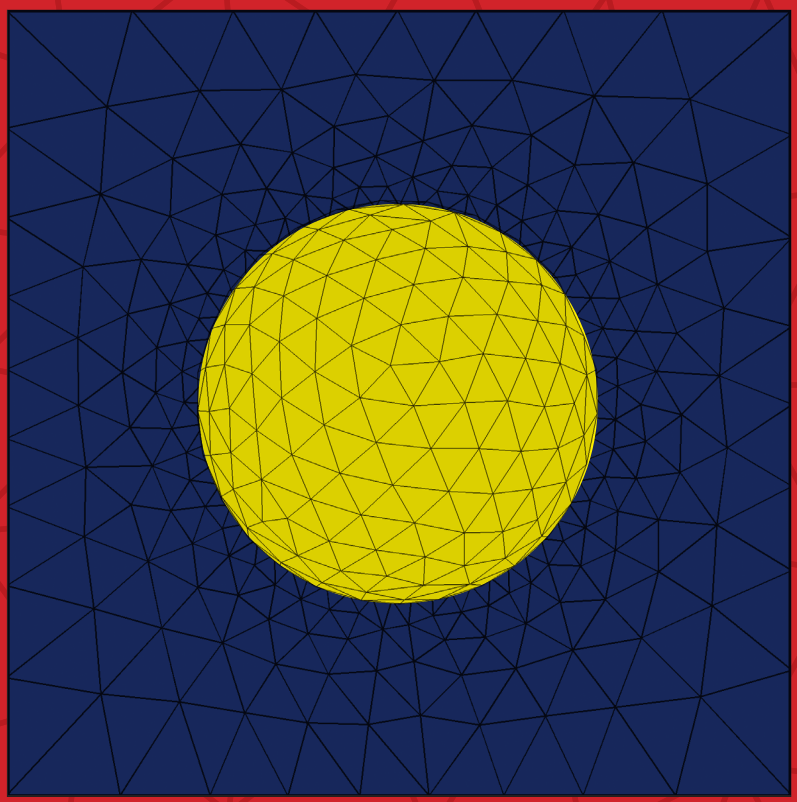
Introduction to Numerical Electrostatics Using MATLAB®



Lawrence N. Dworsky

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Preface

My graduate work was in the area of microwave oscillation mechanisms in semiconductor devices. My contribution was the prediction, analysis, and verification of yet another mode of semiconductor microwave oscillation. After graduation, I taught for a brief period (2 years) and worked part-time for a local electronics firm designing positive–intrinsic–negative (PIN) diode attenuators for their line of microwave signal generators. Then, in 1974, I went to work for Motorola, Inc., in southern Florida.

The part of Motorola that I joined designed and manufactured two-way portable radios (for police, fire department, etc.) and radio pagers, along with the supporting infrastructure systems. The developmental push at the time was to extend the product base up to the (then new) 900-MHz bands.

This organization had no interest whatsoever in my semiconductor physics background. They were, however, keenly interested in the skills that I had acquired during my graduate and part-time work designing stripline and microstrip (transmission line) circuitry. Motorola needed capability with stripline and microstrip filters, interconnects, materials, and so on, and my job was to help develop this capability. I never returned to semiconductor physics, and 40 years after finishing graduate school, I think it's safe to say that I never will.

Transmission line circuit design consists of two parts: (1) the actual circuit design based on the transmission line parameters and (2) the relationships between these transmission line parameters and the physical structure and materials.

As will be explained in Chapter 2, transmission line parameters can be described in terms of their DC (direct current; i.e., electrostatic) capacitance. A significant part of my effort therefore was devoted to performing electrostatic analyses of stripline and microstrip structures to predict the electric fields and capacitances in these structures. I didn't realize it at the time, but I was developing a skill that I would continue using and improving for the rest of my career.

While working on stripline and microstrip circuits, I also worked on piezoelectric (quartz) resonator and filter technology. The piezoelectric device models involved coupling of mechanical motion to electric fields, but the very high ratio of acoustic to electric wave

velocities in the materials of interest (approximately 10^5) allows the electrical part of the analysis to consist of electrostatic analysis.

The 1980s saw the introduction of analog cellular telephone technology in the United States. These cellular telephones and base stations required a complex two-filter system called a *duplexer* that would enable a cellular telephone to transmit and receive simultaneously on two nearby frequencies using the same antenna. (The challenge was to keep your own transmitter's signal out of your receiver.) These duplexers were realized using blocks of high-dielectric-constant ceramic, with partially metalized surfaces, acting as interconnected resonators. Once again, electrostatic modeling was required, and new modeling programs and approaches had to be perfected.

