Frontiers in the History of Science

Michael Friedman

Ramified Surfaces

On Branch Curves and Algebraic Geometry in the 20th Century

Birkhäuser

Frontiers in the History of Science

Birkhäuser

Frontiers in the History of Science is designed for publications of up-to-date research results encompassing all areas of history of science, primarily with a focus on the history of mathematics, physics, and their applications. Graduates and post-graduates as well as scientists will benefit from the selected and thoroughly peer-reviewed publications at the research frontiers of history of sciences and at interdisciplinary “frontiers”: history of science crossing into neighboring fields such as history of epistemology, history of art, or history of culture. The series is curated by the Series Editor with the support of an international group of Associate Editors.

* * *

Michael Friedman

Ramified Surfaces

On Branch Curves and Algebraic Geometry in the 20th Century

Cluster of Excellence Matters of Activity Image Space Material
Funded by DFG
Birkhäuser

Michael Friedman
The Cohn Institute for the History and Philosophy of Science and Ideas, Humanities Faculty
Tel Aviv University
Tel Aviv, Israel

The author acknowledges the support of the Cluster of Excellence "Matters of Activity. Image Space Material" funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Gemany's Excellence Strategy—EXC 2025—390648296.

ISSN 2662-2564 ISSN 2662-2572 (electronic)
Frontiers in the History of Science
ISBN 978-3-031-05719-9 ISBN 978-3-031-05720-5 (eBook)
https://doi.org/10.1007/978-3-031-05720-5

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Acknowledgements

I wish to thank and acknowledge the many scholars who have helped me in the process of writing this book. Vincenzo de Risi accompanied the conception and creation of this book from the beginning till its completion. Both Vincenzo de Risi and Moritz Epple gave extremely valuable comments and advice concerning the content and structure of the book, which helped to bring clarity, shape and form to it. I am also extremely grateful to Wolfgang Schäffner, who provided me with the space and time to pursue the research for this book at the Cluster of Excellence 'Matters of Activity', where a large part of it was written.

Moreover, the research on Boris Moishezon would not have been possible without the help of Mina Teicher; the numerous conversations with her led to thoughtful insights not only on his work and their mutual discoveries but also on the history of the research on branch curves in general, and I warmly thank her for that. I am also very grateful for the help given to me by the Tel Aviv University Archives, and especially from Ella Meirson and Gedalya Zhagov. Special thanks must also go to Donu Arapura, Ron Livne and Vitali Milman for the conversations on the work and life of Moishezon. Moreover, during my research on the Italian school of algebraic geometry I had various conversations with numerous scholars: Maria Dedò, Antonio Lanteri, Anatoly Libgober and Piera Manara helped me greatly understand the various mathematical configurations that were shaped not only during this period but also during other periods. I would also like to thank warmly Eliot Borenstein, Leo Corry, Christopher Hollings, François Lê, Klaus Volkert, Fernando Zalamea and the anonymous referees for important insights and advice. If I had forgotten anyone here, I deeply apologize.

Last but certainly not least, I thank Michael Lorber, who supported me constantly during the creation and writing of this book. Without his support, my own thought would have probably remained ramified and branched. Thank you.

Contents

Introduction

1

> Monge, 1785: "The projection of a body's shadow on any surface is [. . .] the figure that the extensions of the rays of light tangent to the body's surface end on that surface. [. . .] In the following operations we will geometrically determine only the projections of the contours of the pure shadows, they are *the only ones* that it is necessary to have exactly in the drawings."[1]

At the end of the eighteenth century, the mathematician Gaspard Monge emphasized that to investigate surfaces properly, the "projections of the contours of the pure shadows" are the only curves necessary to draw accurately, and in a certain sense, whose properties are the only ones one should know exactly. An example of what should be drawn is given by Monge, as can be seen in Fig. 1.1, when the source of "light" is either a point or a spherical body. Monge's surfaces were real surfaces, that is, defined over the real numbers, and he also termed the "contour of the pure shadow" as "apparent contour."

Jumping to the twentieth century, when one takes these "surfaces" as complex and algebraic surfaces, embedded in a three-dimensional complex space, then the projection of this contour is called the 'branch curve.' Taking into consideration the fact that in the twenty-first century, Monge's requirements seem almost irrelevant, looking at the current research of complex algebraic surfaces, the question arises: What happened? How was this curve researched over decades, and how did its epistemic status change, especially during the twentieth century, in the then flourishing domain of algebraic geometry?

[1] "La projection de l'ombre d'un corps sur une surface quelconque est donc la figure que terminent sur cette surface les prolongements des rayons de lumière tangents à la surface du corps [. . .] Dans les opérations suivantes nous ne déterminerons géométriquement que les projections des contours des ombres pures, *ce sont les seules* qu'il soit nécessaire d'avoir exactement dans les dessins." (Monge 1847 [1785], p. 27, 29).

