

Bayesian Methods in Finance

SVETLOZAR T. RACHEV
JOHN S. J. HSU
BILIANA S. BAGASHEVA
FRANK J. FABOZZI



John Wiley & Sons, Inc.

Bayesian Methods in Finance

THE FRANK J. FABOZZI SERIES

Fixed Income Securities, Second Edition by Frank J. Fabozzi
Focus on Value: A Corporate and Investor Guide to Wealth Creation by James L. Grand and James A. Abater
Handbook of Global Fixed Income Calculations by Dragomir Krgin
Managing a Corporate Bond Portfolio by Leland E. Crabbe and Frank J. Fabozzi
Real Options and Option-Embedded Securities by William T. Moore
Capital Budgeting: Theory and Practice by Pamela P. Peterson and Frank J. Fabozzi
The Exchange-Traded Funds Manual by Gary L. Gastineau
Professional Perspectives on Fixed Income Portfolio Management, Volume 3 edited by Frank J. Fabozzi
Investing in Emerging Fixed Income Markets edited by Frank J. Fabozzi and Efstathia Pilarinu
Handbook of Alternative Assets by Mark J. P. Anson
The Exchange-Trade Funds Manual by Gary L. Gastineau
The Global Money Markets by Frank J. Fabozzi, Steven V. Mann, and Moorad Choudhry
The Handbook of Financial Instruments edited by Frank J. Fabozzi
Collateralized Debt Obligations: Structures and Analysis by Laurie S. Goodman and Frank J. Fabozzi
Interest Rate, Term Structure, and Valuation Modeling edited by Frank J. Fabozzi
Investment Performance Measurement by Bruce J. Feibel
The Handbook of Equity Style Management edited by T. Daniel Coggin and Frank J. Fabozzi
The Theory and Practice of Investment Management edited by Frank J. Fabozzi and Harry M. Markowitz
Foundations of Economics Value Added: Second Edition by James L. Grant
Financial Management and Analysis: Second Edition by Frank J. Fabozzi and Pamela P. Peterson
Measuring and Controlling Interest Rate and Credit Risk: Second Edition by Frank J. Fabozzi, Steven V. Mann, and Moorad Choudhry
Professional Perspectives on Fixed Income Portfolio Management, Volume 4 edited by Frank J. Fabozzi
The Handbook of European Fixed Income Securities edited by Frank J. Fabozzi and Moorad Choudhry
The Handbook of European Structured Financial Products edited by Frank J. Fabozzi and Moorad Choudhry
The Mathematics of Financial Modeling and Investment Management by Sergio M. Focardi and Frank J. Fabozzi
Short Selling: Strategies, Risk and Rewards edited by Frank J. Fabozzi
The Real Estate Investment Handbook by G. Timothy Haight and Daniel Singer
Market Neutral: Strategies edited by Bruce I. Jacobs and Kenneth N. Levy
Securities Finance: Securities Lending and Repurchase Agreements edited by Frank J. Fabozzi and Steven V. Mann
Fat-Tailed and Skewed Asset Return Distributions by Svetlozar T. Rachev, Christian Menn, and Frank J. Fabozzi
Financial Modeling of the Equity Market: From CAPM to Cointegration by Frank J. Fabozzi, Sergio M. Focardi, and Petter N. Kolm
Advanced Bond Portfolio management: Best Practices in Modeling and Strategies edited by Frank J. Fabozzi, Lionel Martellini, and Philippe Priaulet
Analysis of Financial Statements, Second Edition by Pamela P. Peterson and Frank J. Fabozzi
Collateralized Debt Obligations: Structures and Analysis, Second Edition by Douglas J. Lucas, Laurie S. Goodman, and Frank J. Fabozzi
Handbook of Alternative Assets, Second Edition by Mark J. P. Anson
Introduction to Structured Finance by Frank J. Fabozzi, Henry A. Davis, and Moorad Choudhry
Financial Econometrics by Svetlozar T. Rachev, Stefan Mittnik, Frank J. Fabozzi, Sergio M. Focardi, and Teo Jasic
Developments in Collateralized Debt Obligations: New Products and Insights by Douglas J. Lucas, Laurie S. Goodman, Frank J. Fabozzi, and Rebecca J. Manning
Robust Portfolio Optimization and Management by Frank J. Fabozzi, Peter N. Kolm, Dessislava A. Pachamanova, and Sergio M. Focardi
Advanced Stochastic Models, Risk Assessment, and Portfolio Optimizations by Svetlozar T. Rachev, Stogan V. Stoyanov, and Frank J. Fabozzi
How to Select Investment Managers and Evaluate Performance by G. Timothy Haight, Stephen O. Morrell, and Glenn E. Ross
Bayesian Methods in Finance by Svetlozar T. Rachev, John S. J. Hsu, Biliana S. Bagasheva, and Frank J. Fabozzi