In the late 1980s we did exploratory work in the newly emerging field of micromachined electromechanical devices. Using semiconductor industry processing technologies, it was becoming possible to build extremely small accelerometers, switches, and resonators whose operation is based on electrostatic forces. This was a new area of electrostatic modeling for me. Everything I had done before had involved electrodes that stayed in place, and we never cared about the physical forces involved. Now, we had to calculate the forces and keep track of the fields and forces as the electrodes moved. Again, I was extending my experience base in electrostatic modeling.

In the early 1990s we became interested in vacuum microelectronics, particularly in a structure called the *field emission display*, an electronic display whose operation is based on electron emission from millions of very small, sharp, metal tips due to high local electric fields. Once again, I was extending my electrostatic modeling experience to include structures with vastly different scales (submicromillimeter resolution near the tips to millimeter resolution near the screen). Structural capacitances were of interest in that they could limit circuit switching speeds, but the principal issues were the magnitude and uniformity of the fields at the emitter tips and then the electron trajectory control (both desired and undesired) due to these fields as the electrons traveled to the screen, striking the light emitting phosphors when they arrived. In these models the electrodes remained immobile, so the fields didn't change; electron trajectories were the principal subject of interest.

Putting my history together, although I didn't realize it at the time, I have spent more time creating and working with electrostatic analyses than with any single other electrical engineering discipline. These analyses were never a goal unto themselves. They were an engineering tool. The simplest approach that could do the job was always the chosen approach.

The philosophy of this book follows from my personal experience. There is an incredibly long list of mathematical approaches to numerical electrostatic modeling, but in terms of learning the electrostatics and choosing a modeling approach to study a given situation, I try to avoid using more exotic schemes simply "because they're there." This doesn't mean that all of the approaches in the literature aren't interesting, important, and valuable, but in any given circumstance the simplest tool that can do a job is probably the best tool for that job.

LAWRENCE N. DWORSKY

Introduction

An introductory treatment of electrostatics usually begins with Coulomb's law, the concepts of charge, the electric field and energy stored in the field, potential, capacitance, and so on. Poisson's and Laplace's equations soon appear.

Unfortunately, almost no real world problem can be solved in closed form using the latter equations. The most basic of electrostatic device analysis, the electric fields surrounding and the capacitance of a simple parallel plate capacitor, cannot be found.

Typically, a few interesting solution techniques, such as the separation of variables, are presented. Then the author has to choose a path. Other solution techniques such as conformal mapping can be shown; if the book is to be more than an introductory text, more formal materials such as Greene's functions can be introduced. In any case, the practitioner with real world geometries to be analyzed has been abandoned.

A book about numerical analysis techniques typically presents just that – numerical analysis techniques. The few examples presented are usually based as much on the ease of their presentation as on their ultimate usefulness.

My goal in writing this book is to present enough basic electrostatic theory as necessary to get into real world problems, then to present several of the available numerical techniques that are applicable to these problems, and finally to present numerous, detailed, examples showing how these techniques are applied. In other words, I am presenting the basics of electrostatics and several relevant numerical analysis techniques, with the emphasis on practical geometries.

The numerical analysis of problems in fields such as electrostatics typically have three distinct phases:

1. *Pre-processing.* The conversion of the physical description of the problem to a data set that is meaningfully digestible to the numerical analysis program chosen for the job.
2. *Numerical analysis.* The calculations, based upon the data set describing the geometry and the chosen boundary conditions, resulting (typically) in an approximate solution for the voltage distribution over the chosen space.

3 *Post-processing.* Calculations and programming necessary to provide summary data such as capacitance and computer visualization of the results.

This book will concentrate on item 2, the numerical analysis. Three fundamental techniques – method of moments, finite difference, and finite elements – are introduced. Sample problems are presented and computer code that solves the problems is developed.

In order to accomplish the above, some pre-processing capability is necessary. Rather than develop this capability or simply request that the reader "get it done," several freely available packages are introduced and basic tutorials on their usage are presented. These tutorials are not exhaustive; they introduce enough of the capabilities of these packages to allow the student to follow, replicate, and extensively modify the examples to the student's own needs. No claims are being made that the packages chosen are the best possible choices for the job. They are, however, choices that work well.