M. Friedman, *Ramified Surfaces*, Frontiers in the History of Science,
https://doi.org/10.1007/978-3-031-05720-5_1

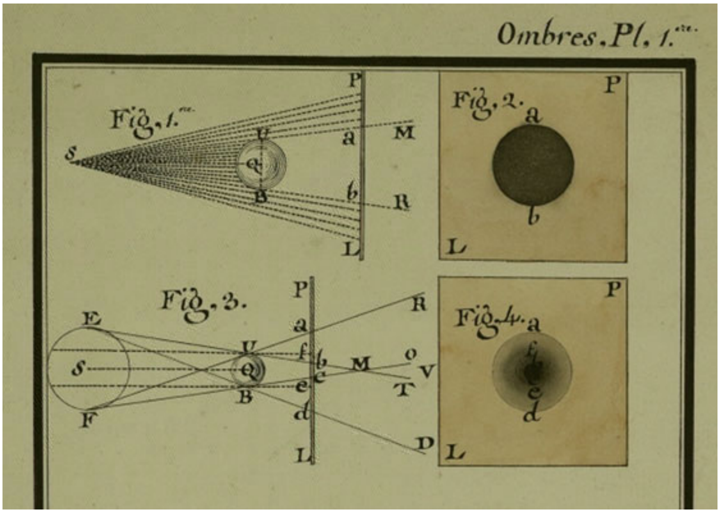

Fig. 1.1 A part of the first plate from Monge's *Ombres*, from the 1780s. The circumference of the shape on Fig. 2 of the plate is an example of Monge's "contour of the pure shadow," today called the "branch curve." © Bibliothèque de l'Ecole polytechnique /Collections Ecole polytechnique/SABIX

This book will aim to answer these questions by presenting a certain cross section of the history of algebraic geometry during the twentieth century. I aim to show that the problem of how to define and characterize the branch curve has not only given rise to novel ways to consider algebraic surfaces and singular curves, but has also prompted research with new mathematical configurations. But in order to explicate what this curve is, which is the object of this book, let me take a step back, and instead of looking directly at complex algebraic surfaces, I will start by looking at complex algebraic curves. The next section—Sect. 1.1—will be somewhat technical, starting with the mathematical definition of branch points and branch curves, as I consider it essential to present, at the outset, a definition of the object of this book. Therefore, I ask the reader to bear with me while reading the next four pages.

1.1 On Branch Points and Branch Curves

In his 1851 dissertation, Bernhard Riemann (1826–1866) introduced the now well-known *Riemann surface*, defined as a covering of the complex (affine or projective) line for multi-valued analytical functions. In the following, I will present Riemann's results in *modern* mathematical language; below I will discuss Riemann's own formulations and present a more precise mathematical description (see Sect. 1.2.2).

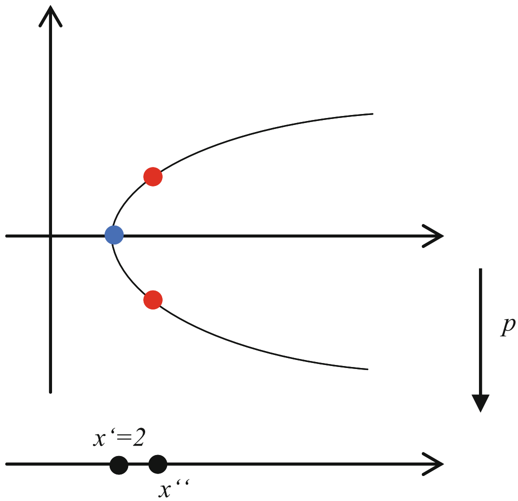

Fig. 1.2 The real part of the curve $y^2 = x - 2$, the *ramification* point (2,0) (blue point), and the preimages with respect to the projection p of point x'', being close to the *branch* point $x' = 2$; these preimages are the two red points 'above' point x'' on the x-axis; one can imagine how these two red points 'coincide' into the blue point when x'' approaches x'. Hence one says that there are two *branches*, which 'come together' at the ramification point. The problem with this visualization is that it does not show how the complex part (or complex preimages) of the curve looks

Riemann considered complex algebraic functions of two variables as ramified coverings over a complex line. To consider this complex algebraic function—today considered as an algebraic curve—as a covering, means to look at an algebraic curve defined by $f(x, y) = 0$ of degree n, and to consider its projection p to the x-axis:

$$p : \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \to \mathbb{C}, (x, y) \mapsto x.$$

A simple example would be to consider the function $y^2 = x - 2$, a function of degree 2; this function and its projection can be seen in Fig. 1.2. Note that for every $x' \neq 2$, the equation $(y')^2 = x' - 2$ has two solutions. Another way to formulate this is that with respect to the projection p, any point $x' \in \mathbb{C}$ (besides $x' = 2$) has two different preimages:[2] these are the points (x', y_1) and $(x', y_2) \in \mathbb{C}^2$ when $(y_1)^2 = x' - 2$ and $(y_2)^2 = x' - 2$.

However, for $x' = 2$ the number of preimages is less than two—explicitly, there is only one preimage: $(2, 0)$, as can also be seen in Fig. 1.2. One might say that when considering the points $x'' \in \mathbb{C}$ which are close to $x' = 2$, the two preimages of x'' 'coincide' into one point when x'' approaches x'. These points, whose number of preimages is lower than the

[2]For $a \in \mathbb{C}$, the preimages $p^{-1}(a)$ of a are all the points (a, b) on the curve which are projected via p to a. For example, $x' = 3$ has two preimages: $(3, 1)$ and $(3, -1)$, since $y' = \sqrt{3 - 2} = \sqrt{1} = \pm 1$.

expected one—when the expected number of points is n—are called *branch points*; the points on the curve for which a few of the preimages 'coincide' are called—in current terminology—*ramification* points. These ramification points were named by Riemann either "Windungspunkte"[3] or "Verzweigungspunkte,"[4] a terminology that I will also discuss below. In modern terms, a branch point of an algebraic curve $f(x, y) = 0$ can also be defined as the set of points $x \in \mathbb{C}$ for which the derivative df/dx vanishes or does not exist.

Obviously one can consider more complicated functions—below I examine the curve $y^3 = x - 2$—but hopefully, a general idea of what a branch point is has been conveyed. Concerning how branch points can characterize the Riemann surface, several important results were proved during the second half of the nineteenth century, which will be also discussed below, and these results were fundamental to understanding algebraic curves and Riemann surfaces: the Riemann–Hurwitz formula, the computation of the moduli of Riemann surfaces, or the determination of the number of Riemann surfaces, branched along a given number of branch points.

