THE FRANK J. FABOZZI SERIES

Financial Models with Lévy Processes and Volatility Clustering

SVETLOZAR T. RACHEV • YOUNG SHIN KIM
MICHELE LEONARDO BIANCHI • FRANK J. FABOZZI

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STR

To my grandchildren Iliana, Zoya, and Svetlozar

YSK

To my wife Myung-Ja and my son Minseob

MLB

To my wife Giorgia

FJF

To my wife Donna and my children Francesco, Patricia, and Karly

Preface

Carl Frederick Gauss, born in 1777, is one of the foremost mathematicians the world has known. Labeled the "prince of mathematicians" and viewed by some as on par with Sir Isaac Newton, the various works of Gauss have influenced a wide range of fields in mathematics and science. Although very few in the finance profession are familiar with his great contributions and body of work—which are published by the by the Royal Society of Göttingen in seven quatro volumes—most are familiar with his important work in probability theory that bears his name: the Gaussian distribution. The more popular name for this distribution is the normal distribution and was also referred to as the "bell curve" in 1733 by Abraham de Moivre, who first discovered this distribution based on his empirical work. Every finance professional who has taken a probability and statistics course has had a heavy dose of the Gaussian distribution and probably can still recite some properties of this distribution.

The normal distribution has found many applications in the natural sciences and social sciences. However, there are those who have long warned about the misuse of the normal distribution, particularly in the social sciences. In a 1981 article in *Humanity and Society*, Ted Goertzel and Joseph Fashing ("The Myth of the Normal Curve: A Theoretical Critique and Examination of its Role in Teaching and Research") argue that

> *The myth of the bell curve has occupied a central place in the theory of inequality ... Apologists for inequality in all spheres of social life have used the theory of the bell curve, explicitly and implicitly, in developing moral rationalizations to justify the status quo. While the misuse of the bell curve has perhaps been most frequent in the*

field of education, it is also common in other areas of social science and social welfare.

A good example is in the best-selling book by Richard Herrnstein and Charles Murray, *The Bell Curve*, published in 1994 with the subtitle *Intelligence and Class Structure in American Life*. The authors argue based on their empirical evidence that in trying to predict an individual's income or job performance, intelligence is a better predictor than the educational level or socioeconomic status of that individual's parents. Even the likelihood to commit a crime or to exhibit other antisocial behavior is better predicted by intelligence, as measured by IQ, than other potential explanatory factors. The policy implications drawn from the book are so profound that they set off a flood of books both attacking and supporting the findings of Herrnstein and Murray.

In finance, where the normal distribution was the underlying assumption in describing asset returns in major financial theories such as the capital asset pricing theory and option pricing theory, the attack came in the early 1960s from Benoit Mandelbrot, a mathematician at IBM's Thomas J. Watson Research Center. Although primarily known for his work in fractal geometry, the finance profession was introduced to his study of returns on commodity prices and interest rate movements that strongly rejected the assumption that asset returns are normally distributed. The mainstream financial models at the time relied on the work of Louis Bachelier, a French mathematician who at the beginning of the 20th century was the first to formulate random walk models for stock prices. Bachelier's work assumed that relative price changes followed a normal distribution. Mandelbrot, however, was not the first to attack the use of the normal distribution in finance. As he notes, Wesley Clair Mitchell, an American economist who taught at Columbia University and founded the National Bureau of Economic Research, was the first to do so in 1914. The bottom line is that the findings of Mandelbrot that empirical distributions do not follow a normal distribution led a leading financial economist, Paul Cootner of MIT, to warn the academic community that Mandelbrot's finding may mean that “past econometric work is meaningless.”

The overwhelming empirical evidence of asset returns in real-world financial markets is that they are not normally distributed. In commenting on the normal distribution in the context of its use in the social sciences, “Earnest Ernest” wrote the following in the November 10, 1974, in the *Philadelphia Inquirer*:

> *Surely the hallowed bell-shaped curve has cracked from top to bottom. Perhaps, like the Liberty Bell, it should be enshrined somewhere as a memorial to more heroic days.*

Finance professionals should heed the same advice when using the normal distribution in asset pricing, portfolio management, and risk management.

In Mandelbrot's attack on the normal distribution, he suggested that asset returns are more appropriately described by a non-normal stable distribution referred to as a stable Paretian distribution or alpha-stable distribution (α-stable distribution), so-named because the tails of this distribution have Pareto power-type decay. The reason for describing this distribution as “non-normal stable” is because the normal distribution is a special case of the stable distribution. Because of the work by Paul Lévy, a French mathematician who introduced and characterized the non-normal stable distribution, this distribution is also referred to as the Lévy stable distribution and the Pareto-Lévy stable distribution. (There is another important contribution to probability theory by Lévy that we apply to financial modeling in this book. More specifically, we will apply the Lévy processes, a continuous-stochastic process.)