Post-processing of numerical analysis results, in this book, is done on an ad-hoc basis. Calculating the capacitance of a structure is often very useful because, if the example structure has been analyzed by other techniques and the results published, accuracy of results can be compared. This comparison allows for a convenient, one number figure of merit for choices of resolution, approximate boundary conditions, and so on. Often, graphical interpretations of field, voltage, and/or charge distributions are presented. These are useful as a quick visual check on the boundary conditions and on gaining insight into the electrostatic properties (high field points, etc.) of the structure being studied.

With only one exception, all the numerical analysis, post-processing and graphics were created using MATLAB®. This type of work is what MATLAB is designed to do. The analysis techniques presented convert partial differential equations into sets of linear equations with coefficients and variables represented by matrices and vectors. MATLAB is a scientific programming language designed with the matrix as the fundamental data type. The language is expressive, the available function list extensive and the easily used graphics superb.

MATLAB is fundamentally an interpreted computer language. It achieves impressive processing speeds by providing a liberal assortment of precompiled functions. Preparing user-written code for these functions involves a procedure that MATLAB calls "vectorizing." Vectorizing means not writing explicit loops to process array elements because the language itself allows processing of all of the array elements simultaneously. From an authoring point of view, this raises a question: Should demonstration code be vectorized as much as possible in the name of program execution speed or should demonstration code be written to explicitly parallel the derivations of the formulas in the text in the name of pedagogical clarity?

There is no absolutely right answer to the above question. In every case in this book judgment calls were made – vectorized code is shown when the algorithm seems "clear enough." This compromise, like all compromises, won't please everybody every time; ideally, it will please enough readers enough of the time.

The problems at the end of the chapters were written, as much as possible, to be extensions of the chapters, rather than "verify XXX or put numbers into YYYY." Many of the problems involve modifying existing or writing new MATLAB code, leading to capabilities that were not presented in the chapters' materials. When the thrust of the problem isn't to extend the modeling capability, it is to make a point about the electrostatics issues involved in the example being treated. In all cases, solving the problems and reading the solution discussions will be an integral part of the learning process.

Solutions to end-of-chapter exercises may be found at the book companion site, `www.wiley.com/go/numerelectrostatics`. Additional resources may be found at `www.lawrencedworsky.com`.

Acknowledgments

I certainly could not have done this job alone. I thank Kari Capone and George Telecki (retired) at John Wiley & Sons: Kari for her very good natured help in teaching me how to prepare my manuscript then reviewing it, and George for his initial support and encouragement. The folks at MathWorks were very generous in providing me with MATLAB software and support. Without the love, patience, and support of my wife Suzanna, I might never have started this project and I certainly never would have finished it.

L. N. D.

1

A Review of Basic Electrostatics

Electric and magnetic phenomena, including electromagnetic wave propagation, are described by Maxwell's equations.[1] When nothing is changing with time, that is, when all derivatives with respect to time are zero, the electric and the magnetic phenomena decouple and become separate electric and magnetic phenomena. These are referred to respectively as *electrostatics*, which describes the properties of systems with separated static regions of positive and negative electric charge (although the entire system is charge-neutral), and *magnetostatics*, which describes the properties of systems with electric currents and/or magnetized materials.

In this book we shall consider only electrostatics. This subset of a subset of topics describes a vast number of real-world situations. Chapter 2 describes some practical needs and uses of electrostatic analyses, the remainder of the book will be dedicated to examining several techniques for performing these analyses.

The materials to follow are intended to be a quick review of the relationships that will be used throughout this book. The intent here is to provide a consistent set of notation using all the relationships that will be needed going forward. Many of these relationships are stated without derivation or proof. A more complete electrostatics theory text is recommended for newcomers to the subject. There are very many excellent texts available. The references list at the end of this chapter is certainly not exhaustive, but the texts cited are considered standards in the field.

