i} \;\Big|\; N\Big]\Big] \\ &= \mathbb{E}\Big[\prod_{i=1}^{N} M_Y(r)\Big] = \mathbb{E}\big[(M_Y(r))^N\big] = \mathbb{E}\Big[\mathrm{e}^{N\log(M_Y(r))}\Big] \\ &= M_N(\log(M_Y(r)))\,,\end{aligned}$$

where $M_N(r)$ is the moment generating function of N. The coefficient of skewness $\mathbb{E}[(S-\mathbb{E}[S])^3]/(\mathrm{Var}[S])^{3/2}$ can be calculated from the moment generating function by using the formula

$$\mathbb{E}[(S-\mathbb{E}[S])^3] = \frac{\mathrm{d}^3}{\mathrm{d}r^3}\log(M_S(r))\,\Big|_{r=0}\,.$$

1.2 The Compound Binomial Model

We model $N \sim \mathrm{B}(n, p)$ for some $n \in \mathbb{N}$ and $p \in (0, 1)$. We obtain

$$\mathbb{E}[S] = np\mu\,,$$

$$\mathrm{Var}[S] = np(1-p)\mu^2 + np(\mu_2 - \mu^2) = np(\mu_2 - p\mu^2)$$

and

$$M_S(r) = (pM_Y(r) + 1 - p)^n .$$

Let us consider another characteristic of the distribution of S, the skewness. We need to compute $\mathbb{E}[(S - \mathbb{E}[S])^3]$.

$$\begin{aligned}\frac{\mathrm{d}^3}{\mathrm{d}r^3} n \log(pM_Y(r) + 1 - p) &= n\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{pM_Y'(r)}{pM_Y(r) + 1 - p}\right)\\ &= n\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{pM_Y''(r)}{pM_Y(r) + 1 - p} - \frac{p^2 M_Y'(r)^2}{(pM_Y(r) + 1 - p)^2}\right)\\ &= n\left(\frac{pM_Y'''(r)}{pM_Y(r) + 1 - p} - \frac{3p^2 M_Y''(r)M_Y'(r)}{(pM_Y(r) + 1 - p)^2} + \frac{2p^3(M_Y'(r))^3}{(pM_Y(r) + 1 - p)^3}\right) .\end{aligned}$$

For $r = 0$ we get

$$\mathbb{E}[(S - \mathbb{E}[S])^3] = n(p\mu_3 - 3p^2\mu_2\mu + 2p^3\mu^3) ,$$

from which the coefficient of skewness can be calculated.

Example 1.1 Assume that the claim amounts are deterministic, y_0 say. Then

$$\mathbb{E}[(S - \mathbb{E}[S])^3] = ny_0^3(p - 3p^2 + 2p^3) = 2ny_0^3 p(\frac{1}{2} - p)(1 - p) .$$

Thus

$$\mathbb{E}[(S - \mathbb{E}[S])^3] \gtreqless 0 \iff p \lesseqgtr \frac{1}{2} .$$

■

1.3 The Compound Poisson Model

In addition to the compound binomial model we assume that n is large and p is small. Let $\lambda = np$. Because

$$B(n, \lambda/n) \longrightarrow \text{Pois}(\lambda) \qquad \text{as } n \to \infty$$

it is natural to model

$$N \sim \text{Pois}(\lambda) .$$

We get

$$\mathbb{E}[S] = \lambda\mu ,$$

$$\text{Var}[S] = \lambda\mu^2 + \lambda(\mu_2 - \mu^2) = \lambda\mu_2$$

and

$$M_S(r) = \exp\{\lambda(M_Y(r) - 1)\}\,.$$

Let us also compute the coefficient of skewness.

$$\frac{\mathrm{d}^3}{\mathrm{d}r^3}\log(M_S(r)) = \frac{\mathrm{d}^3}{\mathrm{d}r^3}(\lambda(M_Y(r) - 1)) = \lambda M_Y'''(r)$$

and thus

$$\mathbb{E}[(S - \mathbb{E}[S])^3] = \lambda\mu_3\,.$$

The coefficient of skewness is

$$\frac{\mathbb{E}[(S - \mathbb{E}[S])^3]}{(\mathrm{Var}[S])^{3/2}} = \frac{\mu_3}{\sqrt{\lambda\mu_2^3}} > 0\,.$$

Problem: The distribution of S is always positively skewed.