There are two other facts about asset return distributions that have been supported by empirical evidence. First, distributions have been observed to be skewed or nonsymmetric. That is, unlike in the case of the normal distribution where there is a mirror imaging of the two sides of the probability distribution, typically in a skewed distribution, one tail of the distribution is much longer (i.e., has greater probability of extreme values occurring) than the other tail of the probability distribution. Probability distributions with this attribute are referred to as having fat tails or heavy tails. The second finding is the tendency of large changes in asset prices (either positive or negative) to be followed by large changes, and small changes to be followed by small changes. This attribute of asset return distributions is referred to as volatility clustering.

In this book, we consider these well-established facts about asset return distributions in providing a framework for modeling the behavior of stock returns. In particular, we provide applications to the financial modeling used in asset pricing, option pricing, and portfolio/risk management. In addition to explaining how one can employ non-normal distributions, we also provide coverage of several topics that are of special interest to finance professionals.

We begin by explaining the need for better financial modeling, followed by the basics of probability distributions—the different types of probability distributions (discrete and continuous), specific types of probability distributions, parameters of a probability distribution, and joint probability distributions. The definition of the stable Pareto distribution (we adopted the term α-stable distribution in this book) that Mandelbrot suggested is described. Although this distribution has certain desirable properties and is superior to the normal distribution, it is not suitable in certain financial modeling applications such as the modeling of option prices because the mean, variance, and exponential moments of the return distribution have to exist. For this reason, we introduce distributions that we believe are better suited for financial modeling, distributions obtained by tempering the tail properties of the α-stable distribution: the smoothly truncated stable distribution and various types of tempered stable distributions. Because of their important role in the applications in this book, we review continuous-time stochastic processes with emphasis on Lévy processes.

There are chapters covering the so-called exponential Lévy model, and we study this continuous-time option pricing model and analyze the change of measure problem. Prices of plain vanilla options are calculated with both analytical and Monte Carlo methods.

After examples dealing with the simulation of non-normal random numbers, we study two multivariate settings that are suitable to explain joint extreme events. In the first approach, we describe a multivariate random variable for joint extreme events, and in the second we model the joint behavior of log-returns of stocks by considering a feasible dependence structure together with marginals able to explain volatility clutering.

Then we get into the core of the book where we deal with examples of discrete-time option pricing models. Starting from the classic normal model with volatility clustering, we progress to the more recent models that jointly consider volatility clustering and heavy tails. We conclude with a non-normal GARCH model to price American options.

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Chapter 1

Introduction

1.1 The Need for Better Financial Modeling of Asset Prices

Major debacles in financial markets since the mid-1990s such as the Asian financial crisis in 1997, the bursting of the dot-com bubble in 2000, the subprime mortgage crisis that began in the summer of 2007, and the days surrounding the bankruptcy of Lehman Brothers in September 2008 are constant reminders to risk managers, portfolio managers, and regulators of how often extreme events occur. These major disruptions in the financial markets have led researchers to increase their efforts to improve the flexibility and statistical reliability of existing models that seek to capture the dynamics of economic and financial variables. Even if a catastrophe cannot be predicted, the objective of risk managers, portfolio managers, and regulators is to limit the potential damages.

The failure of financial models has been identified by some market observers as a major contributor—indeed some have argued that it is the single most important contributor—for the latest global financial crisis. The allegation is that financial models used by risk managers, portfolio managers, and even regulators simply did not reflect the realities of real-world financial markets. More specifically, the underlying assumption regarding asset returns and prices failed to reflect real-world movements of these quantities.

Pinpointing the criticism more precisely, it is argued that the underlying assumption made in most financial models is that distributions of prices and returns are normally distributed, popularly referred to as the “normal model.” This probability distribution—also referred to as the *Gaussian distribution* and in lay terms the “bell curve”—is the one that dominates the teaching curriculum in probability and statistics courses in all business schools. Despite its popularity, the normal model flies in the face of what has been well documented regarding asset prices and returns. The preponderance of the empirical evidence has led to the following three stylized facts regarding financial time series for asset returns: (1) they have *fat tails* (*heavy tails*), (2) they may be *skewed*, and (3) they exhibit *volatility clustering*.

The “tails” of the distribution are where the extreme values occur. Empirical distributions for stock prices and returns have found that the extreme values are more likely than would be predicted by the normal distribution. This means that between periods where the market exhibits relatively modest changes in prices and returns, there will be periods where there are changes that are much higher (i.e., crashes and booms) than predicted by the normal distribution. This is not only of concern to financial theorists, but also to practitioners who are, in view of the frequency of sharp market down turns in the equity markets noted earlier, troubled by, in the words of hoppe (1999), the “… compelling evidence that something is rotten in the foundation of the statistical edifice … used, for example, to produce probability estimates for financial risk assessment.” Fat tails can help explain larger price fluctuations for stocks over short time periods than can be explained by changes in fundamental economic variables as observed by shiller (1981).

The normal distribution is a *symmetric distribution*. That is, it is a distribution where the shape of the left side of the probability distribution is the mirror image of the right side of the probability distribution. For a skewed distribution, also referred to as a *nonsymmetric distribution*, there is no such mirror imaging of the two sides of the probability distribution. Instead, typically in a skewed distribution one tail of the distribution is much longer (i.e., has greater probability of *extreme values* occurring) than the other tail of the probability distribution, which, of course, is what we referred to as fat tails. Volatility clustering behavior refers to the tendency of large changes in asset prices (either positive or negative) to be followed by large changes, and small changes to be followed by small changes.