1.1 CHARGE, FORCE, AND THE ELECTRIC FIELD

Electric charges exert forces on one another. This is the basis of electrostatics. The characteristics of these forces are summarized in Coulomb's law:

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1. Electric charge carries a polarity, or sign. The choice of sign was originally arbitrary, but now is established by tradition—the electron, the most common charged subatomic particle, carries a negative charge.
2. For point charges q_1 and q_2, measured in coulombs, the (coulomb) force, measured in newtons, in a uniform medium, is given by

$$\overrightarrow{F} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\overrightarrow{a_r} \tag{1.1}$$

3. In equation (1.1) and elsewhere ε is the permittivity of the material in farads per meter (F/m). In free space, $\varepsilon = \varepsilon_0 = 8.854$ F/m. For other linear, isotropic, homogeneous materials, $\varepsilon = k\varepsilon_0$, where k is the relative permittivity, the relative dielectric constant, or sometimes simply the dielectric constant, of the material. *Farads* per se are defined as coulombs per volt (C/V). In this book we shall consider only linear, isotropic, homogeneous dielectric materials, and going forward this will be assumed.
4. In equation (1.1) r is the distance between q_1 and q_2.
5. Also, $\overrightarrow{a_r}$ is a unit vector along the line connecting q_1 and q_2. If q_1 and q_2 have the same sign, then $\overrightarrow{F}$ is pushing q_1 and q_2 apart. If q_1 and q_2 have opposite signs, $\overrightarrow{F}$ is pulling them together.

Equation (1.1) is expressed in the rationalized meter-kilogram-second (mks) system of units. The derivation of this set of units is an interesting discussion in itself.[2]

When a test charge is in the area of a collection of charges and the magnitude of these latter charges is sufficient, relative to the test charge, to render negligible any perturbation of the situation due to the test charge, then the force on the test charge divided by its charge is defined to be the electric field at that point (typically called the *field point*). The electric field at (the field point) p due to a charge q is therefore

$$\overrightarrow{E_p} \equiv \frac{\overrightarrow{F_p}}{q_p} = \frac{q}{4\pi\varepsilon r^2}\overrightarrow{a_r} \tag{1.2}$$

where $\overrightarrow{a_r}$ is the unit vector along the line from charge q to point p and r is the distance from charge q to point p. The values of $\overrightarrow{E}$ are expressed in volts per meter (V/m).

Since the test charge at p in the preceding example doesn't disturb the electric field, the electric field is considered to be a consequence of q; in other words, the test charge doesn't have to be present for the field to exist.

The term $\overrightarrow{E}$ is a vector with both magnitude and direction. The direction of $\overrightarrow{E}$ anywhere in space is identically the direction of the force that would be experienced by a (positive) test charge at that point. We can look at the *field lines* of $\overrightarrow{E}$ as a representative of the direction of the force on a test charge due to q. For a single-point charge, the field lines are simply radial lines pointing away from the charge. The lines point away because a positive test charge placed anywhere would feel a force pushing it away from the (source) charge. The magnitude of the field decreases with the square of the distance from the charge.

For a collection of charges, the electric field at any point is the sum of the contributions of all of the charges in the collection. Figure 1.1, for example, shows electric field lines in the

