Bayesian Methods in Finance

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S.T.R.
To Iliana and Zoya

J.S.J.H.
To Serene, Justin, and Andrew

B.S.B.
To my mother, Gökhan, and my other loved ones

F.J.F.
To my wife Donna and my children Francesco,
Patricia, and Karly

Contents

Preface

This book provides the fundamentals of Bayesian methods and their applications to students in finance and practitioners in the financial services sector. Our objective is to explain the concepts and techniques that can be applied in real-world Bayesian applications to financial problems. While statistical modeling has been used in finance for the last four or five decades, recent years have seen an impressive growth in the variety of models and modeling techniques used in finance, particularly in portfolio management and risk management. As part of this trend, Bayesian methods are enjoying a rediscovery by academics and practitioners alike and growing in popularity. The choice of topics in this book reflects the current major developments of Bayesian applications to risk management and portfolio management.

Three fundamental factors are behind the increased adoption of Bayesian methods by the financial community. Bayesian methods provide (1) a theoretically sound framework for combining various sources of information; (2) a robust estimation setting that incorporates explicitly estimation risk; and (3) the flexibility to handle complex and realistic models. We believe this book is the first of its kind to present and discuss Bayesian financial applications. The fundamentals of Bayesian analysis and Markov Chain Monte Carlo are covered in Chapters 2 through 5 and the applications are introduced in the remaining chapters. Each application presentation begins with the basics, works through the frequentist perspective, followed by the Bayesian treatment.

The applications include:

- The Bayesian approach to mean-variance portfolio selection and its advantages over the frequentist approach (Chapters 6 and 7).
- A general framework for reflecting degrees of belief in an asset pricing model when selecting the optimal portfolio (Chapters 6 and 7).
- Bayesian methods to portfolio selection within the context of the Black-Litterman model and extensions to it (Chapter 8).
- Computing measures of market efficiency and the way predictability influences optimal portfolio selection (Chapter 9).

- Volatility modeling (ARCH-type and SV models) focusing on the various numerical methods available for Bayesian estimation (Chapters 10, 11, and 12).
- Advanced techniques for model selection, notably in the setting of nonnormality of stock returns (Chapter 13).
- Multifactor models of stock returns, including risk attribution in both an analytical and a numerical setting (Chapter 14).

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Bayesian Methods in Finance

CHAPTER 1

Introduction

Quantitative financial models describe in mathematical terms the relationships between financial random variables through time and/or across assets. The fundamental assumption is that the model relationship is valid independent of the time period or the asset class under consideration. Financial data contain both meaningful information and random noise. An adequate financial model not only extracts optimally the relevant information from the historical data but also performs well when tested with new data. The uncertainty brought about by the presence of data noise makes imperative the use of statistical analysis as part of the process of financial model building, model evaluation, and model testing.

Statistical analysis is employed from the vantage point of either of the two main statistical philosophical traditions—"frequentist" and "Bayesian." An important difference between the two lies with the interpretation of the concept of probability. As the name suggests, advocates of frequentist statistics adopt a *frequentist* interpretation: The probability of an event is the limit of its long-run relative frequency (i.e., the frequency with which it occurs as the amount of data increases without bound). Strict adherence to this interpretation is not always possible in practice. When studying rare events, for instance, large samples of data may not be available and in such cases proponents of frequentist statistics resort to theoretical results. The Bayesian view of the world is based on the *subjectivist* interpretation of probability: Probability is subjective, a degree of belief that is updated as information or data are acquired.[1]

[1]The concept of subjective probability is derived from arguments for rationality of the preferences of agents. It originated in the 1930s with the (independent) works of Bruno de Finetti and Frank Ramsey, and was further developed by Leonard Savage and Dennis Lindley. The subjective probability interpretation can be traced back to the Scottish philosopher and economist David Hume, who also had philosophical influence over Harry Markowitz (by Markowitz's own words in his autobiography

Closely related to the concept of probability is that of uncertainty. Proponents of the frequentist approach consider the source of uncertainty to be the randomness inherent in realizations of a random variable. The probability distributions of variables are not subject to uncertainty. In contrast, Bayesian statistics treats probability distributions as uncertain and subject to modification as new information becomes available. Uncertainty is implicitly incorporated by probability updating. The probability beliefs based on the existing knowledge base take the form of the *prior probability*. The *posterior probability* represents the updated beliefs.