* * *

This short, very partial and ahistorical overview of branch points implies that these points already played a central role in the research on coverings of the complex line $\mathbb{C}$. The question hence arises: What happens to the function and the epistemic status of these branch points when one increases the complex dimension by 1, that is, when one looks not at complex algebraic curves, but rather at complex algebraic *surfaces*, and considers their projection to the complex plane $\mathbb{C}^2$ and the resulting *branch curve*?[5] The answer is naively surprising: things get complicated.

But why? To answer this question, it might be worth noting that on the face of it, the definitions of a branch point and branch curve are surprisingly similar. Indeed, one may take a complex algebraic surface S of degree n defined by $f(x, y, z) = 0$,[6] and consider its projection p to the complex plane $\mathbb{C}^2$:

$$p : \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = 0\} \to \mathbb{C}^2, (x, y, z) \mapsto (x, y).$$

[3] In English: "turning points". The term was coined in 1851; see: Riemann (1851, p. 8). Note that the translation into English in 2004 translated "Windungspunkte" into "branch points" (Baker et al. 2004, p. 6).

[4] In English: "branch points". The term was coined in 1857; see e.g. Riemann (1857, p. 107). The different sheets of the function in the neighborhood of such a point are called "branches [Zweige]".

[5] A similar discussion can be presented regarding the complex projective plane. I will also discuss shortly below the situation when increasing the *real* dimension by 1, i.e. when one considers covers either of $\mathbb{R}^3$ or of the 3-sphere.

[6] The surface is considered as embedded in the 3-dimensional complex space. Similar definitions exist for algebraic surfaces embedded in an m-dimensional (projective) complex space.

The branch curve B is defined as the set of all points on $\mathbb{C}^2$ for which some of the preimages on the algebraic surface S 'coincide':

$$B = \{(x, y) \in \mathbb{C}^2 : (x, y) \text{ has less than } n \text{ preimages on } S\}.$$

Over the real numbers, when considering the surface as embedded in the three-dimensional *real* space, one may view such projections as a modern reformulation of Monge's proposal from 1785, to examine a "projection of a body's shadow" and its "contours." In more modern terms, given the surface $f(x, y, z) = 0$, one defines the branch curve B as the set of all points (x, y) in $\mathbb{C}^2$ for which the derivative df/dz vanishes or does not exist. Or, one can define the *ramification curve* R as the intersection of $\{f = 0\}$ and $\{df/dz = 0\}$, and then the *branch curve* B as the image of R under p—as can be seen in Fig. 1.3a. Moreover, the curve B, as it turns out, is *singular*, having (generically) only nodes and cusps as singularities.[7] An example of a branch curve of a cubic surface is depicted in Fig. 1.3b.

So far, the definitions of branch point and branch curve are similar. Therefore, why do 'things get complicated'? Indeed, already at the end of the nineteenth century, with the research of Italian algebraic geometers Guido Castelnuovo and Federigo Enriques, it became evident, as I will elaborate below, that one cannot 'transfer' in an analogous way results from the research on algebraic curves to algebraic surfaces; or, more concretely, one cannot assume that results about branch points can be simply 'transferred' to branch curves.

So how is one to characterize complex algebraic surfaces? And more importantly, can the branch curve help with this inquiry in the same way that branch points illuminate essential aspects of Riemann surfaces and algebraic curves? In contrast to the situation with Riemann surfaces, where the central role of branch points was made clear by Riemann in the early years of his research, how branch curves were considered and what role they played have a more complicated history. Indeed, the definition of the branch curve given above was not the only one employed during the nineteenth and twentieth centuries; this implies—and this is one of the core claims of this book—that there was not only one, single 'branch curve' that was merely presented and researched in various ways over the decades.

This claim should be explicated: since *branch points* were considered as a way to characterize Riemann surfaces, one might have thought that *branch curves* could be used to characterize complex algebraic surfaces. However, as it turned out during the twentieth century, such a simple characterization was neither obvious nor immediate, partially because algebraic surfaces turned out to be more complicated objects than algebraic curves, and partially because branch curves—being curves with nodes and cusps—were discovered to be a special subset in the set of nodal–cuspidal plane curves. The present book aims to show that the problem of how to characterize this curve and its relation to the corresponding branched surface has given rise not only to novel ways of considering

[7] At a *node*, one has locally a curve of the form $xy = 0$. At a *cusp*, one has locally a curve of the form $y^2 = x^3$.

