Understanding the Mathematics of Personal Finance

An Introduction to Financial Literacy

Lawrence N. Dworsky

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To all the people struggling to understand the calculations behind the various financial instruments they encounter: I hope this book helps.

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Preface

What is personal finance? An informal definition is "how you interact with money." Among the subcategories of personal finance are topics such as budgeting, saving, borrowing, investing, gambling, and buying and selling real estate. Many books, courses, professional advisors, and software programs are available to help you optimize your path through your financial life.

This book is about various forms of borrowing and saving money, and includes some discussion of investing money. Borrowing money takes many forms, including home mortgage loans, auto loans, and credit card debt. Saving money includes putting money under your mattress, depositing it into a savings bank, and buying certificates of deposit (CDs). Insurance policies can be thought of as a special kind of pooled savings plan whereby many people put money into the same savings account, and this money becomes available to these people when a specified special need (illness, repairing a car, death benefit) unexpectedly arises. Investing is an opportunity to earn more money with your money than a savings bank will give you, but with less certainty about the earnings and, for that matter, less certainty about maintaining your original money than a government-insured savings account would give you.

When you borrow money or, equivalently, take a loan from a person, a bank, a mortgage company, or elsewhere, you will be expected to pay a fee for the use of this money. The amount you borrow is called the *principal* of your loan and the fee you pay for borrowing the money is called the *interest*. The amount of interest you have to pay is based upon the principal, the amount of time you have the money, and the prevailing financial conditions. The longer you have this money, the more interest you can expect to pay. In common situations such as a home mortgage or a car loan, you usually repay the loan gradually over a period of time. In this case, calculating the interest gets a little messy because the amount you owe at any given time (the *balance*) is being reduced due to your payments, while it is simultaneously being increased by the accrual of interest based on your balance at that time. In a properly structured loan, your payments are large enough that the balance decreases after each payment and eventually goes to 0, so that your loan is *paid off*.

The concepts and calculations for a simple one repayment loan and for multiple payment loans such as mortgages and car loans are the same; it's just that in the latter cases you have to repeat the same calculations many times. Before the era of spreadsheets on personal computers and the Internet, the complexity of the multiple calculations was so significant that only banks and mortgage companies and other large financial institutions could undertake them. When you took a loan, you would be provided with a table of payment due dates and loan balances (an *amortization* table) for your loan. Comparing different loan opportunities was very difficult unless you wanted to spend a lot of time in the library working with books of loan tables.

Today, everybody can easily calculate loan details themselves. Pocket calculators with all the necessary financial functions built-in are inexpensive and easy to use. Users of spreadsheet programs on personal computers can generate their own amortization tables based on the financial functions built into these spreadsheets and/or can build up these formulas from basic principles. Most common financial calculations are available on the Internet ("online") in the form of simple calculators designed specifically for a single type of problem.

My goal in writing the book is to explain how even the most involved loan scenarios can be understood just by repeated application of the fundamental concept of *compound interest*, which is the subject of Chapter 2. I'll show how to calculate everything involved with these loans using a computer spreadsheet program, and whenever possible, I'll reference some online calculators—particularly those on my own website.

I should mention here that I'm using "loan" as a generic term for one party letting another party use his or her money for some time and expecting interest as compensation. When you take a mortgage loan on a home, you are borrowing the money from somebody. When you put money into a savings bank or purchase a CD ("invest" your money), the bank is borrowing money from you. In terms of the mathematics involved, these are identical situations—you just have to keep track of which way dollars are flowing.

If you *loan* me money, then I am *borrowing* money from you and vice versa. In terms of usage, I often see that the terms *loan* and *borrow* are used interchangeably. In many situations that you encounter, you'll simply have to pull the correct meaning out of context. This is unfortunate because each term has a specific meaning; they're not interchangeable. I will admit that the correct usage can sometimes be confusing—when I *take a loan*, I'm *borrowing* money. The person or company that loaned me the money is the *lender*, and once I've borrowed the money I am the *debtor*.

This is not a book that gives investment or borrowing strategies. I won't offer suggestions on how to plan for retirement, whether or not you want a reverse mortgage, how to allocate your savings, and so on. My goal is to provide the tools for you to be able to calculate the real costs and/or profits involved in using these various financial instruments and therefore to put you in a position to see for yourself what the best deals are and/or how you could sometimes get yourself into a financial mess.