Example 1.2 An insurance company models claims from fire insurance as $\mathrm{LN}(m,\, \sigma^2)$. Let us first compute the moments

$$\mu_n = \mathbb{E}[Y_1^n] = \mathbb{E}[\mathrm{e}^{n\log Y_1}] = M_{\log Y_1}(n) = \exp\{\sigma^2 n^2/2 + nm\}\,.$$

Thus

$$\mathbb{E}[S] = \lambda\exp\{\sigma^2/2 + m\}\,,$$

$$\mathrm{Var}[S] = \lambda\exp\{2\sigma^2 + 2m\}$$

and

$$\frac{\mathbb{E}[(S - \mathbb{E}[S])^3]}{(\mathrm{Var}[S])^{3/2}} = \frac{\exp\{9\sigma^2/2 + 3m\}}{\sqrt{\lambda\exp\{6\sigma^2 + 6m\}}} = \frac{\exp\{3\sigma^2/2\}}{\sqrt{\lambda}}\,.$$

■

The computation of the characteristics of a risk is much easier for the compound Poisson model than for the compound binomial model. Using a compound Poisson model also has another great advantage. Assume that a portfolio consists of several independent single risks $S^{(1)}, S^{(2)}, \dots, S^{(j)}$, each modelled as compound Poisson. For simplicity we use $j = 2$ in the following calculation. We want to find the moment generating function of $S^{(1)} + S^{(2)}$.

$$\begin{aligned} M_{S^{(1)}+S^{(2)}}(r) &= M_{S^{(1)}}(r)M_{S^{(2)}}(r) \\ &= \exp\{\lambda^{(1)}(M_{Y^{(1)}}(r) - 1)\}\exp\{\lambda^{(2)}(M_{Y^{(2)}}(r) - 1)\} \\ &= \exp\Bigl\{\lambda\Bigl(\frac{\lambda^{(1)}}{\lambda}M_{Y^{(1)}}(r) + \frac{\lambda^{(2)}}{\lambda}M_{Y^{(2)}}(r) - 1\Bigr)\Bigr\}\,, \end{aligned}$$

where $\lambda = \lambda^{(1)} + \lambda^{(2)}$. It follows that $S^{(1)} + S^{(2)}$ is compound Poisson distributed with Poisson parameter λ and claim size distribution function

$$G(x) = \frac{\lambda^{(1)}}{\lambda} G^{(1)}(x) + \frac{\lambda^{(2)}}{\lambda} G^{(2)}(x)\,.$$

The claim size can be obtained by choosing it from the first risk with probability $\lambda^{(1)}/\lambda$ and from the second risk with probability $\lambda^{(2)}/\lambda$.

Let us now split the claim amounts into different classes. Choose some disjoint sets $A_1, A_2, \ldots, A_m$ with $\mathbb{P}[Y_1 \in \cup_{k=1}^m A_k] = 1$. Let $p_k = \mathbb{P}[Y_1 \in A_k]$ be the probability that a claim is in claim size class k. We can assume that $p_k > 0$ for all k. We denote by N_k the number of claims in claim size class k. Because the claim amounts are independent it follows that, given $N = n$, the vector $(N_1, N_2, \ldots, N_m)$ is conditionally multinomial distributed with parameters $n, p_1, p_2, \ldots, p_m$. We now want to find the unconditioned distribution of $(N_1, N_2, \ldots, N_m)$. Let $n_1, n_2, \ldots, n_m$ be natural numbers and $n = n_1 + n_2 + \cdots + n_m$.

$$\begin{aligned}
&\mathbb{P}[N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m] \\
&\quad = \mathbb{P}[N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m, N = n] \\
&\quad = \mathbb{P}[N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m \mid N = n]\mathbb{P}[N = n] \\
&\quad = \frac{n!}{n_1! n_2! \cdots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \frac{\lambda^n}{n!} \mathrm{e}^{-\lambda} = \prod_{k=1}^m \frac{(\lambda p_k)^{n_k}}{n_k!} \mathrm{e}^{-\lambda p_k}\,.
\end{aligned}$$