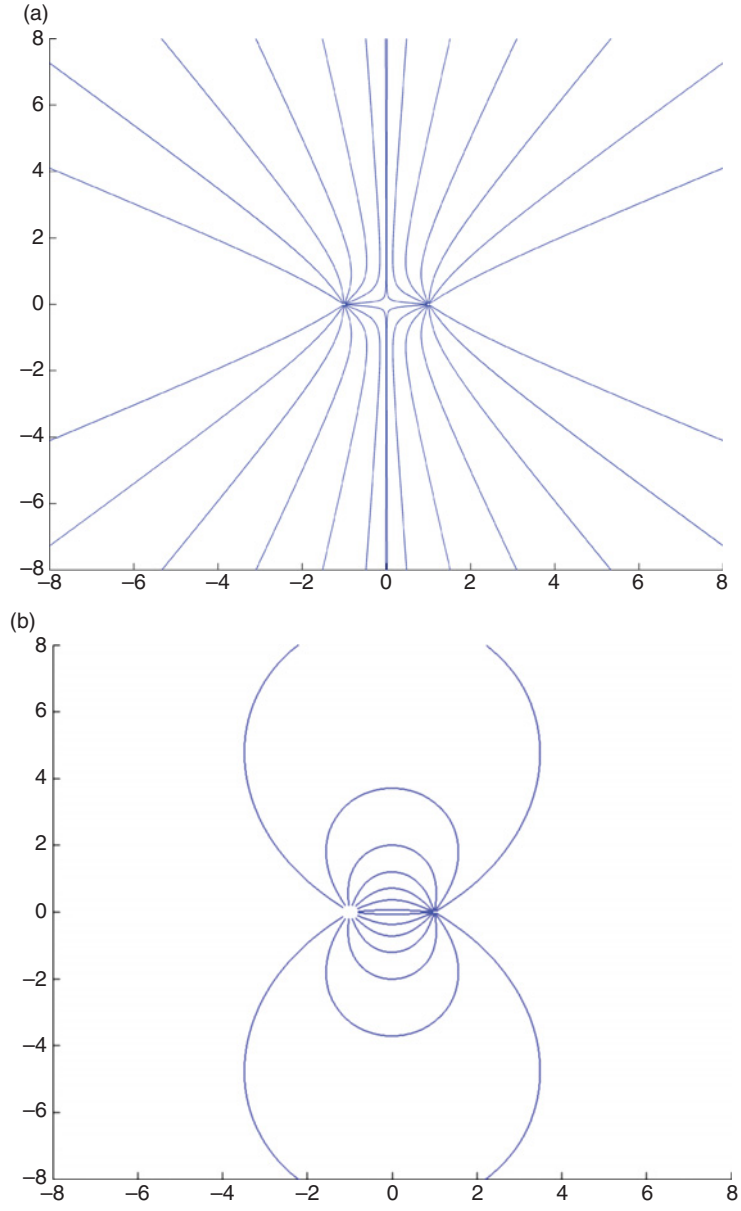

FIGURE 1.1 Electric field lines for point charges at (−1,0,0) and (1,0,0).

X–*Y* plane for two point charges placed at (−1,0,0) and (+1,0,0). In part (a) the two charges are identical; in part (b) they are the same in magnitude but opposite in sign.

In order to calculate $\overrightarrow{E}$ directly we must keep track of the vector components of every charge contributing to it. Continuing with the example of Figure 1.1, the simple MATLAB function `charges.m` shown here calculates the field anywhere in the *X*–*Y* plane. Calculation of the field components from the geometry is shown in the equations in this program. Setting q_2 to +1 or −1 produces the two cases discussed above.

```
function [ Ex, Ey, Emag ] = charges(x,y)
% This function calculates the electric field for the 2 charge
%  layout of Figure 1.1

q2 = -1;          % Set this to +1 or -1 as needed
eps = 8.854;      % pFd/m in free space

theta1 = atan2(y,x + 1);          theta2 = atan2(y,x-1);
rsq1 = (x + 1).^2 + y.^2;           rsq2 = (x-1).^2 + y.^2
Emag1 = 1./(4*pi*eps*rsq1);       Emag2 = 1./(4*pi*eps*rsq2);
Ex1 = Emag1.*cos(theta1);          Ex2 = q2*Emag2.*cos(theta2);
Ey1 = Emag1.*sin(theta1);          Ey2 = q2*Emag2.*sin(theta2);

Ex = Ex1 + Ex2;        Ey = Ey1 + Ey2;
Emag = sqrt(Ex.^2 + Ey.^2);

end
```

If both charges are equal to +1 in this example, then along the y axis E_x must always be zero. This can be deduced from the symmetry of the situation without consulting the equations. On the other hand E_y is zero only at $y = 0$ and must be an odd function of y. $E_y(0,y,0)$ is shown in Figure 1.2.

If the right-hand charge (in Figure 1.2) is changed to −1, then, along the y axis E_y must always be zero—again, from symmetry considerations. E_x in this case is an even positive function of y, as shown in Figure 1.3.

If a small charged mass such as an electron is placed near charge(s), as in part (a) or (b) of Figure 1.1, it would immediately start moving. Its trajectory would not be along a field line.