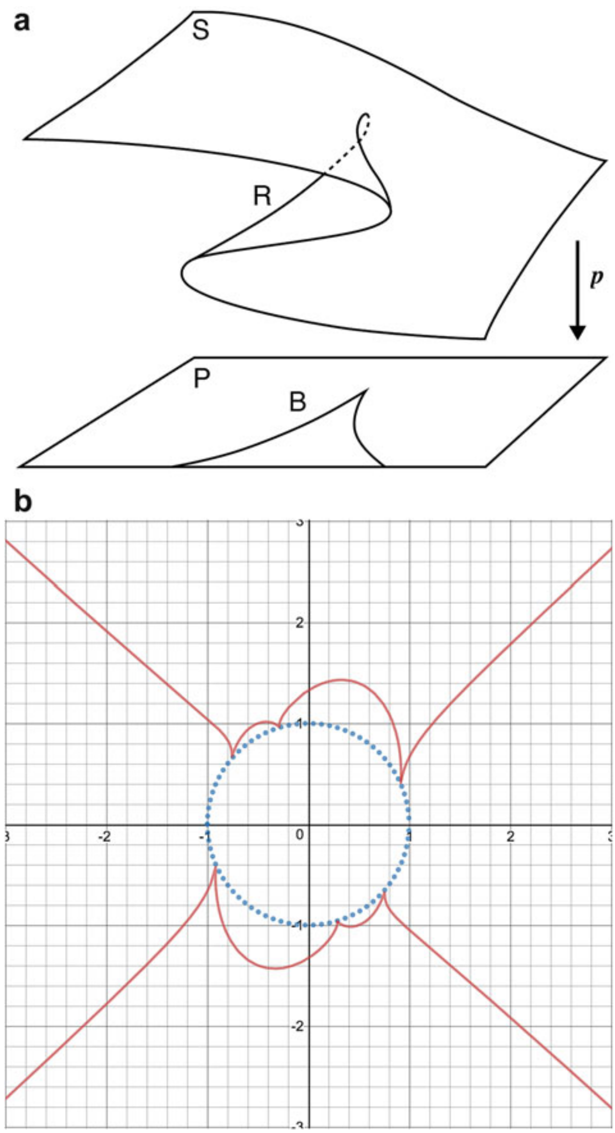

Fig. 1.3 (**a**) A surface S, its projection to a complex plane $P = \mathbb{C}^2$ and the cuspidal branch (resp. ramification) curve B (resp. R). (**b**) Given the projection p, the figure presents the real part of the branch curve (in red) of a smooth complex surface of degree 3: $f(z) = z^3 - 3az + b$, where $a = \{x^2 + y^2 - 1\}$, $b = \{y - 5(x^3 - 3x/4)\}$. The branch curve has six cusps, and these cusps lie on a conic $\{a = 0\}$ (in blue) (drawn by M.F. with https://www.desmos.com/calculator). This drawing (or any attempt to sketch this curve and the special position of the cusps) did not appear in any of the papers dealing with the subject during the twentieth century

algebraic surfaces and nodal–cuspidal curves, but has also prompted research with and within several mathematical domains that were not previously connected to the research on branch points and Riemann surfaces. In this sense, one can consider the research on branch curves as a cross-section through the history of algebraic geometry of the twentieth century, one that shows some of the main transitions and developments in this field. As will be explicated below, the object itself called ‘branch curve’ has changed through the decades, and it has been relocated and redefined within various mathematical settings, or more precisely, within various mathematical configurations. Hence, and this is essential to remember throughout this book, to talk about ‘the’ branch curve is highly misleading.

1.2 Dynamics of a Mathematical Object

> Enriques, 1949: “[. . .] it used to be said that while algebraic curves (already composed in a harmonic theory) are created by God, surfaces, instead, are the work of the devil. Now, on the contrary, it is clear that God chose to create for surfaces an order of more hidden harmonies, where a wonderful beauty shines forth [. . .]”[8]

As I have already implied, the research on algebraic surfaces turned out not to be analogous to the one on algebraic curves. The citation above from Enriques’s *Le superficie algebriche* exemplifies this all too well, and we will see a few examples of this situation below. Moreover, the search for the “hidden harmonies” of surfaces was, one might say, in a constant process of being reformulated and reshaped. Concentrating on the branch curve as one of the objects used to detect those harmonies, I aim to show that while a plurality of approaches for investigating this curve was employed, those various approaches did not necessarily lead to the anticipated harmony, but rather forced shifts in context (or themselves underwent such shifts), sometimes overshadowing each other—or even prompting a marginalization of the original object of study, that is, of the branch curve itself. These shifts can be considered to have occurred in both the body of mathematical knowledge and the image of mathematical knowledge.

Here I employ the distinction introduced by Leo Corry between body and image of mathematical knowledge:[9] statements included in the body of knowledge are about the subject matter of the discipline involved, where these may be theories, conjectures, methods, problems, proofs, etc. Statements belonging to the image of knowledge function

[8] Enriques (1949, p. 464): “[. . .] si soleva dire che, mentre le curve algebriche (già composte in una teoria armonica) sono create da Dio, le superfici sono opera del Demonio. Ora si palesa invece che piacquea Dio di creare per le superficie un ordine di armonie più riposte ove rifulge una meravigliosa bellezza [. . .]”

[9] See: Corry (1989, 2004).

as "guiding principles or selectors,"[10] and answer questions about the discipline as such. These questions may be about authority, the correct and valid methods and proofs that can be used, methodology, and what, how and with whom one should investigate.[11] Though this distinction is essential for analytical purposes, Corry stresses not only that the "body and the image of mathematics appear as organically interconnected domains in the actual history of the discipline,"[12] but also that one should analyze "the subsequent transformations in both the body and the images of mathematics."[13] To examine these "subsequent transformations," I will frame the various research settings of the branch curve in terms of ephemeral local epistemic configurations—a term that I will explain further on. But first, to see how the branch curve reflects those shifts and transitions in the field of algebraic geometry, a field famous for being rewritten at least twice during the twentieth century—first by Oscar Zariski and André Weil and then by Alexander Grothendieck—it is instructive to look at several of the definitions given for the ramification and branch curves starting in the middle of the nineteenth century.

The following three definitions will be explicated throughout the various chapters of this book; the point of bringing them up now is not to demand or expect that the reader understand them mathematically, but rather to present the variety of mathematical settings in which the object called 'branch curve' was introduced and defined.