It follows that $N_1, N_2, \ldots, N_m$ are independent and that N_k is Pois(λp_k) distributed. Because the claim sizes are independent of N the risks

$$S_k = \sum_{i=1}^N Y_i \mathbb{1}_{\{Y_i \in A_k\}}$$

are compound Poisson distributed with Poisson parameter λp_k and claim size distribution

$$G_k(x) = \mathbb{P}[Y_1 \le x \mid Y_1 \in A_k] = \frac{\mathbb{P}[Y_1 \le x, Y_1 \in A_k]}{\mathbb{P}[Y_1 \in A_k]}\,.$$

Moreover, the sums $\{S_k\}$ are independent. In the special case where $A_k = (t_{k-1}, t_k]$ we get

$$G_k(x) = \frac{G(x) - G(t_{k-1})}{G(t_k) - G(t_{k-1})} \qquad (t_{k-1} \le x \le t_k).$$

1.4 The Compound Mixed Poisson Model

As mentioned before it is a disadvantage of the compound Poisson model that the distribution is always positively skewed. In practice, it also often turns out that the model does not allow enough fluctuations. For instance, $\mathbb{E}[N] = \operatorname{Var}[N]$. A simple way to allow for more fluctuation is to let the parameter λ be stochastic. Let H denote the distribution function of λ.

$$\mathbb{P}[N = n] = \mathbb{E}[\mathbb{P}[N = n \mid \lambda]] = \mathbb{E}\Big[\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\Big] = \int_0^\infty \frac{\ell^n}{n!}\mathrm{e}^{-\ell}\,\mathrm{d}H(\ell)\;.$$

The moments are

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S \mid \lambda]] = \mathbb{E}[\lambda\mu] = \mathbb{E}[\lambda]\mu\;, \tag{1.1a}$$
$$\mathbb{E}[S^2] = \mathbb{E}[\mathbb{E}[S^2 \mid \lambda]] = \mathbb{E}[\lambda\mu_2 + \lambda^2\mu^2] = \mathbb{E}[\lambda^2]\mu^2 + \mathbb{E}[\lambda]\mu_2 \tag{1.1b}$$

and

$$\begin{aligned}\mathbb{E}[S^3] &= \mathbb{E}[\mathbb{E}[(\{S - \lambda\mu\} + \lambda\mu)^3 \mid \lambda]] \\ &= \mathbb{E}[\lambda^3]\mu^3 + 3\mathbb{E}[\lambda^2]\mu_2\mu + \mathbb{E}[\lambda]\mu_3\;. \end{aligned} \tag{1.1c}$$

Thus the variance is

$$\operatorname{Var}[S] = \operatorname{Var}[\lambda]\mu^2 + \mathbb{E}[\lambda]\mu_2$$

and the third centralised moment becomes

$$\begin{aligned}\mathbb{E}[(S - \mathbb{E}[\lambda]\mu)^3] &= \mathbb{E}[\lambda^3]\mu^3 + 3\mathbb{E}[\lambda^2]\mu_2\mu + \mathbb{E}[\lambda]\mu_3 \\ &\quad -3(\mathbb{E}[\lambda^2]\mu^2 + \mathbb{E}[\lambda]\mu_2)\mathbb{E}[\lambda]\mu + 2\mathbb{E}[\lambda]^3\mu^3 \\ &= \mathbb{E}[(\lambda - \mathbb{E}[\lambda])^3]\mu^3 + 3\operatorname{Var}[\lambda]\mu_2\mu + \mathbb{E}[\lambda]\mu_3\;.\end{aligned}$$

We can see that the coefficient of skewness can also be negative.

It remains to compute the moment generating function

$$M_S(r) = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{rS} \mid \lambda\right]\right] = \mathbb{E}[\exp\{\lambda(M_Y(r) - 1)\}] = M_\lambda(M_Y(r) - 1)\;. \tag{1.2}$$

1.5 The Compound Negative Binomial Model

Let us first consider an example of the compound mixed Poisson model. Assume that $\lambda \sim \Gamma(\gamma, \beta)$. Then

$$M_N(r) = M_\lambda(e^r - 1) = \Big(\frac{\beta}{\beta - (e^r - 1)}\Big)^\gamma = \Big(\frac{\frac{\beta}{\beta+1}}{1 - (1 - \frac{\beta}{\beta+1})e^r}\Big)^\gamma .$$

In this case N has a negative binomial distribution. Actuaries started using the negative binomial distribution for the number of claims a long time ago. They recognised that a Poisson distribution rarely fits the real data. Estimating the parameters in a negative binomial distribution yielded satisfactory results.