Bayesian Methods in Finance

SVETLOZAR T. RACHEV
JOHN S. J. HSU
BILIANA S. BAGASHEVA
FRANK J. FABOZZI



John Wiley & Sons, Inc.

Copyright © 2008 by John Wiley & Sons, Inc. All rights reserved.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.

Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning, or otherwise, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4470, or on the Web at www.copyright.com. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, (201) 748-6011, fax (201) 748-6008, or online at <http://www.wiley.com/go/permission>.

Limit of Liability/Disclaimer of Warranty: While the publisher and author have used their best efforts in preparing this book, they make no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives or written sales materials. The advice and strategies contained herein may not be suitable for your situation. You should consult with a professional where appropriate. Neither the publisher nor author shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

For general information on our other products and services or for technical support, please contact our Customer Care Department within the United States at (800) 762-2974, outside the United States at (317) 572-3993, or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books. For more information about Wiley products, visit our Web site at www.wiley.com.

ISBN: 978-0-471-92083-0

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

S.T.R.
To Iliana and Zoya

J.S.J.H.
To Serene, Justin, and Andrew

B.S.B.
To my mother, Gökhan, and my other loved ones

F.J.F.
*To my wife Donna and my children Francesco,
Patricia, and Karly*

Contents

Preface	xv
About the Authors	xvii
CHAPTER 1	
Introduction	1
A Few Notes on Notation	3
Overview	4
CHAPTER 2	
The Bayesian Paradigm	6
The Likelihood Function	6
The Poisson Distribution Likelihood Function	7
The Normal Distribution Likelihood Function	9
The Bayes' Theorem	10
Bayes' Theorem and Model Selection	14
Bayes' Theorem and Classification	14
Bayesian Inference for the Binomial Probability	15
Summary	21
CHAPTER 3	
Prior and Posterior Information, Predictive Inference	22
Prior Information	22
Informative Prior Elicitation	23
Noninformative Prior Distributions	25
Conjugate Prior Distributions	27
Empirical Bayesian Analysis	28
Posterior Inference	30
Posterior Point Estimates	30
Bayesian Intervals	32
Bayesian Hypothesis Comparison	32
Bayesian Predictive Inference	34

Illustration: Posterior Trade-off and the Normal Mean	
Parameter	35
Summary	37
Appendix: Definitions of Some Univariate and Multivariate	
Statistical Distributions	38
The Univariate Normal Distribution	39
The Univariate Student's t -Distribution	39
The Inverted χ^2 Distribution	39
The Multivariate Normal Distribution	40
The Multivariate Student's t -Distribution	40
The Wishart Distribution	41
The Inverted Wishart Distribution	41
CHAPTER 4	
Bayesian Linear Regression Model	43
The Univariate Linear Regression Model	43
Bayesian Estimation of the Univariate Regression	
Model	45
Illustration: The Univariate Linear Regression Model	53
The Multivariate Linear Regression Model	56
Diffuse Improper Prior	58
Summary	60
CHAPTER 5	
Bayesian Numerical Computation	61
Monte Carlo Integration	61
Algorithms for Posterior Simulation	63
Rejection Sampling	64
Importance Sampling	65
MCMC Methods	66
Linear Regression with Semiconjugate Prior	77
Approximation Methods: Logistic Regression	82
The Normal Approximation	84
The Laplace Approximation	89
Summary	90
CHAPTER 6	
Bayesian Framework For Portfolio Allocation	92
Classical Portfolio Selection	94
Portfolio Selection Problem Formulations	95

Mean-Variance Efficient Frontier	97
Illustration: Mean-Variance Optimal Portfolio with Portfolio Constraints	99
Bayesian Portfolio Selection	101
Prior Scenario 1: Mean and Covariance with Diffuse (Improper) Priors	102
Prior Scenario 2: Mean and Covariance with Proper Priors	103
The Efficient Frontier and the Optimal Portfolio	105
Illustration: Bayesian Portfolio Selection	106
Shrinkage Estimators	108
Unequal Histories of Returns	110
Dependence of the Short Series on the Long Series	112
Bayesian Setup	112
Predictive Moments	113
Summary	116