The attack on the normal model is by no means recent. The first fundamental attack on the assumption that price or return distribution are not normally distributed was in the 1960s by mandelbrot (1963). He strongly rejected normality as a distributional model for asset returns based on his study of commodity returns and interest rates. Mandlebrot conjectured that financial returns are more appropriately described by a non-normal stable distribution. Since a normal distribution is a special case of the stable distribution, to distinguish between Gaussian and non-Gaussian stable distributions, the latter are often referred to as *stable Paretian* distributions or *Lévy stable* distributions.[1] We will describe these distributions later in this book.

Mandelbrot's early investigations on returns were carried further by Fama (1963a, 1963b, among others, and led to a consolidation of the hypothesis that asset returns can be better described as a stable Paretian distribution. However, there was obviously considerable concern in the finance profession by the findings of Mandelbrot and Fama. In fact, shortly after the publication of the Mandelbrot paper, cootner (1964) expressed his concern regarding the implications of those findings for the statistical tests that had been published in prominent scholarly journals in economics and finance. He warned that (Cootner, 1964 p. 337):

> Almost without exception, past econometric work is meaningless. Surely, before consigning centuries of work to the ash pile, we should like to have some assurance that all our work is truly useless. If we have permitted ourselves to be fooled for as long as this into believing that the Gaussian assumption is a workable one, is it not possible that the Paretian revolution is similarly illusory?

Although further evidence supporting Mandelbrot's empirical work was published, the "normality" assumption remains the cornerstone of many central theories in finance. The most relevant example for this book is the pricing of options or, more generally, the pricing of contingent claims. In 1900, the father of modern option pricing theory, Louis Bachelier, proposed using *Brownian motion* for modeling stock market prices.[2] Inspired by his work, samuelson (1965) formulated the log-normal model for stock prices that formed the basis for the well-known Black-Scholes option pricing model. black (1973) and merton (1974) introduced pricing and hedging theory for the options market employing a stock price model based on the *exponential Brownian motion*. The model greatly influences the way market participants price and hedge options; in 1997, Merton and Scholes were awarded the Nobel Prize in Economic Science.

Despite the importance of option theory as formulated by Black, Scholes, and Merton, it is widely recognized that on Black Monday, October 19, 1987, the Black-Scholes formula failed. The reason for the failure of the model particularly during volatile periods is its underlying assumptions necessary to generate a closed-form solution to price options. More specifically, it is assumed that returns are normally distributed and that return volatility is constant over the option's life. The latter assumption means that regardless of an option's strike price, the implied volatility (i.e., the volatility implied by the Black-Scholes model based on observed prices in the options market) should be the same. Yet, it is now an accepted fact that in the options market, implied volatility varies depending on the strike price. In some options markets, for example, the market for individual equities, it is observed that, for options, implied volatility decreases with an option's strike price. This relationship is referred to as *volatility skew*. In other markets, such as index options and currency options, it is observed that at-the-money options tend to have an implied volatility that is lower than for both out-of-the-money and in-the-money options. Since graphically this relationship would show that implied volatility decreases as options move from out-of-the-money options to at-the-money options and then increase from at-the-money options to in-the-money options, this relationship between strike price and implied volatility is called *volatility smile*. Obviously, both volatility skew and volatility smile are inconsistent with the assumption of a constant volatility.

Consequently, since the mid-1990s there has been growing interest in non-normal models not only in academia but also among financial practitioners seeking to try to explain extreme events that occur in financial markets. Furthermore, the search for proper models to price complex financial instruments and to calibrate the observed prices of those instruments quoted in the market has motivated studies of more complex models. There is still a good deal of work to be done on financial modeling using alternative non-normal distributions that have recently been proposed in the finance literature. In this book, we explain these univariate and multivariate models (both discrete and continuous) and then show their applications to explaining stock price behavior and pricing options.

In the balance of this chapter we describe some background information that is used in the chapters ahead. At the end of the chapter we provide an overview of the book.

1.2 The Family of Stable Distribution and its Properties

As noted earlier, Mandelbrot and Fama observed fat tails for many asset price and return data. For assets whose returns or prices exhibit fat-tail attributes, non-normal distribution models are required to accurately model the tail behavior and compute probabilities of extreme returns. The candidates for non-normal distributions that have been proposed for modeling extreme events in addition to the α-stable Paretian distribution include mixtures of two or more normal distributions, Student *t*-distributions, hyperbolic distributions, and other scale mixtures of normal distributions, gamma distributions, extreme value distributions. The class of stable Paretian distributions (which includes α-stable Paretian distribution as a special case) are simply referred to as *stable distributions*.

Although we cover the stable distribution in considerable detail in Chapter 3, here we only briefly highlight the key features of this distribution.

1.2.1 Parameterization of the Stable Distribution

In only three cases does the density function of a stable distribution have a closed-form expression. In the general case, stable distributions are described by their