Let now $N \sim \mathrm{NB}(\alpha, p)$. Then

$$\mathbb{E}[S] = \frac{\alpha(1-p)}{p}\mu$$

and

$$\mathrm{Var}[S] = \frac{\alpha(1-p)}{p^2}\mu^2 + \frac{\alpha(1-p)}{p}(\mu_2 - \mu^2) .$$

The moment generating function is

$$M_S(r) = \Big(\frac{p}{1-(1-p)M_Y(r)}\Big)^\alpha .$$

The third centralised moment can be computed as

$$\mathbb{E}[(S - \mathbb{E}[S])^3] = \alpha\Big(\frac{1}{p} - 1\Big)\mu_3 + 3\alpha\Big(\frac{1}{p} - 1\Big)^2\mu_2\mu + 2\alpha\Big(\frac{1}{p} - 1\Big)^3\mu^3 .$$

Note that the compound negative binomial distribution is always positively skewed. Thus the compound negative binomial distribution does not satisfy all the desired properties. Nevertheless, in practice almost all risks are positively skewed.

1.6 A Note on the Individual Model

Assume that a portfolio consist of m independent possibly not identically distributed individual contracts $(S^{(i)})_{i \le m}$. There can be at most one claim for each of the contracts. Such a claim occurs with probability $p^{(i)}$. Its size has distribution function $F^{(i)}$ and moment generating function $M^{(i)}(r)$. Let $\lambda = \sum_{i=1}^m p^{(i)}$. The moment generating function of the aggregate claims from the portfolio is

$$M_S(r) = \prod_{i=1}^m (1 + p^{(i)}(M^{(i)}(r) - 1)) .$$

The term $p^{(i)}(M^{(i)}(r) - 1)$ is small for r not too large. Consider the logarithm

$$\log(M_S(r)) = \sum_{i=1}^{m} \log(1 + p^{(i)}(M^{(i)}(r) - 1))$$
$$\approx \sum_{i=1}^{m} p^{(i)}(M^{(i)}(r) - 1) = \lambda\Big(\sum_{i=1}^{m} \frac{p^{(i)}}{\lambda}(M^{(i)}(r) - 1)\Big).$$

The last expression turns out to be the logarithm of the moment generating function of a compound Poisson distribution. In fact, this derivation is the reason why the compound Poisson model is very popular amongst actuaries.

The compound model can be used for a large, quite homogeneous model, where the number of claims is small compared to the number of contracts. If we imagine the possible claims from the different contracts are in an urn, and after a claim, we draw one of these claims, then it is very unlikely that we will draw the same claim twice.

1.7 A Note on Reinsurance

For most insurance companies the premium volume is not big enough to carry the complete risk. This is especially the case for large claim sizes, like insurance against damages caused by hurricanes or earthquakes. Therefore the insurers try to share part of the risk with other companies. Sharing the risk is done via reinsurance. Let S^I denote the part of the risk taken by the insurer and S^R the part taken by the reinsurer. Reinsurance can act on the individual claims or it can act on the whole risk S. Let f be an increasing function with $f(0) = 0$ and $f(x) \le x$ for all $x \ge 0$. A reinsurance form acting on the individual claims is

$$S^I = \sum_{i=1}^{N} f(Y_i), \qquad S^R = S - S^I.$$

The most common reinsurance forms are

- proportional reinsurance $\quad f(x) = \alpha x, \quad (0 < \alpha < 1)$,
- excess of loss reinsurance $\quad f(x) = \min\{x, M\}, \quad (M > 0)$.

We will consider these two reinsurance forms in the sequel. A reinsurance form acting on the whole risk is

$$S^I = f(S), \qquad S^R = S - S^I.$$

The most common example of this reinsurance form is the

- stop loss reinsurance $\quad f(x) = \min\{x, M\}, \quad (M > 0)$.