The most important concept to hold in your mind is that because of interest accruing on borrowed money, the amount of money you owe (or are owed) has a *time value* to it. One thousand dollars to be paid to you today is worth more than \$1,000 to be paid to you a year from now. One thousand dollars to be paid to you a year from now is worth more than \$2,000 to be paid to you 20 years from now. You must learn to work with concepts such as *future value*, which is the amount that some number of dollars today will be worth on a specific date in the future, and also the *present value*, which is the amount that some number of dollars on some specific date in the future is worth today.

In this book, you will find descriptions of various financial instruments (mortgages, credit card purchases, cash advances, etc.) You will also see how these financial instruments work and how to use the proper analysis tools (primarily the computer spreadsheet) well enough that you can tackle a new situation and come up with the right answers.

There are many computer spreadsheet programs available. Fortunately, they are all very similar in structure, and the instructions I give for my spreadsheets will work on all popular spreadsheet programs.

The spreadsheet calculators used in this book are all available on my website (www.lawrencedworsky.com). Chapter 15 shows you how to get a free spreadsheet program if you need one, how to get to the spreadsheets I'm providing, and a general introduction on how I'm setting them up and how to use and maintain them.

In a sense, this book will never be finished. My website will always be changing. I will improve the existing spreadsheets, adding examples and explanations as well as new capabilities. I will have an up-to-date *errata* section (that hopefully will be very, very, short). Also, my website has the typical Contact Me capability. This is how I will learn what I haven't explained well, what relevant facts or scenarios I have overlooked, and so on. I will address all of these matters and put my work on the website as quickly as possible. Interesting problems may become additional problems for the book, posted on the website.

Chapter 1 contains a review of the basic mathematics necessary to understand the book. Most readers shouldn't find this math difficult. The only new information presented is that the notation isn't usually what was taught to you in high school. I'll go through this slowly and carefully. There are powerful notations to properly express calculations that you probably already know how to do. These notations are important because they can describe involved calculations clearly and concisely.

Included in the book are a few sections of mathematical nature that delve a bit more deeply into a topic than does most of the book. These sections are not necessary for a good understanding of the book or use of the calculator spreadsheets and can be skipped if you wish. I'll clearly state at the beginning of each of these sections that you can skip the section if you don't want to wrestle with the mathematics.

Lawrence N. Dworsky

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I thank you all.

List of Abbreviations

ADB	Average Daily Balance
APR	Annual Percentage Rate
ARM	Adjustable Rate Mortgage
ATM	Automatic Teller Machine
CD	Certificate of Deposit
Cmpds	Compoundings
CORR	Corrected
COL	Cost of Living
DB	Daily Balance
EAPR	Effective Annual Percentage Rate
HECM	Home Energy Conversion Mortgage
HUD	(Department of) Housing and Urban Development
IAWPC	Immediate Annuity with Period Certain
INC	Income
Int or INT	Interest
IOU	I Owe You (a promissory note)
IRA	Individual Retirement Account
Mnth	Month
NAPR	Nominal Annual Percentage Rate—same as APR
Nr	Number
NrPmts	Number of Payments
PMT	Payment
PV	Present Value
SEC	(U.S.) Securities and Exchange Commission
SEP	Single Employer Pension
Tot or TOT	Total
Vol	Volume (number of shares traded in a given time)

Chapter 1

Background Mathematics

1.1 ARITHMETIC, NOTATION, AND FORMULAS

Almost all of the mathematics used in this book involve only the four basic operations of addition, subtraction, multiplication, and division. If you can comfortably read about and then actually perform calculations using these four operations, you have all the math background you need. If you have a pocket calculator or a computer with a spreadsheet program, then you have the "machine power" to do whatever you need to do without resorting to pencil and paper.

Mathematical *notation*, the way we express what we want to calculate, can sometimes be confusing. Mathematical notation is the vocabulary of the language of mathematical concepts. Often a student will think he or she doesn't understand a concept when he or she simply is not familiar with the notation. It's like being given driving directions in a foreign language when you don't know the words for "turn left" or "turn right." To further complicate things, there is almost always more than one way to write a particular mathematical expression. Often the choice is a matter of style and/or convenience. In this section, I'll go through the various ways of writing different expressions involving only the four basic operations and explain why I will choose what I choose when I choose it.