1. As we will see in Sect. 2.2, one of the common definitions of the branch curve was obtained by considering a surface S, defined by an equation $\{f = 0\}$, embedded in $\mathbb{C}^3$ (resp. $\mathbb{CP}^3$), and projecting it to the complex (projective) plane $\mathbb{C}^2$ (resp. $\mathbb{CP}^2$) from a point. The point was usually not on the surface, though one also considered projections of surfaces when the point of projection was on those surfaces. The surface itself could be singular, having either isolated singularities (such as nodes) or even a double curve. A concrete example of the calculation of the branch curve was given in Fig. 1.3b for a smooth cubic surface, by taking the projection of the intersection of $\{f = 0\}$ and $\{df/dz = 0\}$. During the 1930s, for example, Zariski took this method as the definition of the ramification curve.[14]

One indeed may think of this projection, and specifically, of the ramification curve, by considering drawing tangent lines to the surface, exiting from the given point. This method was made explicit by Monge, as we saw above. The ramification curve is then the

[10] Corry (1989, p. 411).

[11] Corry claims moreover that in "the particular case of mathematics, it brings to the fore a peculiar trait of this discipline, which will be of special interest for the discussion advanced in the present book: the possibility of formulating and proving metastatements about the discipline of mathematics, from within the body of mathematical knowledge." (Corry 2004, p. 4) Corry takes and extends this distinction from (Elkana 1981).

[12] Corry (2004, p. 5).

[13] Ibid.

[14] See: Zariski (1929, p. 306; 1935, p. 160).

collection of all tangent points.[15] Moreover, there are two special kinds of tangent lines: the first are tangent to the surface at two different points of it, hence corresponding to nodes of the branch curve; the second are also tangent to the ramification curve, hence corresponding to cusps of the branch curve (see Fig. 1.3a). Moreover, if the surface has isolated singularities, the branch curve has these singularities as well.

2. Other definitions slowly began to emerge during the 1950s, reflecting the algebraic rewriting of algebraic geometry. In 1958, Zariski gave a purely algebraic definition of a ramified point (with respect to a covering), using the machinery of ring and ideal theory, working with "an absolutely irreducible, *r*-dimensional normal algebraic variety" denoted by *V*, defined over "an arbitrary ground field" *k*. Taking V^* as the normalization of *V*, Zariski then defined the following:

> Let P^* be an arbitrary point of V^* [. . .], and let *P* be the corresponding point of *V*. We denote by $\mathfrak{o}$ the local ring of *P* on *V*/*k* and by $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. Let $\mathfrak{o}^*$ and $\mathfrak{m}^*$ have a similar meaning for P^* and V^*/k^*. It is well known that: (1) $\mathfrak{o}^*\mathfrak{m}$ is a primary ideal, with $\mathfrak{m}^*$ as associated prime ideal; (2) the residue field $k^*(P^*)$ $(= \mathfrak{o}^*/\mathfrak{m}^*)$ is a-finite algebraic extension of the field $k(P)$ $(= \mathfrak{o}/\mathfrak{m})$.
>
> *Definition*: The point P^* is said to be *unramified* (with respect to *V*) if the following conditions are satisfied:
>
> (a) $o^*\mathrm{m} = \mathrm{m}^*$;
> (b) $k^*(P^*)$ is a separable extension of $k(P)$.
>
> In the contrary case P^* is said to be *ramified* (with respect to *V*).[16]

Any trace of the older definition of ramification (point, curve or variety) is hardly to be noticed at first glance.

3. In 1984, one finds the following definition in the book *Compact Complex Surfaces*, given now in another language, that of sheaf theory and canonical sections. The definition itself is as follows: for $p : X \to Y$ a covering, "let us assume that both *X* and *Y* are manifolds. Then the set of ramification points is the zero divisor *R* of the canonical section in $\mathrm{Hom}(p^*(\mathcal{K}_Y), \mathcal{K}_X)$, i.e. $\mathcal{K}_X = p^*(\mathcal{K}_Y) \otimes \mathcal{O}_X(R)$,"[17] where, for a variety *V*, $\mathcal{K}_V$ is defined by "$\mathcal{K}_V = \bigwedge^n \mathcal{T}_V^\vee$ is the canonical bundle of *V*."[18]

[15] Note that the tangent lines may also intersect the surface at other points. Note also that when the surface has a double curve, then this double curve does not count as a component of the ramification curve.

[16] Zariski (1958, p. 791; cursive by M.F.)

[17] Barth / Peters / Van de Ven (1984, p. 41).

[18] Ibid., p. 1.

There are and were other definitions as well,[19] along with variations of the above definitions; one of the common ways at the beginning of the twentieth century to construct surfaces with certain desired properties was to begin with a curve $f(x, y) = 0$, usually singular, and consider the double cover $z^2 = f(x, y)$ of the plane (also called "multiple plane"), ramified over this curve, or even the cyclic cover $z^n = f(x, y)$. Here one did not need to define the branch curve—it was already given by the construction, being the curve $\{f = 0\}$.

1.2.1 Ephemeral Epistemic Configurations and the Identity of the Mathematical Objects

What does this plurality of definitions imply? First, one should note that it implies, to follow one of the main insights of Lakatos,[20] that mathematical concepts and objects are always in a process of development and change, and no specific definition can capture the "essence" of the object (in our case, the branch curve). If such a mathematical object had a well-defined, static definition, one would be able to use it as a technical thing, to follow Hans-Jörg Rheinberger's differentiation between technical and epistemic things, the latter being objects of research.[21] If one focuses on mathematical research, it is instructive to note Moritz Epple's reflections on the various definitions of knots during the nineteenth and twentieth centuries. Concerning these definitions, Epple claims, it is clear that not only "[n]one of these definitions makes sense in mathematical practice without a technical framework,"[22] but also that "[t]he dynamics of the epistemic objects of mathematical

[19]See for example the definition of Grothendieck from the 1960s (Sect. 5.1). Here is the definition given in (Vakil 2017, p. 588): "Suppose $\pi : X \to Y$ is a morphism of schemes. The support of the quasicoherent sheaf $\Omega_\pi = \Omega_{X/Y}$ is called the ramification locus, and the image of its support, $\pi(Supp\ \Omega_{X/Y})$ is called the *branch locus*. If $\Omega_\pi = 0$, we say that π is formally unramified, and if π is also furthermore locally of finite type, we say π is unramified."