1.7.1 Proportional Reinsurance

For proportional reinsurance we have $S^I = \alpha S$, and thus

$$\begin{aligned} \mathbb{E}[S^I] &= \alpha\mathbb{E}[S]\,, \\ \mathrm{Var}[S^I] &= \alpha^2\mathrm{Var}[S]\,, \\ \frac{\mathbb{E}[(S^I - \mathbb{E}[S^I])^3]}{(\mathrm{Var}[S^I])^{3/2}} &= \frac{\mathbb{E}[(S - \mathbb{E}[S])^3]}{(\mathrm{Var}[S])^{3/2}} \end{aligned}$$

and

$$M_{S^I}(r) = \mathbb{E}\left[\mathrm{e}^{r\alpha S}\right] = M_S(\alpha r)\,.$$

We can see that the coefficient of skewness does not change, but the variance is much smaller. The following considerations also suggest that the risk has decreased. Let the premium charged for this contract be p and assume that the insurer gets αp, the reinsurer $(1-\alpha)p$. Let the initial capital of the company be u. The probability that the company gets ruined after one year is

$$\mathbb{P}[\alpha S > \alpha p + u] = \mathbb{P}[S > p + u/\alpha]\,.$$

The effect for the insurer is thus like having a larger initial capital.

Remark 1.3 The computations for the reinsurer can be obtained by replacing α by $(1-\alpha)$. ■

1.7.2 Excess of Loss Reinsurance

Under excess of loss reinsurance we cannot give formulae of the above type for the cumulants and the moment generating function. They all have to be computed from the new distribution function of the claim sizes $Y_i^I = \min\{Y_i, M\}$. An indication that the risk has decreased for the insurer is that the claim sizes are bounded. This is wanted especially in the case of large claims.

For the calculation of the expected value of the payments, note that $\mathbb{P}[Y_i^R > x] = \mathbb{P}[Y > x + M] = 1 - G(x+M)$. Thus

$$\mathbb{E}[Y_i^R] = \int_0^\infty (1 - G(x+M))\,\mathrm{d}x = \int_M^\infty (1 - G(z))\,\mathrm{d}z\,.$$

In particular,

$$\mathbb{E}[Y_i^I] = \mathbb{E}[Y_i - Y_i^R] = \int_0^M (1 - G(z))\,\mathrm{d}z\,.$$

Example 1.4 Let S be the compound Poisson model with parameter λ and $\mathrm{Pa}(\alpha, \beta)$ distributed claim sizes. Assume that $\alpha > 1$, i.e. that $\mathbb{E}[Y_i] < \infty$. Let us compute the expected value of the outgo paid by the insurer;

$$\mathbb{E}[Y_i^I] = \int_0^M \Bigl(\frac{\beta}{\beta + x}\Bigr)^\alpha \,\mathrm{d}x = \frac{\beta^\alpha}{\alpha - 1}\Bigl(\frac{1}{\beta^{\alpha-1}} - \frac{1}{(\beta + M)^{\alpha-1}}\Bigr)$$
$$= \Bigl(1 - \Bigl(\frac{\beta}{\beta + M}\Bigr)^{\alpha-1}\Bigr)\frac{\beta}{\alpha - 1} = \Bigl(1 - \Bigl(\frac{\beta}{\beta + M}\Bigr)^{\alpha-1}\Bigr)\mathbb{E}[Y_i]\,.$$

For $\mathbb{E}[S^I]$ it follows that

$$\mathbb{E}[S^I] = \Bigl(1 - \Bigl(\frac{\beta}{\beta + M}\Bigr)^{\alpha-1}\Bigr)\mathbb{E}[S]\,.$$

■

Let

$$N^R = \sum_{i=1}^N \mathbb{1}_{\{Y_i > M\}}$$

denote the number of claims the reinsurer has to pay for. We denote by $q = \mathbb{P}[Y_i > M]$ the probability that a claim amount exceeds the level M. What is the distribution of N^R? We first note that the moment generating function of $\mathbb{1}_{\{Y_i > M\}}$ is $q\mathrm{e}^r + 1 - q$.

(i) Let $N \sim \mathrm{B}(n, p)$. The moment generating function of N^R is

$$M_{N^R}(r) = (p(q\mathrm{e}^r + 1 - q) + 1 - p)^n = (pq\mathrm{e}^r + 1 - pq)^n\,.$$

Thus $N^R \sim \mathrm{B}(n, pq)$.