Since the beginning of last century, when quantitative methods and models became a mainstream tool to aid in understanding financial markets and formulating investment strategies, the framework applied in finance has been the frequentist approach. The term "frequentist" usually refers to the Fisherian philosophical approach named after Sir Ronald Fisher. Strictly speaking, "Fisherian" has a broader meaning as it includes not only frequentist statistical concepts such as unbiased estimators, hypothesis tests, and confidence intervals, but also the maximum likelihood estimation framework pioneered by Fisher. Only in the last two decades has Bayesian statistics started to gain greater acceptance in financial modeling, despite its introduction about 250 years ago by Thomas Bayes, a British minister and mathematician. It has been the advancements of computing power and the development of new computational methods that has fostered the growing use of Bayesian statistics in finance.

On the applicability of the Bayesian conceptual framework, consider an excerpt from the speech of former chairman of the Board of Governors of the Federal Reserve System, Alan Greenspan:

> *The Federal Reserve's experiences over the past two decades make it clear that uncertainty is not just a pervasive feature of the monetary policy landscape; it is the defining characteristic of that landscape. The term "uncertainty" is meant here to encompass both "Knightian uncertainty," in which the probability distribution of outcomes is unknown, and "risk," in which uncertainty of outcomes is delimited by a known probability distribution. [...] This conceptual framework emphasizes understanding as much as possible the many sources of risk and uncertainty that policymakers face, quantifying those risks when possible, and assessing the costs associated with each of the risks. In essence, the risk management*

published in *Les Prix Nobel* (1991)). Holton (2004) provides a historical background of the development of the concepts of risk and uncertainty.

> *approach to monetary policymaking is an application of Bayesian [decision-making].*[2]

The three steps of Bayesian decision making that Alan Greenspan outlines are:

1. Formulating the prior probabilities to reflect existing information.
2. Constructing the quantitative model, taking care to incorporate the uncertainty intrinsic in model assumptions.
3. Selecting and evaluating a utility function describing how uncertainty affects alternative model decisions.

While these steps constitute the rigorous approach to Bayesian decision-making, applications of Bayesian methods to financial modeling often only involve the first two steps or even only the second step. This tendency is a reflection of the pragmatic Bayesian approach that researchers of empirical finance often favor and it is the approach that we adopt in this book.

The aim of the book is to provide an overview of the theory of Bayesian methods and explain their applications to financial modeling. While the principles and concepts explained in the book can be used in financial modeling and decision making in general, our focus will be on portfolio management and market risk management since these are the areas in finance where Bayesian methods have had the greatest penetration to date.[3]

A FEW NOTES ON NOTATION

Throughout the book, we follow the convention of denoting vectors and matrices in boldface.

We make extensive use of the proportionality symbol, '$\propto$', to denote the cases where terms constant with respect to the random variable of interest have been dropped from that variable's density function. To illustrate, suppose that the random variable, $\boldsymbol{X}$, has a density function

$$p(x) = 2x. \tag{1.1}$$

[2]Alan Greenspan made these remarks at the Meetings of the American Statistical Association in San Diego, California, January 3, 2004.

[3]Bayesian methods have been applied in corporate finance, particularly in capital budgeting. An area of Bayesian methods with potentially important financial applications is Bayesian networks. Bayesian networks have been applied in operational risk modeling. See, for example, Alexander (2000) and Neil, Fenton, and Tailor (2005).

Then, we can write

$$p(x) \propto x. \tag{1.2}$$

Now suppose that we take the logarithm of both sides of (1.2). Since the logarithm of a product of two terms is equivalent to the sum of the logarithms of those terms, we obtain

$$\log(p(x)) = \text{const} + \log(x), \tag{1.3}$$

where *const* $= \log(2)$ in this case. Notice that it would not be precise to write $\log(p(x)) \propto \log(x)$. We come across the transformation in (1.3) in Chapters 10 through 14, in particular.