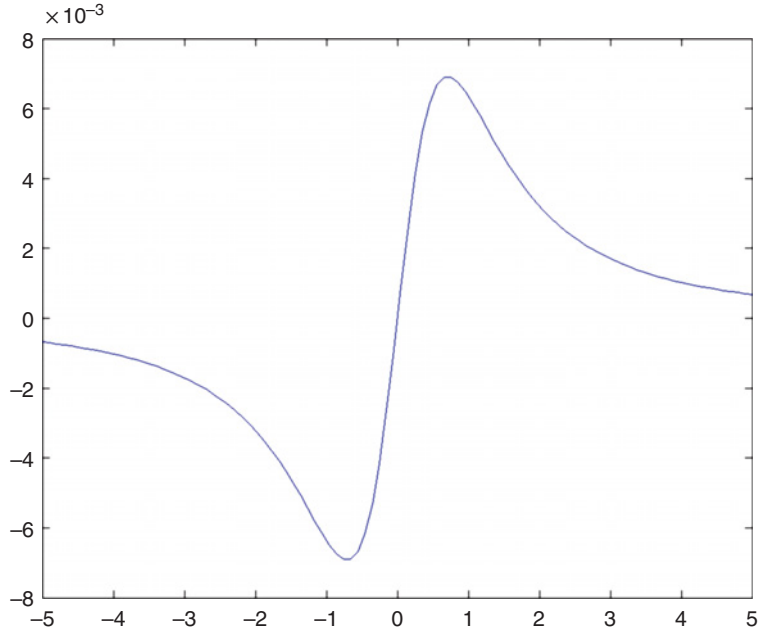

FIGURE 1.2 $E_y(0,y,0)$ for two identical positive charges.

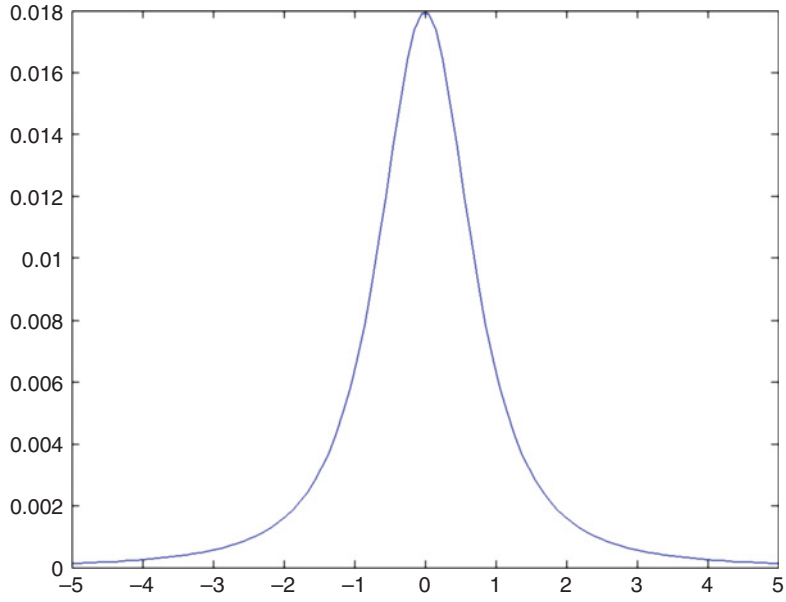

FIGURE 1.3 $E_x(0,y,0)$ for two charges of opposite sign.

Since the electron has mass, it gathers momentum as it moves and a proper description of its motion requires solving Newton's equation with the electric field as the driving force. Electron trajectories in various electric field profiles will be examined in Chapter 17.

Inspection of Figure 1.1a shows that the field lines emanating from both charges start out radially. They then bend rather than cross and leave the region, going instead to infinity. This characteristic is identical to the radial field lines from a single charge which also go to infinity. In Figure 1.1b, however, each field line travels from the positive (left-hand) charge to the negative (right-hand) charge and terminates. This is characteristic of an electrically neutral structure, and we can extract a general rule: Electric field lines originate at and terminate at charge; a neutral structure will have no field lines going to infinity. This will be expressed as a mathematical relationship in Section 1.2.

1.2 ELECTRIC FLUX DENSITY AND GAUSS'S LAW

Let us define a (vector) quantity $\overrightarrow{D}$ as follows:

$$\overrightarrow{D} = \varepsilon \overrightarrow{E} \tag{1.3}$$

Combining this definition with equation (1.2), we obtain

$$\overrightarrow{D} = \frac{q}{4\pi r^2}\overrightarrow{a_r} \tag{1.4}$$

which is independent of the dielectric constant.

The $\overrightarrow{D}$ in these equations is the *electric flux density*. The rationale for using this term will become clear shortly. Consider a point charge q surrounded by a virtual spherical shell of radius r_0. The surface area of this shell is $4\pi r_0^2$. Since $\overrightarrow{D}$ is a function only of r, it is a constant

everywhere on this shell; also, since it is pointed radially outward everywhere, it is normal to the shell at intersection. The integral of (the magnitude of) $\overrightarrow{D}$ over the surface of the shell is

$$\iint_s |\overrightarrow{D}|\, ds = \iint_s \frac{q}{4\pi r^2} 4\pi r^2 = q \tag{1.5}$$

By examining the situation for an arbitrary collection of charges and an arbitrary surface surrounding them, we can generalize this result to Gauss's law[3]

$$q = \iint_s \overrightarrow{D} \cdot ds \tag{1.6}$$

where the integral is over the entire surface. $\overrightarrow{ds}$ is a differential area with vector direction normal to the plane of the area and q is the total charge enclosed.