(ii) Let $N \sim \mathrm{Pois}(\lambda)$. The moment generating function of N^R is

$$M_{N^R}(r) = \exp\{\lambda((q\mathrm{e}^r + 1 - q) - 1)\} = \exp\{\lambda q(\mathrm{e}^r - 1)\}\,.$$

Thus $N^R \sim \mathrm{Pois}(\lambda q)$.

(iii) Let $N \sim \mathrm{NB}(\alpha, p)$. The moment generating function of N^R is

$$M_{N^R}(r) = \Bigl(\frac{p}{1 - (1 - p)(q\mathrm{e}^r + 1 - q)}\Bigr)^\alpha = \Biggl(\frac{\frac{p}{p+q-pq}}{1 - \Bigl(1 - \frac{p}{p+q-pq}\Bigr)\mathrm{e}^r}\Biggr)^\alpha\,.$$

Thus $N^R \sim \mathrm{NB}(\alpha, \frac{p}{p+q-pq})$.

1.8 Computation of the Distribution of S in the Discrete Case

Let us consider the case where the claim sizes $\{Y_i\}$ have an arithmetic distribution. We can assume that $\mathbb{P}[Y_i \in \mathbb{N}] = 1$ by choosing the monetary unit appropriately. We define $p_k = \mathbb{P}[N = k]$, $f_k = \mathbb{P}[Y_i = k]$ and $g_k = \mathbb{P}[S = k]$. For simplicity let us assume that $f_0 = 0$. Let $f_k^{*n} = \mathbb{P}[Y_1 + Y_2 + \cdots Y_n = k]$ denote the convolutions of the claim size distribution. Note that

$$f_k^{*(n+1)} = \sum_{i=1}^{k-1} f_i^{*n} f_{k-i} \, .$$

We get the following identities:

$$g_0 = \mathbb{P}[S = 0] = \mathbb{P}[N = 0] = p_0 \, ,$$

$$g_n = \mathbb{P}[S = n] = \mathbb{E}[\mathbb{P}[S = n \mid N]] = \sum_{k=1}^{n} p_k f_n^{*k} \, .$$

We have explicit formulae for the distribution of S, but the computation of the f_n^{*k}'s is messy. An easier procedure is called for. Let us now make an assumption on the distribution of N.

Assumption 1.5 Assume that there exist real numbers a and b such that

$$p_r = \Bigl(a + \frac{b}{r}\Bigr) p_{r-1}$$

for $r \in \mathbb{N} \setminus \{0\}$.

Let us check the assumption for the distributions of the models we have considered so far.

(i) **Binomial** $\mathrm{B}(n,p)$

$$\frac{p_r}{p_{r-1}} = \frac{\frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}}{\frac{n!}{(r-1)!(n-r+1)!} p^{r-1} (1-p)^{n-r+1}} = \frac{(n-r+1)p}{r(1-p)}$$
$$= -\frac{p}{1-p} + \frac{(n+1)p}{r(1-p)} \, .$$

Thus

$$a = -\frac{p}{1-p} \, , \qquad b = \frac{(n+1)p}{1-p} \, .$$

(ii) **Poisson** Pois(λ)

$$\frac{p_r}{p_{r-1}} = \frac{\frac{\lambda^r}{r!}e^{-\lambda}}{\frac{\lambda^{r-1}}{(r-1)!}e^{-\lambda}} = \frac{\lambda}{r}\,.$$

Thus

$$a = 0\,, \qquad b = \lambda\,.$$

(iii) **Negative binomial** NB(α, p)

$$\frac{p_r}{p_{r-1}} = \frac{\frac{\Gamma(\alpha+r)}{r!\Gamma(\alpha)}p^\alpha(1-p)^r}{\frac{\Gamma(\alpha+r-1)}{(r-1)!\Gamma(\alpha)}p^\alpha(1-p)^{r-1}} = \frac{(\alpha+r-1)(1-p)}{r}$$
$$= 1 - p + \frac{(\alpha-1)(1-p)}{r}\,.$$

Thus

$$a = 1 - p\,, \qquad b = (\alpha-1)(1-p)\,.$$

These are in fact the only distributions satisfying the assumption. If we choose $a = 0$ we get by induction $p_r = p_0 b^r/r!$, which is the Poisson distribution.