[20]Lakatos (1976).

[21]According to Rheinberger epistemic objects possess an "irreducible vagueness." They are the object of research, and as they are in the process of being researched, their "vagueness is inevitable because, paradoxically, epistemic things embody what one does not yet know." (Rheinberger 1997, p. 28) These objects, their purpose, or the field of research that they open and the questions that they may propose are not yet defined or not yet canonically categorized. This is exactly what makes them into an epistemic object, being in a process of becoming 'well-defined' or 'stable' objects, while at the same time presenting an epistemic openness, in the sense that the questions resulting from the research on them are still open. In contrast, experimental conditions and technical objects "tend to be characteristically determined within the given standards of purity and precision. [. . .] they restrict and constrain" the scientific objects (ibid., p. 29). But while it seems that there is a clear distinction between the not yet defined epistemic object and the clearly defined technical one, Rheinberger stresses that the "difference between experimental conditions and epistemic things [. . .] is functional rather than structural." (Ibid., p. 30)

[22]Epple (2011, p. 485).

research [. . .] are secondary to the dynamics of the epistemic configurations as a whole."[23] What is meant here by "epistemic configuration"?

An epistemic mathematical configuration, to follow Epple,[24] is an array of epistemic mathematical objects, researched by a group of mathematicians in a certain, specific temporal and geographical setting, as well as of techniques developed to study those objects. An epistemic configuration is hence dependent on time and place; this, in turn, underlines two aspects: first, the *dynamic* character of the research done, dynamism being very much inherent to the epistemic objects themselves; second, the *local* nature of these configurations, which can be temporal, geographical or social. To stress the first aspect, the dynamical process does not necessarily entail the continuous development of mathematical knowledge: it may result in dead ends, unsolved problems or supplying wrong proofs, or in declarations about the image of the researched configuration that, for example, older research domains are obsolete. Following from this explanation of local epistemic configurations, it becomes clear that each such configuration should be considered an interweaving of local bodies and images of mathematical knowledge. While it may seem that epistemic configurations concentrate only on statements within the body of mathematical knowledge, each particular configuration should be considered with its accompanying image of knowledge. Moreover, every local configuration may have its own (sometimes or often implicit) criteria of what can and should count as a mathematical proof, which may lead to proofs that either have gaps or introduce fallible reasoning in some mathematical arguments.[25] Explicitly, these criteria may prompt the fallibility of either a proof or a justification, or a consideration of judgments of proofs as invalid—and indeed, we will see these phenomena in several of our configurations.

The emphasis on the dynamics of these configurations highlights that they are neither fixed nor static. This also indicates that the history unfolded here, which can be described as the history of various epistemic configurations of branch curves, should not be viewed as an 'internal' one; that is, the 'external' material, social and political conditions and events must also be taken into account when considering how those epistemic configurations transformed and emerged, since these conditions are sometimes the reason for their transformation. This is why, for some of the mathematicians discussed here, the unique social circumstances of the corresponding epistemic configurations are described, as they were very much relevant to the emergence and development of those mathematical configurations. As will become clear throughout the course of this book, several transformations between these configurations were also prompted by forced social and political changes: exile, captivity or emigration.

---

[23] Ibid., p. 488.

[24] See: (Epple 2004).

[25] On fallibilist account of mathematics, see the latest paper of Silvia de Toffoli (2021), who argues for "an alternative to the standard position in the philosophy of mathematics according to which justification requires a genuine proof and is therefore infallible." (ibid., p. 842)

If we return to the three definitions of branch curve given above, one could certainly claim that they are couched in different epistemic configurations. Taken separately, it may seem that none of those definitions of the branch curve has any connection with the other. But it is precisely the above three definitions that exemplify the shifts and transitions of the configurations: it is clear that without understanding "the dynamics of the epistemic configurations," that is, *not only* how frameworks of research on surfaces and curves in particular and algebraic geometry in general changed, *but also* how the image of algebraic geometry was transformed, one cannot comprehend the modifications occurring in the definitions of the branch curve, or the change in its status.

* * *

This discussion on the changes of mathematical objects and configurations is of course not new, as can be seen with Epple's analysis. This dynamic nature is expressed by Fernando Zalamea, who suggests the following, concerning how to think on the history of a mathematical object: "It is not that there exists 'one' fixed mathematical object that could be brought to life independently of the others [. . .]; instead [there] is the plural existence of webs incessantly evolving as they connect with new universes of mathematical interpretation."[26] Norbert Schappacher highlights this "plural existence of [mathematical] webs" when he describes the "explicit transformations of epistemic objects and techniques." Schappacher stresses the "notion of rewriting [. . .] [at a] microhistorical level,"[27] a notion which already underlines the processes of transformation of epistemic configurations. The emphasis on changes detected when focusing on microhistory is also noted by Catherine Goldstein: "the focus of recent history of mathematics has been much more on localised issues, short-term interests and ephemeral situations, on 'the era which produced' the mathematics in question; and moreover it has centred on diversity, differences and changes."[28] Ephemeral configuration, she continues, "implies links with social situations which have their own time scale and are most certainly not 'eternal truths'."[29] That being said, it must therefore be emphasized that two ephemeral configurations researching the 'same' object do not have to refer to each other, even if the second is subsequent to the first. Recognizing this, one may think of the interlacements of epistemic configurations as a braid of threads linking different times and contexts. The metaphor of the braid, as noted by Frédéric Brechenmacher, raises the notion of reconstructing the dynamics of knowledge through the multiplicity of its ephemeral configurations. This multiplicity amounts to establishing a mathematical object, concept, theorem, etc., historically by asking, again and again, about their identity.[30] To explicate: the focus on ephemeral configurations deconstructs the question, or rather the