CHAPTER 7

Prior Beliefs and Asset Pricing Models 118

Prior Beliefs and Asset Pricing Models	119
Preliminaries	119
Quantifying the Belief About Pricing Model Validity	121
Perturbed Model	121
Likelihood Function	122
Prior Distributions	123
Posterior Distributions	124
Predictive Distributions and Portfolio Selection	126
Prior Parameter Elicitation	127
Illustration: Incorporating Confidence about the Validity of an Asset Pricing Model	128
Model Uncertainty	129
Bayesian Model Averaging	131
Illustration: Combining Inference from the CAPM and the Fama and French Three-Factor Model	134
Summary	135
Appendix A: Numerical Simulation of the Predictive Distribution	135
Sampling from the Predictive Distribution	136
Appendix B: Likelihood Function of a Candidate Model	138

CHAPTER 8**The Black-Litterman Portfolio Selection Framework 141**

Preliminaries	142
Equilibrium Returns	142
Investor Views	144
Distributional Assumptions	144
Combining Market Equilibrium and Investor Views	146
The Choice of τ and Ω	147
The Optimal Portfolio Allocation	148
Illustration: Black-Litterman Optimal Allocation	149
Incorporating Trading Strategies into the Black-Litterman Model	153
Active Portfolio Management and the Black-Litterman Model	154
Views on Alpha and the Black-Litterman Model	157
Translating a Qualitative View into a Forecast for Alpha	158
Covariance Matrix Estimation	159
Summary	161

CHAPTER 9**Market Efficiency and Return Predictability 162**

Tests of Mean-Variance Efficiency	164
Inefficiency Measures in Testing the CAPM	167
Distributional Assumptions and Posterior Distributions	168
Efficiency under Investment Constraints	169
Illustration: The Inefficiency Measure, Δ^R	170
Testing the APT	171
Distributional Assumptions, Posterior and Predictive Distributions	172
Certainty Equivalent Returns	173
Return Predictability	175
Posterior and Predictive Inference	177
Solving the Portfolio Selection Problem	180
Illustration: Predictability and the Investment Horizon	182
Summary	183
Appendix: Vector Autoregressive Setup	183

CHAPTER 10	
Volatility Models	185
Garch Models of Volatility	187
Stylized Facts about Returns	188
Modeling the Conditional Mean	189
Properties and Estimation of the GARCH(1,1) Process	190
Stochastic Volatility Models	194
Stylized Facts about Returns	195
Estimation of the Simple SV Model	195
Illustration: Forecasting Value-at-Risk	198
An Arch-Type Model or a Stochastic Volatility Model?	200
Where Do Bayesian Methods Fit?	200
CHAPTER 11	
Bayesian Estimation of ARCH-Type Volatility Models	202
Bayesian Estimation of the Simple GARCH(1,1) Model	203
Distributional Setup	204
Mixture of Normals Representation of the Student's t -Distribution	206
GARCH(1,1) Estimation Using the Metropolis-Hastings Algorithm	208
Illustration: Student's t GARCH(1,1) Model	211
Markov Regime-switching GARCH Models	214
Preliminaries	215
Prior Distributional Assumptions	217
Estimation of the MS GARCH(1,1) Model	218
Sampling Algorithm for the Parameters of the MS GARCH(1,1) Model	222
Illustration: Student's t MS GARCH(1,1) Model	222
Summary	225
Appendix: Griddy Gibbs Sampler	226
Drawing from the Conditional Posterior Distribution of ν	227
CHAPTER 12	
Bayesian Estimation of Stochastic Volatility Models	229
Preliminaries of SV Model Estimation	230
Likelihood Function	231
The Single-Move MCMC Algorithm for SV Model Estimation	232

Prior and Posterior Distributions	232
Conditional Distribution of the Unobserved Volatility	233
Simulation of the Unobserved Volatility	234
Illustration	236
The Multimove MCMC Algorithm for SV Model Estimation	237
Prior and Posterior Distributions	237
Block Simulation of the Unobserved Volatility	239
Sampling Scheme	241
Illustration	241
Jump Extension of the Simple SV Model	241
Volatility Forecasting and Return Prediction	243
Summary	244
Appendix: Kalman Filtering and Smoothing	244
The Kalman Filter Algorithm	244
The Smoothing Algorithm	246