If $a < 0$, then because $a + b/r$ is negative for r large enough there must be an $n_0 \in \mathbb{N}$ such that $b = -an_0$ in order that $p_r \geq 0$ for all r. Letting $n = n_0 - 1$ and $p = -a(1-a)$ we get the binomial distribution.

If $a > 0$ we need $a + b \geq 0$ in order that $p_1 \geq 0$. The case $a + b = 0$ can be considered as the degenerate case of a Poisson distribution with $\lambda = 0$. Suppose therefore that $a + b > 0$. In particular, $p_r > 0$ for all r. Let $k \in \mathbb{N} \setminus \{0\}$. Then

$$\sum_{r=1}^k rp_r = \sum_{r=1}^k (ar+b)p_{r-1} = a\sum_{r=1}^k (r-1)p_{r-1} + (a+b)\sum_{r=1}^k p_{r-1}$$
$$= a\sum_{r=1}^{k-1} rp_r + (a+b)\sum_{r=0}^{k-1} p_r\,.$$

This can be written as

$$kp_k = (a-1)\sum_{r=1}^{k-1} rp_r + (a+b)\sum_{r=0}^{k-1} p_r\,.$$

Suppose $a \geq 1$. Then $kp_k \geq (a+b)p_0$ and therefore $p_k \geq (a+b)p_0/k$. In particular,

$$1 - p_0 = \sum_{r=1}^\infty p_r \geq (a+b)p_0\sum_{r=1}^\infty \frac{1}{r}\,.$$

This is a contradiction. Thus $a < 1$. If we now let $p = 1 - a$ and $\alpha = a^{-1}(a + b)$ we get the negative binomial distribution.

We will use the following lemma.

Lemma 1.6 *Let $n \geq 2$. Then*

(i)
$$\mathbb{E}\Big[Y_1 \,\Big|\, \sum_{i=1}^n Y_i = r\Big] = \frac{r}{n}\,,$$

(ii)
$$p_n f_r^{*n} = \sum_{k=1}^{r-1}\Big(a + \frac{bk}{r}\Big) f_k p_{n-1} f_{r-k}^{*(n-1)}\,.$$

Proof (i) Noting that the $\{Y_i\}$ are iid. we get

$$\mathbb{E}\Big[Y_1 \,\Big|\, \sum_{i=1}^n Y_i = r\Big] = \frac{1}{n}\sum_{j=1}^n \mathbb{E}\Big[Y_j \,\Big|\, \sum_{i=1}^n Y_i = r\Big]$$
$$= \frac{1}{n}\mathbb{E}\Big[\sum_{j=1}^n Y_j \,\Big|\, \sum_{i=1}^n Y_i = r\Big] = \frac{r}{n}\,.$$

(ii) Note that $f_0^{*(n-1)} = 0$. Thus

$$p_{n-1}\sum_{k=1}^{r-1}\Big(a + \frac{bk}{r}\Big) f_k f_{r-k}^{*(n-1)} = p_{n-1}\sum_{k=1}^{r}\Big(a + \frac{bk}{r}\Big) f_k f_{r-k}^{*(n-1)}$$
$$= p_{n-1}\sum_{k=1}^{r}\Big(a + \frac{bk}{r}\Big)\mathbb{P}\Big[Y_1 = k, \sum_{j=2}^n Y_j = r - k\Big]$$
$$= p_{n-1}\sum_{k=1}^{r}\Big(a + \frac{bk}{r}\Big)\mathbb{P}\Big[Y_1 = k, \sum_{j=1}^n Y_j = r\Big]$$
$$= p_{n-1}\sum_{k=1}^{r}\Big(a + \frac{bk}{r}\Big)\mathbb{P}\Big[Y_1 = k \,\Big|\, \sum_{j=1}^n Y_j = r\Big] f_r^{*n}$$
$$= p_{n-1}\mathbb{E}\Big[a + \frac{bY_1}{r} \,\Big|\, \sum_{j=1}^n Y_j = r\Big] f_r^{*n}$$
$$= p_{n-1}\Big(a + \frac{b}{n}\Big) f_r^{*n} = p_n f_r^{*n}\,.$$

□

We now use the second formula to find a recursive expression for g_r. We know already that $g_0 = p_0$.