[26] Zalamea (2012, p. 272–273).
[27] Schappacher (2011, p. 3260).
[28] Goldstein (2018, p. 489).
[29] Ibid., p. 490.
[30] Brechenmacher (2006, p. 1).

presupposition, of the identity of the mathematical object. Goldstein emphasizes the difficulty involved with such historiographic constructions in establishing the identity of a mathematical object, as they presuppose the category of the 'same' (as if in very different expressions one can recognize the same truths), not only at a given time, but also between periods.[31] The paradox is that the possibility of writing a history of a mathematical discovery is always based on the identification of certain identities between mathematical objects or domains: "writing the history of algebra presupposes the identification of what algebra is, or at least what could be part of this particular history."[32] Or, writing the history of 'the' branch curve presupposes the identification of what a 'branch curve' is and which methods, modes of reasoning and techniques one may be permitted to use. But it is precisely this identification that is a part of the image of the configuration, an image which is itself also ephemeral and subject to change.

There are numerous works on the history of mathematics that deconstruct this alleged retrospective identity of the mathematical object. Christian Gilain examines two versions of the fundamental theorem of algebra which are considered the same, and notes that the two "are fundamentally situated on distinct historical axes; their stories have neither the same origin, nor the same rhythm, nor the same duration."[33] Frédéric Brechenmacher analyses the dispute between Leopold Kronecker and Camille Jordan in 1874 about what we now see as the *same* reduction theorem for matrices.[34] Goldstein, in her work on a certain theorem of Fermat and its readers,[35] frames the question of identity by considering two proofs of this theorem, stressing that one should not consider mathematical knowledge as fixed and persistent. The presupposition of identity should be considered, according to Goldstein, as a problem, and not as a tautology; only in this way do the various epistemic configurations attest to the various practices involved.[36] Moreover, Goldstein stresses that seeing identity as tautology hinders understanding the process of mathematical creation.[37]

These studies hence pose the question of the identity of a mathematical object as it persists and is resituated in various ephemeral epistemic configurations. In the context of the current research, I claim that one can talk about ephemeral configurations of the research on branch curves, as most of the configurations studied here were in a process of transformation and re-evaluation, and were in a sense precarious, for a number of

[31] Goldstein (2010, p. 138–139).

[32] Ibid., p. 139.

[33] Gilain (1991, p. 121).

[34] Brechenmacher (2007).

[35] The theorem is that there is no right-angled triangle with rational sides whose area is a rational square.

[36] Goldstein (1995, p. 16).

[37] Ibid., p. 178. Other studies which pose the question of identity of a theorem or a concept at the centre are for example the study of Hourya Sinaceur on two versions of Sturm's theorem (Sinaceur 1991), or the recent research of François Lê (2020) on the different, though not unrelated terms: *genre*, *Geschlecht* and *connectivity* of an algebraic curve during the 1870s and the 1880s.

reasons that are unique to each particular configuration. Here, precarious configuration does not necessarily mean short-lived or marginal (though this may indeed be the case); it means much more, that each local configuration possesses its own temporality: "the time of mathematics, far from being a linear succession of events, possesses a dynamics."[38] Moreover, even when a relatively big community shares results, practices and objects associated to a certain domain—and hence "mathematics acquires its immanence only once when this process is completed"—this does not mean that the theory stops developing and that local configurations stop emerging or being transformed.[39]

Returning to the question of the identity of the mathematical object, as noted above, this question becomes secondary to the dynamic of the epistemic configuration. In other words, the formation of concepts is shaped and conditioned by reorganizations of ephemeral epistemic configurations. In the process of the consolidation of epistemic configurations, mathematical objects are being individuated and also consolidated as such (i.e., as objects) and as participating in the configuration. This process is essentially dynamic: epistemic configurations can be transformed and reorganized, or they can be extended or reduced when encountering new techniques, new notations or new images of knowledge, which in turn may lead to a reorganization of the very configuration in question.[40]

Understanding the dynamics of those configurations, in which the research on branch curves was situated, also implies that the goal of the current book is not to write a 'long-term' history of 'the' branch curve, as if there were a fixed, technical object, found beyond the various transformations of the local configurations.[41] One can think of the study presented here as considering the branch curve in a cross-section through the history of algebraic geometry of the twentieth century. Or rather, to employ a different, more geographical metaphor, the branch curve can be considered as found at the intersection of several research approaches; this also emphasizes the fact that there were various methods, existing at the same time, to investigate this curve. This can be seen when one notices that the branch curve sometimes functioned as an object of research, but at other times was used as a tool to research other objects.

To return to the metaphor of a geological cross-section, this cross-section does not imply capturing all layers in the history of algebraic geometry. This is also emphasized by concentrating on local ephemeral configurations, since locality does not necessarily imply that every aspect of a broader transformation in the field of algebraic geometry—to give one example, the algebraic rewriting project of algebraic geometry of Weil and Zariski during the 1940s and the 1950s—will be present in every local configuration examined here during this time period. Just as one may not take into account every geographical area when performing a cross-section, so too the branch curve is not present

[38] Ehrhardt (2012, p. 252).

[39] Ibid., p. 252–253.

[40] Cf. Feest / Sturm (2011, p. 294).