I'll begin with the definition of a *variable*. A variable, simply speaking, is a letter or a name that represents a number. If I want to say, for example, that an item in a catalog costs the price listed in the catalog plus a $10 shipping and handling fee, I can write, "If the cost of an item is X dollars, then you must pay $X+10$ dollars if you want to order any item from this catalog." It's a way of generalizing a relationship rather than having to recreate the relationship for each example.

I can get fancier and say that if Y represents the amount you must pay, then

$$Y = X + 10.$$

Some people like to use letters from the end of the alphabet; some like to use letters from the beginning of the alphabet; some like to use Greek letters. It doesn't matter which letters are used, as long as you're clearly told what number each of

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these letters represent. Some authors of books and computer programs use variables that are case sensitive. That is, X and x represent different numbers. I don't do this either in this book or in my spreadsheets.

In some situations, it's convenient to use a whole word as the variable. In the above example, instead of letting Y be the cost, I could have written

$$\text{Cost} = X + 10.$$

The expression

$$Y = X + 10$$

is called a *formula*. A formula is a mathematician's version of a recipe. You put in X (in this case the catalog price for the item) and you get out Y (in this case the amount you must pay to have the item appear on your doorstep). It's conventional, but not necessary, for the variable that you're calculating to appear on the left-hand side of the equal sign and the variable(s) that you're supplying to appear on the right-hand side of the equal sign.

Typically, numbers that don't change are shown as numbers, such as the 10 in the above formula, and numbers that depend on your particular situation, such as X and Y in the above formula, are represented by letters. This isn't a law; it's just a common practice.

If, for example, the shipping cost depends on the item's weight, I could say that

$$Y = X + Z,$$

where Z is the shipping cost. If I do this, then I must refer you to a table or to another formula that explains how to calculate or look up the shipping cost before you can use this formula.

I'll start my discussion of the four basic operations with addition, spelling out some things that are probably obvious. I'm doing this in order to be able to draw a contrast with the other basic functions and also to start explaining the use of parentheses.

When adding numbers, it doesn't matter what order you do things in:

$$3 + 7 + 5 = 3 + 5 + 7 = 7 + 5 + 3 = \ldots$$

Note also the use of the expression "…". This means "and so on" and hopefully will be obvious in its intent when I use it.

I could also write the above addition example using parentheses to group some of the operations:

$$(3 + 7) + 5 = .$$

Writing it this way means "add 3 to 7 first, then add the result to 5." In this example, the parentheses don't contribute any value since the order of the additions doesn't matter. On the other hand, they don't introduce any error. In short, in this example, while the parentheses are harmless, they're also pointless.

Now, let's look at subtraction:

$$7 - 3 = 4.$$

This is pretty clear so far.

However, while 2 + 6 is the same as 6 + 2,

$$6-2=4 \text{ and } 2-6=-4$$

are clearly not the same.

Getting a little more complicated,

$$(7-3)+2=6.$$

The parentheses here mean first evaluate 7 − 3 (= 4) and then add the result to 2, yielding 6.

This is not the same as

$$7-(3+2)=2.$$

In this case, the instructions are to first add 3 + 2 (= 5) and then subtract the result from 7, yielding 2.

Subtraction differs from addition in the importance of notation because the order in which things are calculated matters.

If I were to just write

$$7-3+2=??,$$

I wouldn't know how to evaluate this because without the instructions added by the parentheses, I just don't know what to do first.[1]

Moving on to multiplication, the simplest notation (and one that's hardly ever used) is to use an "×" to indicate multiplication. Using this notation, it's clear to see that, as in the case of addition, order doesn't matter:

$$3\times2\times6=3\times6\times2=6\times2\times3=\ldots=36$$

One good reason why the "×" is hardly used to signify multiplication is that, once you're expecting formulas, you don't know whether this × means multiplication or is itself a variable representing another number.

For better or worse, there are many notations for multiplication. The important consideration is that the chosen notation must be clear and unambiguous.