CHAPTER 13

Advanced Techniques for Bayesian Portfolio Selection 247

Distributional Return Assumptions Alternative to Normality	248
Mixtures of Normal Distributions	249
Asymmetric Student's t -Distributions	250
Stable Distributions	251
Extreme Value Distributions	252
Skew-Normal Distributions	253
The Joint Modeling of Returns	254
Portfolio Selection in the Setting of Nonnormality:	
Preliminaries	255
Maximization of Utility with Higher Moments	256
Coskewness	257
Utility with Higher Moments	258
Distributional Assumptions and Moments	259
Likelihood, Prior Assumptions, and Posterior	
Distributions	259
Predictive Moments and Portfolio Selection	262
Illustration: HLLM's Approach	263
Extending The Black-Litterman Approach: Copula Opinion	
Pooling	263
Market-Implied and Subjective Information	264
Views and View Distributions	265
Combining the Market and the Views: The Marginal	
Posterior View Distributions	266

Views Dependence Structure:The Joint Posterior View	
Distribution	267
Posterior Distribution of the Market Realizations	267
Portfolio Construction	268
Illustration: Meucci's Approach	269
Extending The Black-Litterman Approach:Stable	
Distribution	270
Equilibrium Returns Under Nonnormality	270
Summary	272
APPENDIX A: Some Risk Measures Employed in Portfolio	
Construction	273
APPENDIX B: CVaR Optimization	276
APPENDIX C: A Brief Overview of Copulas	277

CHAPTER 14

Multifactor Equity Risk Models

280

Preliminaries	281
Statistical Factor Models	281
Macroeconomic Factor Models	282
Fundamental Factor Models	282
Risk Analysis Using a Multifactor Equity Model	283
Covariance Matrix Estimation	283
Risk Decomposition	285
Return Scenario Generation	287
Predicting the Factor and Stock-Specific Returns	288
Risk Analysis in a Scenario-Based Setting	288
Conditional Value-at-Risk Decomposition	289
Bayesian Methods for Multifactor Models	292
Cross-Sectional Regression Estimation	293
Posterior Simulations	293
Return Scenario Generation	294
Illustration	294
Summary	295

References

298

Index

311

Preface

This book provides the fundamentals of Bayesian methods and their applications to students in finance and practitioners in the financial services sector. Our objective is to explain the concepts and techniques that can be applied in real-world Bayesian applications to financial problems. While statistical modeling has been used in finance for the last four or five decades, recent years have seen an impressive growth in the variety of models and modeling techniques used in finance, particularly in portfolio management and risk management. As part of this trend, Bayesian methods are enjoying a rediscovery by academics and practitioners alike and growing in popularity. The choice of topics in this book reflects the current major developments of Bayesian applications to risk management and portfolio management.

Three fundamental factors are behind the increased adoption of Bayesian methods by the financial community. Bayesian methods provide (1) a theoretically sound framework for combining various sources of information; (2) a robust estimation setting that incorporates explicitly estimation risk; and (3) the flexibility to handle complex and realistic models. We believe this book is the first of its kind to present and discuss Bayesian financial applications. The fundamentals of Bayesian analysis and Markov Chain Monte Carlo are covered in Chapters 2 through 5 and the applications are introduced in the remaining chapters. Each application presentation begins with the basics, works through the frequentist perspective, followed by the Bayesian treatment.

The applications include:

- The Bayesian approach to mean-variance portfolio selection and its advantages over the frequentist approach (Chapters 6 and 7).
- A general framework for reflecting degrees of belief in an asset pricing model when selecting the optimal portfolio (Chapters 6 and 7).
- Bayesian methods to portfolio selection within the context of the Black-Litterman model and extensions to it (Chapter 8).
- Computing measures of market efficiency and the way predictability influences optimal portfolio selection (Chapter 9).

- Volatility modeling (ARCH-type and SV models) focusing on the various numerical methods available for Bayesian estimation (Chapters 10, 11, and 12).
- Advanced techniques for model selection, notably in the setting of nonnormality of stock returns (Chapter 13).
- Multifactor models of stock returns, including risk attribution in both an analytical and a numerical setting (Chapter 14).