OVERVIEW

The book is organized as follows. In Chapters 2 through 5, we provide an overview of the theory of Bayesian methods. The depth and scope of that overview are subordinated to the methodological requirements of the Bayesian applications discussed in later chapters and, therefore, in certain instances lacks the theoretical rigor that one would expect to find in a purely statistical discussion of the topic.

In Chapters 6 and 7, we discuss the Bayesian approach to mean-variance portfolio selection and its advantages over the frequentist approach. We introduce a general framework for reflecting degrees of belief in an asset pricing model when selecting the optimal portfolio. We close Chapter 7 with a description of Bayesian model averaging, which allows the decision maker to combine conclusions based on several competing quantitative models.

Chapter 8 discusses an emblematic application of Bayesian methods to portfolio selection—the Black-Litterman model. We then show how the Black-Litterman framework can be extended to active portfolio selection and how trading strategies can be incorporated into it.

The focus of Chapter 9 is market efficiency and predictability. We analyze and illustrate the computation of measures of market inefficiency. Then, we go on to describe the way predictability influences optimal portfolio selection. We base that discussion on a Bayesian *vector autoregressive* (VAR) framework.

Chapters 10, 11, and 12 deal with volatility modeling. We devote Chapter 10 to an overview of volatility modeling. We introduce the two types of volatility models—*autoregressive conditionally heteroskedastic* (ARCH)-type models and *stochastic volatility* (SV) models—and discuss some of their important characteristics, along with issues of estimation within the boundaries of frequentist statistics. Chapters 11 and 12 cover, respectively, ARCH-type and SV Bayesian model estimation. Our focus is on the various numerical methods that could be used in Bayesian estimation.

In Chapter 13, we deal with advanced techniques for model selection, notably, recognizing nonnormality of stock returns. We first investigate an approach in which higher moments of the return distribution are explicitly included in the investor's utility function. We then go on to discuss an extension of the Black-Litterman framework that, in particular, employs minimization of the conditional *value-at-risk* (CVaR). In Appendix A of that chapter, we present an overview of risk measures that are alternatives to the standard deviation, such as *value-at-risk* (VaR) and CVaR.

Chapter 14 is devoted to multifactor models of stock returns. We discuss risk attribution in both an analytical and a numerical setting and examine how the multifactor framework provides a natural setting for a coherent portfolio selection and risk management approach.

CHAPTER 2

The Bayesian Paradigm

Likelihood Function and Bayes' Theorem

One of the basic mechanisms of learning is assimilating the information arriving from the external environment and then updating the existing knowledge base with that information. This mechanism lies at the heart of the Bayesian framework. A Bayesian decision maker learns by revising beliefs in light of the new data that become available. From the Bayesian point of view, probabilities are interpreted as degrees of belief. Therefore, the Bayesian learning process consists of revising of probabilities.[1] Bayes' theorem provides the formal means of putting that mechanism into action; it is a simple expression combining the knowledge about the distribution of the model parameters and the information about the parameters contained in the data.

In this chapter, we present some of the basic principles of Bayesian analysis.

THE LIKELIHOOD FUNCTION

Suppose we are interested in analyzing the returns on a given stock and have available a historical record of returns. Any analysis of these returns, beyond a very basic one, would require that we make an educated guess about (propose) a process that might have generated these return data. Assume that we have decided on some statistical distribution and denote it by

$$p\left(y \mid \theta\right), \tag{2.1}$$

[1]Contrast this with the way probability is interpreted in the classical (frequentist) statistical theory—as the relative frequency of occurrence of an event in the limit, as the number of observations goes to infinity.