When it is clear what I mean, I will just write the two numbers (and/or variables) that I want to multiply next to each other: $3x$, or xy. Obviously this won't work for multiplying 3 by 2, because 32 (or 23) would be interpreted as a two-digit number, not instructions to multiply the two single-digit numbers together.

When multiplying a number by a variable, it's common to put the number first: $3x$ means the same as $x3$ but is almost always written as $3x$.

This example also shows why multiplication is almost never written using an "x" to signify multiplication—the "x" is probably the most common choice of a letter for a variable, and writing $3 \times x$ to mean "multiply 3 by the variable x" is just a confusing mess.

[1] Computer programming languages usually resolve this kind of ambiguity by having a default procedure such as "when there are no explicit instructions (parentheses), work from left to right." I won't assume any such default procedures in this book.

The expression

$$3x(y+7)$$

can be interpreted two ways. The two ways are equivalent and both are valid.

The first interpretation is that you should do what's inside the parentheses first. That is, if y represents some cost or payment or whatever, let's say y = \$15.50, then add y to 7, giving 15.50 + 7 = 22.50. Then we have

$$3x(22.50).$$

This is simply three numbers multiplied together. The parentheses now are used just to keep the 3 from being tangled up with the 22.50. Since numbers multiplied together can be multiplied in any order, we have

$$3x(22.50) = 3(22.50)x = 67.50x$$

At this point we need a value for x or we just have to stop.

In order to do what I just did, I needed a value for the variable y. If I don't have a value for y, I can either leave things as they are for the time being, or I can "expand" the expression. This is the second interpretation: What's outside the parentheses multiplies everything that's inside the parentheses. Therefore,

$$3x(y+7) = 3x(y) + 3x(7) = 3xy + 21x$$

Whether the latter way of writing things is any clearer, or more useful, than the original expression is in the eye of the beholder.

Taking this one step further, what if I have

$$(12+4)(3+6).$$

The same rules apply; you just have to do a little more work:

$$(12+4)(3+6) = (16)(9) = 144.$$

This type of expression, when there are algebraic variables involved, often trips up students. Another correct way of evaluating this expression is to use the second interpretation above: What is outside the parentheses multiplies everything that is inside the parentheses. This time we have to remember that there are two sets of parentheses, so we have

$$\begin{aligned}(12+4)(3+6) &= 12(3+6) + 4(3+6) = 12(3) + 12(6) + 4(3) + 4(6)\\ &= 36 + 72 + 12 + 24 = 144.\end{aligned}$$

This is sometimes called "expanding" the expression.

An example of the same expression with algebraic variables is

$$(a+b)(c+d) = ac + ad + bc + bd.$$

In this book, I'll often be presenting formulas for use in calculating a number, typically a dollar value. This isn't an algebra book. You will have the working knowledge that you need as long as you understand the first interpretation, that is, put in the values for the variables then evaluate what's inside the parentheses (or sets of parentheses).

The last of the four basic operations is division. First, notation: The elementary school notation $6 \div 3$ is pretty much never used. Instead, recognizing that a division expression is the same as a fraction expression, 6 divided by 3 will be written as one of the following:

$$6/3 = \frac{6}{3} = 2.$$

If I want to use the first of these forms for multiple operations, then I have to get involved with parentheses, because order counts. That is,

$$18/6/3 =$$

is ambiguous because I don't know what to do first. My choices are

$$(18/6)/3 = 3/3 = 1$$

or

$$18/(6/3) = 18/2 = 9,$$

and I have no way of knowing which interpretation was intended.

Using the fraction notation, I can finesse the parentheses issue by working with different size fraction lines. That is,

$$\frac{\frac{18}{6}}{3} = \frac{3}{3} = 1$$

while

$$\frac{18}{\frac{6}{3}} = \frac{18}{2} = 9.$$

I could keep going with compound expressions—say, fractions involving sums or differences of numbers and variables inside the parentheses, and so on. But again, this is not an algebra book and my aim is not to trip you up but to give you clear rules for evaluating (finding the numeric value of) a formula when you're presented with it.