ACKNOWLEDGMENTS

We thank several individuals for their assistance in various aspects of this project. Thomas Leonard provided us with guidance on several theoretical issues that we encountered. Doug Steigerwald of the University of California–Santa Barbara directed us in the preparation of the discussion on the efficient methods of moments in Chapter 10.

Svetlozar Rachev gratefully acknowledges research support by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California–Santa Barbara; the Deutschen Forschungsgemeinschaft; and the Deutscher Akademischer Austausch Dienst. Biliana Bagasheva gratefully acknowledges the support of the Fulbright Program at the Institute of International Education and the Department of Statistics and Applied Probability, University of California–Santa Barbara. Lastly, Frank Fabozzi gratefully acknowledges the support of Yale’s International Center for Finance.

Svetlozar T. Rachev
John S. J. Hsu
Biliana S. Bagasheva
Frank J. Fabozzi

About the Authors

Svetlozar (Zari) T. Rachev completed his Ph.D. degree in 1979 from Moscow State (Lomonosov) University and his doctor of science degree in 1986 from Steklov Mathematical Institute in Moscow. Currently, he is chair-professor in statistics, econometrics and mathematical finance at the University of Karlsruhe in the School of Economics and Business Engineering. He is also Professor Emeritus at the University of California–Santa Barbara in the Department of Statistics and Applied Probability. He has published seven monographs, eight handbooks, and special-edited volumes, and over 250 research articles. His recently coauthored books published by John Wiley & Sons in mathematical finance and financial econometrics include *Financial Econometrics: From Basics to Advanced Modeling Techniques* (2007); *Operational Risk: A Guide to Basel II Capital Requirements, Models, and Analysis* (2007); and *Advanced Stochastic Models, Risk Assessment and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures* (2008). Professor Rachev is cofounder of Bravo Risk Management Group specializing in financial risk-management software. Bravo Group was recently acquired by FinAnalytica, for which he currently serves as chief-scientist.

John S. J. Hsu is professor of statistics and applied probability at the University of California, Santa Barbara. He is also a faculty member in the University's Center for Research in Financial Mathematics and Statistics. He obtained his Ph.D. in statistics with a minor in business from the University of Wisconsin–Madison in 1990. Professor Hsu has published numerous papers and coauthored a Cambridge University Press advanced series text, *Bayesian Methods: An Analysis for Statisticians and Interdisciplinary Researchers* (1999), with Thomas Leonard.

Biliana S. Bagasheva completed her Ph.D. in Statistics at the University of California–Santa Barbara. Her research interests include risk management, portfolio construction, Bayesian methods, and financial econometrics. Currently, Biliana is a consultant in London.

Frank J. Fabozzi is Professor in the Practice of Finance in the School of Management at Yale University. Prior to joining the Yale faculty, he was a visiting professor of finance in the Sloan School at MIT. He is a Fellow of the International Center for Finance at Yale University and on the Advisory Council for the Department of Operations Research and

Financial Engineering at Princeton University. Professor Fabozzi is the editor of the *Journal of Portfolio Management*. His recently coauthored books published by John Wiley & Sons in mathematical finance and financial econometrics include *The Mathematics of Financial Modeling and Investment Management* (2004); *Financial Modeling of the Equity Market: From CAPM to Cointegration* (2006); *Robust Portfolio Optimization and Management* (2007); and *Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty and Performance Measures* (2008). He earned a doctorate in economics from the City University of New York in 1972. In 2002, he was inducted into the Fixed Income Analysts Society's Hall of Fame and is the 2007 recipient of the C. Stewart Sheppard Award given by the CFA Institute. He earned the designation of Chartered Financial Analyst and Certified Public Accountant. He has authored and edited numerous books in finance.

Bayesian Methods in Finance

Introduction

Quantitative financial models describe in mathematical terms the relationships between financial random variables through time and/or across assets. The fundamental assumption is that the model relationship is valid independent of the time period or the asset class under consideration. Financial data contain both meaningful information and random noise. An adequate financial model not only extracts optimally the relevant information from the historical data but also performs well when tested with new data. The uncertainty brought about by the presence of data noise makes imperative the use of statistical analysis as part of the process of financial model building, model evaluation, and model testing.