where y is a realization of the random variable Y (stock return) and θ is a parameter specific to the distribution, p. Assuming that the distribution we proposed is the one that generated the observed data, we draw a conclusion about the value of θ. Obviously, central to that goal is our ability to summarize the information contained in the data. The likelihood function is a statistical construct with this precise role. Denote the n observed stock returns by $y_1, y_2, \ldots, y_n$. The joint density function of Y, for a given value of θ, is[2]

$$f(y_1, y_2, \ldots, y_n \mid \theta).$$

We can observe that the function above can also be treated as a function of the unknown parameter, θ, given the observed stock returns. That function of θ is called *the likelihood function*. We write it as

$$L(\theta \mid y_1, y_2, \ldots, y_n) = f(y_1, y_2, \ldots, y_n \mid \theta). \tag{2.2}$$

Suppose we have determined from the data two competing values of θ, θ_1 and θ_2, and want to determine which one is more likely to be the true value (at least, which one is closer to the true value). The likelihood function helps us make that decision. Assuming that our data were indeed generated by the distribution in (2.1), θ_1 is more likely than θ_2 to be the true parameter value whenever $L(y_1, y_2, \ldots, y_n \mid \theta_1) > L(y_1, y_2, \ldots, y_n \mid \theta_2)$. This observation provides the intuition behind the method most often employed in "classical" statistical inference to estimate θ from the data alone—*the method of maximum likelihood*. The value of θ most likely to have yielded the observed sample of stock return data, $y_1, y_2, \ldots, y_n$, is *the maximum likelihood estimate*, $\widehat{\theta}$, obtained from maximizing the likelihood function in (2.2).

To illustrate the concept of a likelihood function, we briefly discuss two examples—one based on the Poisson distribution (a discrete distribution) and another based on the normal distribution (one of the most commonly employed continuous distributions).

The Poisson Distribution Likelihood Function

The Poisson distribution is often used to describe the random number of events occurring within a certain period of time. It has a single parameter,

[2]By using the term "density function," we implicitly assume that the distribution chosen for the stock return is continuous, which is invariably the case in financial modeling.

θ, indicating the rate of occurrence of the random event, that is, how many events happen on average per unit of time. The probability distribution of a Poisson random variable, X, is described by the following expression:[3]

$$p\left(X=k\right)=\frac{\theta^{k}}{k!}e^{-\theta},\qquad k=0,1,2,\ldots. \tag{2.3}$$

Suppose we are interested to examine the annual number of defaults of North American corporate bond issuers and we have gathered a sample of data for the period from 1986 through 2005. Assume that these corporate defaults occur according to a Poisson distribution. Denoting the 20 observations by x_1, x_2, ..., x_{20}, we write the likelihood function for the Poisson parameter θ(the average rate of defaults) as[4]

$$\begin{aligned} L\left(\theta\,|\,x_1,x_2,\ldots,x_{20}\right) &= \prod_{i=1}^{20} p\left(X=x_i\,|\,\theta\right) = \prod_{i=1}^{20}\frac{\theta^{x_i}}{x_i!}e^{-\theta} \\ &= \frac{\theta^{\sum_{i=1}^{20}x_i}}{\prod_{i=1}^{20}x_i!}e^{-20\theta}. \end{aligned} \tag{2.4}$$

As we see in later chapters, it is often customary to retain in the expressions for the likelihood function and the probability distributions only the terms that contain the unknown parameter(s); that is, we get rid of the terms that are constant with respect to the parameter(s). Thus, (2.4) could be written as

$$L\left(\theta\,|\,x_1,x_2,\ldots,x_{20}\right)\propto\theta^{\sum_{i=1}^{20}x_i}e^{-20\theta}, \tag{2.5}$$

where $\propto$ denotes "proportional to." Clearly, for a given sample of data, the expressions in (2.4) and (2.5) are proportional to each other and therefore contain the same information about θ. Maximizing either of them with

[3]The Poisson distribution is employed in the context of finance (most often, but not exclusively, in the areas of credit risk and operational risk) as the distribution of a stochastic process, called *the Poisson process*, which governs the occurrences of random events.

[4]In this example, we assume, perhaps unrealistically, that θ stays constant through time and that the annual number of defaults in a given year is independent from the number of defaults in any other year within the 20-year period. The independence assumption means that each observation of the number of annual defaults is regarded as a realization from a Poisson distribution with the same average rate of defaults, θ; this allows us to represent the likelihood function as the product of the mass function at each observation.