1.2 MINUS (NEGATIVE) SIGNS

The seemingly benign set of rules for manipulating a minus sign nevertheless manages to cause an endless set of headaches. Let's see if I can summarize these rules quickly and clearly:

1. (Not so much a rule as a reminder) When a sign is not shown, a positive sign is implied:

$$34 = +34,$$
$$(35) = (+35) = +(35).$$

2. Subtracting B from A is the same as adding –B to A:

$$7-5=7+(-5)=2.$$

3. As implied above, multiplying a positive number by a negative number yields a negative number:

$$(-3)(6)=-(3)(6)=-18,$$
$$(-6)=-(6)=(-1)(6).$$

4. Multiplying two negative numbers yields a positive number:

$$(-5)(-7)=+35.$$

5. Division rules are the same as multiplication rules. Dividing a positive number by a negative number or dividing a negative number by a positive number yields a negative number. Dividing a negative number by a negative number yields a positive number:

$$\frac{4}{-3}=\frac{-4}{3}=-\left(\frac{4}{3}\right)=-\frac{4}{3}=-1.33,$$
$$\frac{-4}{-3}=\frac{4}{3}=1.33.$$

1.3 LISTS AND SUBSCRIPTED VARIABLES

Throughout this book, I make frequent use of tables. Tables are lists of numbers that relate variables in different situations. This isn't as bad as it first sounds. I'm sure you've all seen this many times—everything from income tax tables that the Internal Revenue Service provides to automobile value depreciation tables.

Table 1.1 is a hypothetical automobile value depreciation table. Don't worry about what kind of car it is—I just made up the numbers for the sake of this example.

Looking from left to right, you see two columns: the age of the car and the car's wholesale price. Looking from top to bottom you see six rows. The top row contains the headings, or descriptions, of what the numbers beneath mean. Then there are

Table 1.1 Hypothetical Automobile Value Depreciation Table

Age of car (years)	Wholesale price ($)
0	32,000
1	26,500
2	21,300
3	18,000
4	15,500
5	13,250

Table 1.2 Hypothetical Automobile Depreciation Table with Air-Conditioning Option

Age of car (years)	Wholesale price ($)	Extra for air-conditioning
0	32,000	1,200
1	26,500	1,050
2	21,300	850
3	18,000	650
4	15,500	550
5	13,250	450

five rows of numbers. The numbers on each row "belong together." For example, when the car is 2 years old, the wholesale price is $21,300.

An important point about the headings is that whenever appropriate, the *units* should be listed. In Table 1.1, the age of the car is expressed in years. If I didn't say so, how would you know I didn't mean months, or decades? The value of the car is expressed in dollars. To be very precise, maybe I should have said U.S. dollars (if that's what I meant). Someone in Great Britain could easily assume that the prices are in pounds if I didn't clearly state otherwise.

Very often a table will have many columns. Table 1.2 is a repeat of Table 1.1, but with a third column added: How much more the car is worth if it has air-conditioning. Notice that I was a little sloppy here. I didn't say that the extra amount was in dollars. In this case, however, a little sloppiness is harmless. Once you know that we're dealing in dollars, you can be pretty sure that things will be consistent.

Again, the numbers in a given row belong together: A 3-year-old car is worth $18,000, and it is worth $650 more if it has air-conditioning.

Tables 1.1 and 1.2 tell you some dollar amounts based on the age of the car. It's therefore typical for the age of the car to appear in the leftmost column. I could have put the car's age in the middle column (of Table 1.2) or in the right column. Even though doing this wouldn't introduce any real errors, it makes things harder to read.

Whenever convenient, columns are organized from left to right in order of decreasing importance. That is, I could have made the air-conditioning increment the second column and the car value the third column (always count columns from the left), but again it's clearer if I put the more important number to the left of the less important number.

Some tables have many, many rows. The Life Tables presented in Chapter 10, the chapter about life insurance, have 102 rows—representing ages from 0 to 100, plus the heading row. The second column in the Life Tables is a number represented by the variable q, the third by the variable l, and so on. Don't worry about what these letters mean now; this is a topic in Chapter 10.

In Table 1.3, I've extracted a piece of the Life Table shown in Table 10.1. As you can see, for every age there are six associated pieces of information. Suppose I wanted to compare the values of q for two different ages, or to make some