Statistical analysis is employed from the vantage point of either of the two main statistical philosophical traditions—“frequentist” and “Bayesian.” An important difference between the two lies with the interpretation of the concept of probability. As the name suggests, advocates of frequentist statistics adopt a *frequentist* interpretation: The probability of an event is the limit of its long-run relative frequency (i.e., the frequency with which it occurs as the amount of data increases without bound). Strict adherence to this interpretation is not always possible in practice. When studying rare events, for instance, large samples of data may not be available and in such cases proponents of frequentist statistics resort to theoretical results. The Bayesian view of the world is based on the *subjectivist* interpretation of probability: Probability is subjective, a degree of belief that is updated as information or data are acquired.¹

¹The concept of subjective probability is derived from arguments for rationality of the preferences of agents. It originated in the 1930s with the (independent) works of Bruno de Finetti and Frank Ramsey, and was further developed by Leonard Savage and Dennis Lindley. The subjective probability interpretation can be traced back to the Scottish philosopher and economist David Hume, who also had philosophical influence over Harry Markowitz (by Markowitz’s own words in his autobiography

Closely related to the concept of probability is that of uncertainty. Proponents of the frequentist approach consider the source of uncertainty to be the randomness inherent in realizations of a random variable. The probability distributions of variables are not subject to uncertainty. In contrast, Bayesian statistics treats probability distributions as uncertain and subject to modification as new information becomes available. Uncertainty is implicitly incorporated by probability updating. The probability beliefs based on the existing knowledge base take the form of the *prior probability*. The *posterior probability* represents the updated beliefs.

Since the beginning of last century, when quantitative methods and models became a mainstream tool to aid in understanding financial markets and formulating investment strategies, the framework applied in finance has been the frequentist approach. The term “frequentist” usually refers to the Fisherian philosophical approach named after Sir Ronald Fisher. Strictly speaking, “Fisherian” has a broader meaning as it includes not only frequentist statistical concepts such as unbiased estimators, hypothesis tests, and confidence intervals, but also the maximum likelihood estimation framework pioneered by Fisher. Only in the last two decades has Bayesian statistics started to gain greater acceptance in financial modeling, despite its introduction about 250 years ago by Thomas Bayes, a British minister and mathematician. It has been the advancements of computing power and the development of new computational methods that has fostered the growing use of Bayesian statistics in finance.

On the applicability of the Bayesian conceptual framework, consider an excerpt from the speech of former chairman of the Board of Governors of the Federal Reserve System, Alan Greenspan:

The Federal Reserve’s experiences over the past two decades make it clear that uncertainty is not just a pervasive feature of the monetary policy landscape; it is the defining characteristic of that landscape. The term “uncertainty” is meant here to encompass both “Knightian uncertainty,” in which the probability distribution of outcomes is unknown, and “risk,” in which uncertainty of outcomes is delimited by a known probability distribution. [...] This conceptual framework emphasizes understanding as much as possible the many sources of risk and uncertainty that policymakers face, quantifying those risks when possible, and assessing the costs associated with each of the risks. In essence, the risk management

published in *Les Prix Nobel* (1991)). Holton (2004) provides a historical background of the development of the concepts of risk and uncertainty.

*approach to monetary policymaking is an application of Bayesian [decision-making].*²

The three steps of Bayesian decision making that Alan Greenspan outlines are:

1. Formulating the prior probabilities to reflect existing information.
2. Constructing the quantitative model, taking care to incorporate the uncertainty intrinsic in model assumptions.
3. Selecting and evaluating a utility function describing how uncertainty affects alternative model decisions.

While these steps constitute the rigorous approach to Bayesian decision-making, applications of Bayesian methods to financial modeling often only involve the first two steps or even only the second step. This tendency is a reflection of the pragmatic Bayesian approach that researchers of empirical finance often favor and it is the approach that we adopt in this book.

The aim of the book is to provide an overview of the theory of Bayesian methods and explain their applications to financial modeling. While the principles and concepts explained in the book can be used in financial modeling and decision making in general, our focus will be on portfolio management and market risk management since these are the areas in finance where Bayesian methods have had the greatest penetration to date.³

A FEW NOTES ON NOTATION

Throughout the book, we follow the convention of denoting vectors and matrices in boldface.

We make extensive use of the proportionality symbol, ‘ \propto ’, to denote the cases where terms constant with respect to the random variable of interest have been dropped from that variable’s density function. To illustrate, suppose that the random variable, X , has a density function

$$p(x) = 2x. \quad (1.1)$$

²Alan Greenspan made these remarks at the Meetings of the American Statistical Association in San Diego, California, January 3, 2004.