[41] See: Goldstein (1995, p. 179).

in every development of algebraic geometry. More to the point, not every study on surfaces in algebraic geometry saw branch curves as a way to characterize them. Moreover, this does not imply that every community of mathematicians dealing with branch curves was a leading community in algebraic geometry, just as a cross-section can also capture minor, smaller or thinner layers between the major ones. In this sense, unfolding the history of the research on branch curves also reveals a history of more minor traditions in algebraic geometry—or even marginal ones, and how they may (or may not) reflect (at the same point in time) or join (at a later point in time) the more major traditions. In this sense, the claim of this book is that the various branch curves and the way they were considered and researched during the twentieth century may serve as a touchstone for several of the changes and revolutions in algebraic geometry.

1.2.2 On Branch Points, Again: on Riemann's Terminology and How (Not) to Transfer Results

It is time to examine more closely the relations between branch points and branch curves and their corresponding configurations. Moreover, to delineate how the various epistemic configurations dealing with branch curves operated and were transformed, it is instructive to first return to Riemann's own research and terminology, and then to briefly examine whether the results and terms were successfully transferred to the research configurations dealing with branch curves.

(i) Riemann's formulation

The definition given in Sect. 1.1 is very much a modern one. From the description presented there, one might already have guessed that this was not the terminology used by Riemann.[42] As François Lê notes, "there is no algebraic curve in Riemann's memoir[s]. [. . .] Riemann only talked about algebraic equations, and never interpreted them as equations defining algebraic curves. Further, his proofs did not involve any other object related to algebraic curves, like tangents or cusps."[43] So how did Riemann describe coverings and branch points, and what did he term them?

Considering an algebraic function with two variables x and y, in 1851, Riemann characterized a covering as follows: "[. . .] we permit x, y to vary only over a finite region. The position of the point 0 is no longer considered as being in the plane A [i.e., on the complex line], but in a surface T spread out over the plane. We choose this wording since it

---

[42] For an extensive survey of Riemann's work and the responses to it, see for example (Gray 2015b, p. 153–194); see also (Bottazzini / Gray 2013, p. 259–341) for a similar discussion. On the development of the concept of manifold in Riemann's work and his concept of covering, see e.g., (Scholz 1999, p. 26–30). See also (Scholz 1980).

[43] Lê (2020, p. 78). Lê refers to Riemann's memoir from 1857 on Abelian functions: (Riemann 1857).

is inoffensive to speak of one surface lying on another, to leave open the possibility that the position of 0 can extend more than once over a given part of the plane [. . .]."[44] By the last sentence, Riemann meant that the degree n of the covering may be greater than 1. Moreover, taking, for example, the multi-valued function $y = \sqrt[n]{(x - x')}$, one notes that the n preimages of points in the neighborhood of x' in fact come together when approaching x'. For example, the function $y = \sqrt[3]{(x - 2)}$ is of degree 3 and has a branch point at $x = 2$. The three preimages of points in the neighborhood of $x' = 2$ come together when approaching $x' = 2$.[45] The question that arises is how to describe the behavior of such a function, viewed as a covering, in the neighborhood of such a (preimage of such a) branch point.[46] An explicit description of this situation was given by Riemann in 1857.

Given a function $F(s, z) = 0$—when the degree of s is n and the degree of z is m and when s is branched,[47] Riemann defined a simple "branch point" and a "branch point of multiplicity $\mu + 1$" as follows:

> "A point of the surface T at which only two branches are connected, so that one branch continues into the other and vice versa around this point, is called a *simple branch point* [*einfacher Verzweigungspunkt*].
>
> A point of the surface around which it winds [*windet*] $\mu + 1$ times can then be regarded as the equivalent of μ coincident (or infinitely near) simple branch points."[48]

Riemann specified that one can associate a permutation to each branch point that describes how the preimages of points (or more accurately, the sheets of the Riemann surface) near branch points "behave" and interchange. According to Riemann, above every point $z' \in \mathbb{C}$ (which is not an image of a branch point) there are n preimages, i.e., n values of s, being

[44] Riemann (1851, p. 7): "Für die folgenden Betrachtungen beschranken wir die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes 0 nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. Wir wählen diese Einkleidung, bei der es unanstössig sein wird, von auf einander liegenden Flächen zu reden, um die Möglichkeit offen zu lassen, dass der Ort des Punktes 0 über denselben Theil der Ebene sich mehrfach erstrecke [. . .]." Translation taken from: (Baker et al. 2004, p. 4).

[45] For example, for $x_0 = 3$, then $y_0 = \sqrt[3]{(3 - 2)} = \sqrt[3]{1}$, an equation which has (over the complex numbers) three solutions: $1, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and hence $p^{-1}(x_0)$ has three preimages.

[46] For Riemann, "branch point" often designated the preimage of the branch point on the surface (i.e. in modern terms, the *ramification* point).

[47] Baker et al. (2004, p. 101): "Let us now suppose that the irreducible equation $F(s^n, z^m) = 0$ has been given and that we have to determine the branching of the function s." (Riemann 1857, p. 110: "Es sei jetzt die irreductible Gleichung $F(s^n, z^m) = 0$ gegeben und die Art der Verzweigung der Function s [. . .] zu bestimmen.")

[48] Ibid.: "Ein Punkt der Fläche T, in welchem nur zwei Zweige einer Function zusammenhängen, so dass sich um diesen Punkt der erste in den zweiten und dieser in jenen fortsetzt, heisse ein *einfacher Verzweigungspunkt*. Ein Punkt der Fläche, um welchen sie sich $(\mu + 1)$ mal windet, kann dann angesehen werden als μ zusammengefallene (oder unendlich nahe) einfache Verzweigungspunkte." Translation taken from: (Baker et al. 2004, p. 101).