³Bayesian methods have been applied in corporate finance, particularly in capital budgeting. An area of Bayesian methods with potentially important financial applications is Bayesian networks. Bayesian networks have been applied in operational risk modeling. See, for example, Alexander (2000) and Neil, Fenton, and Tailor (2005).

Then, we can write

$$p(x) \propto x. \quad (1.2)$$

Now suppose that we take the logarithm of both sides of (1.2). Since the logarithm of a product of two terms is equivalent to the sum of the logarithms of those terms, we obtain

$$\log(p(x)) = \text{const} + \log(x), \quad (1.3)$$

where $\text{const} = \log(2)$ in this case. Notice that it would not be precise to write $\log(p(x)) \propto \log(x)$. We come across the transformation in (1.3) in Chapters 10 through 14, in particular.

OVERVIEW

The book is organized as follows. In Chapters 2 through 5, we provide an overview of the theory of Bayesian methods. The depth and scope of that overview are subordinated to the methodological requirements of the Bayesian applications discussed in later chapters and, therefore, in certain instances lacks the theoretical rigor that one would expect to find in a purely statistical discussion of the topic.

In Chapters 6 and 7, we discuss the Bayesian approach to mean-variance portfolio selection and its advantages over the frequentist approach. We introduce a general framework for reflecting degrees of belief in an asset pricing model when selecting the optimal portfolio. We close Chapter 7 with a description of Bayesian model averaging, which allows the decision maker to combine conclusions based on several competing quantitative models.

Chapter 8 discusses an emblematic application of Bayesian methods to portfolio selection—the Black-Litterman model. We then show how the Black-Litterman framework can be extended to active portfolio selection and how trading strategies can be incorporated into it.

The focus of Chapter 9 is market efficiency and predictability. We analyze and illustrate the computation of measures of market inefficiency. Then, we go on to describe the way predictability influences optimal portfolio selection. We base that discussion on a Bayesian *vector autoregressive* (VAR) framework.

Chapters 10, 11, and 12 deal with volatility modeling. We devote Chapter 10 to an overview of volatility modeling. We introduce the two types of volatility models—*autoregressive conditionally heteroskedastic* (ARCH)-type models and *stochastic volatility* (SV) models—and discuss some of their important characteristics, along with issues of estimation

within the boundaries of frequentist statistics. Chapters 11 and 12 cover, respectively, ARCH-type and SV Bayesian model estimation. Our focus is on the various numerical methods that could be used in Bayesian estimation.

In Chapter 13, we deal with advanced techniques for model selection, notably, recognizing nonnormality of stock returns. We first investigate an approach in which higher moments of the return distribution are explicitly included in the investor's utility function. We then go on to discuss an extension of the Black-Litterman framework that, in particular, employs minimization of the conditional *value-at-risk* (CVaR). In Appendix A of that chapter, we present an overview of risk measures that are alternatives to the standard deviation, such as *value-at-risk* (VaR) and CVaR.

Chapter 14 is devoted to multifactor models of stock returns. We discuss risk attribution in both an analytical and a numerical setting and examine how the multifactor framework provides a natural setting for a coherent portfolio selection and risk management approach.

The Bayesian Paradigm

Likelihood Function and Bayes' Theorem

One of the basic mechanisms of learning is assimilating the information arriving from the external environment and then updating the existing knowledge base with that information. This mechanism lies at the heart of the Bayesian framework. A Bayesian decision maker learns by revising beliefs in light of the new data that become available. From the Bayesian point of view, probabilities are interpreted as degrees of belief. Therefore, the Bayesian learning process consists of revising of probabilities.¹ Bayes' theorem provides the formal means of putting that mechanism into action; it is a simple expression combining the knowledge about the distribution of the model parameters and the information about the parameters contained in the data.

In this chapter, we present some of the basic principles of Bayesian analysis.

THE LIKELIHOOD FUNCTION

Suppose we are interested in analyzing the returns on a given stock and have available a historical record of returns. Any analysis of these returns, beyond a very basic one, would require that we make an educated guess about (propose) a process that might have generated these return data. Assume that we have decided on some statistical distribution and denote it by

$$p(y|\theta), \quad (2.1)$$

¹Contrast this with the way probability is interpreted in the classical (frequentist) statistical theory—as the relative frequency of occurrence of an event in the limit, as the number of observations goes to infinity.

where y is a realization of the random variable Y (stock return) and θ is a parameter specific to the distribution, p . Assuming that the distribution we proposed is the one that generated the observed data, we draw a conclusion about the value of θ . Obviously, central to that goal is our ability to summarize the information contained in the data. The likelihood function is a statistical construct with this precise role. Denote the n observed stock returns by y_1, y_2, \dots, y_n . The joint density function of Y , for a given value of θ , is²

$$f(y_1, y_2, \dots, y_n | \theta).$$

We can observe that the function above can also be treated as a function of the unknown parameter, θ , given the observed stock returns. That function of θ is called *the likelihood function*. We write it as

$$L(\theta | y_1, y_2, \dots, y_n) = f(y_1, y_2, \dots, y_n | \theta). \quad (2.2)$$

Suppose we have determined from the data two competing values of θ , θ_1 and θ_2 , and want to determine which one is more likely to be the true value (at least, which one is closer to the true value). The likelihood function helps us make that decision. Assuming that our data were indeed generated by the distribution in (2.1), θ_1 is more likely than θ_2 to be the true parameter value whenever $L(y_1, y_2, \dots, y_n | \theta_1) > L(y_1, y_2, \dots, y_n | \theta_2)$. This observation provides the intuition behind the method most often employed in “classical” statistical inference to estimate θ from the data alone—*the method of maximum likelihood*. The value of θ most likely to have yielded the observed sample of stock return data, y_1, y_2, \dots, y_n , is *the maximum likelihood estimate*, $\hat{\theta}$, obtained from maximizing the likelihood function in (2.2).

To illustrate the concept of a likelihood function, we briefly discuss two examples—one based on the Poisson distribution (a discrete distribution) and another based on the normal distribution (one of the most commonly employed continuous distributions).

The Poisson Distribution Likelihood Function

The Poisson distribution is often used to describe the random number of events occurring within a certain period of time. It has a single parameter,

²By using the term “density function,” we implicitly assume that the distribution chosen for the stock return is continuous, which is invariably the case in financial modeling.

θ , indicating the rate of occurrence of the random event, that is, how many events happen on average per unit of time. The probability distribution of a Poisson random variable, X , is described by the following expression:³

$$p(X = k) = \frac{\theta^k}{k!} e^{-\theta}, \quad k = 0, 1, 2, \dots \quad (2.3)$$

Suppose we are interested to examine the annual number of defaults of North American corporate bond issuers and we have gathered a sample of data for the period from 1986 through 2005. Assume that these corporate defaults occur according to a Poisson distribution. Denoting the 20 observations by x_1, x_2, \dots, x_{20} , we write the likelihood function for the Poisson parameter θ (the average rate of defaults) as⁴

$$\begin{aligned} L(\theta | x_1, x_2, \dots, x_{20}) &= \prod_{i=1}^{20} p(X = x_i | \theta) = \prod_{i=1}^{20} \frac{\theta^{x_i}}{x_i!} e^{-\theta} \\ &= \frac{\theta^{\sum_{i=1}^{20} x_i}}{\prod_{i=1}^{20} x_i!} e^{-20\theta}. \end{aligned} \quad (2.4)$$

As we see in later chapters, it is often customary to retain in the expressions for the likelihood function and the probability distributions only the terms that contain the unknown parameter(s); that is, we get rid of the terms that are constant with respect to the parameter(s). Thus, (2.4) could be written as

$$L(\theta | x_1, x_2, \dots, x_{20}) \propto \theta^{\sum_{i=1}^{20} x_i} e^{-20\theta}, \quad (2.5)$$

where \propto denotes “proportional to.” Clearly, for a given sample of data, the expressions in (2.4) and (2.5) are proportional to each other and therefore contain the same information about θ . Maximizing either of them with

³The Poisson distribution is employed in the context of finance (most often, but not exclusively, in the areas of credit risk and operational risk) as the distribution of a stochastic process, called *the Poisson process*, which governs the occurrences of random events.

⁴In this example, we assume, perhaps unrealistically, that θ stays constant through time and that the annual number of defaults in a given year is independent from the number of defaults in any other year within the 20-year period. The independence assumption means that each observation of the number of annual defaults is regarded as a realization from a Poisson distribution with the same average rate of defaults, θ ; this allows us to represent the likelihood function as the product of the mass function at each observation.