Returning to equation (1.5), we have

$$\varepsilon \iint_s |\overrightarrow{E}|\, ds = q \tag{1.7}$$

For a spherical shell centered at q, we obtain $\overrightarrow{E} = \overrightarrow{E}(r)$ only, pointing radially outward, and therefore

$$\varepsilon \overrightarrow{E}\left(4\pi r^2\right) = q\,\overrightarrow{a_r} \tag{1.8}$$

which is essentially identical to equation (1.2). In other words, Gauss's and Coulomb's laws are equivalent.

Suppose that we have a sphere of charge of radius a, centered at the origin, of uniform charge density ρ [expressed in coulombs per cubic meter (C/m^3)] (see Figure 1.4).

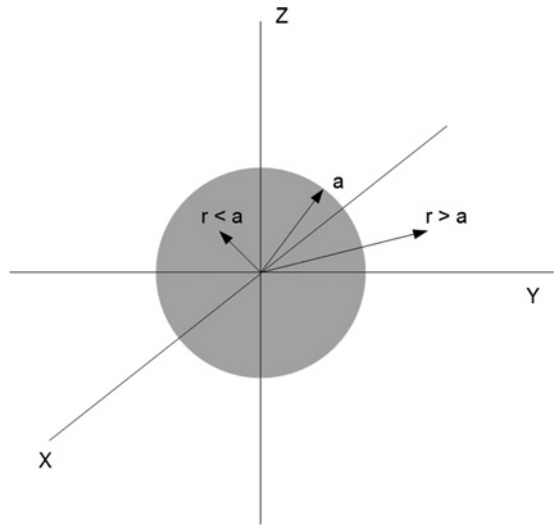

FIGURE 1.4 Sphere of uniform charge density ρ.

From the symmetry of the situation, we know again that $\overrightarrow{E} = \overrightarrow{E}(r)$ only, pointing radially outward. For any $r \le a$, the charge enclosed is

$$Q_{\text{enc}} = \rho \int_0^r \iiint dv = \tfrac{4}{3}\pi r^3 \rho \tag{1.9}$$

Putting this result into Gauss's law, we have

$$Q_{\text{enc}} = \tfrac{4}{3}\pi r^3 \rho = 4\pi r^2 \varepsilon E \tag{1.10}$$

or

$$E = \frac{r\rho}{3\varepsilon} \tag{1.11}$$

The field goes to 0 at $r = 0$, because there is no charge enclosed. It increases with increasing r. At $r = a$ all of the charge is enclosed and again using Gauss' law, for $r \ge a$, we obtain

$$Q_{\text{enc}} = \tfrac{4}{3}\pi a^3 \rho = 4\pi r^2 \varepsilon E \tag{1.12}$$

and then

$$E = \frac{a^3}{3\varepsilon r^2} \tag{1.13}$$

If ρ is not a constant but is instead a function of r (and only r), then it must be brought inside the integral of equation (1.9) and the integral properly evaluated. The electric field outside the sphere of charge ($r \ge a$) depends only on the total charge in the sphere, irrespective of the details of $\rho(r)$. This latter point is significant because it tells us that $E(r)$ (see Figure 1.5) will be the same (again, for $r \ge a$), if all the charge is concentrated at a point at the origin, is spread uniformly through the volume of the sphere, or is distributed in whatever other configuration that can be imagined. An important case we will consider (in Section 1.3) is the case where all of the charge resides in a thin shell at $r = a$.

1.3 CONDUCTORS

An ideal conductor of charge is a material in which the charge carriers are free to move about under the influence of electrostatic forces (Coulomb's law). Good examples of this are metals such as copper and silver—they are not ideal conductors but they are very good conductors. The very mobile charge in metals is the electrons in the outer shell of the metallic atoms; how charge mobility comes about is an important topic of solid-state physics.[4] How charge is arranged in conductors in different situations will be a central theme in discussion of the method of moments (MoM) in later chapters

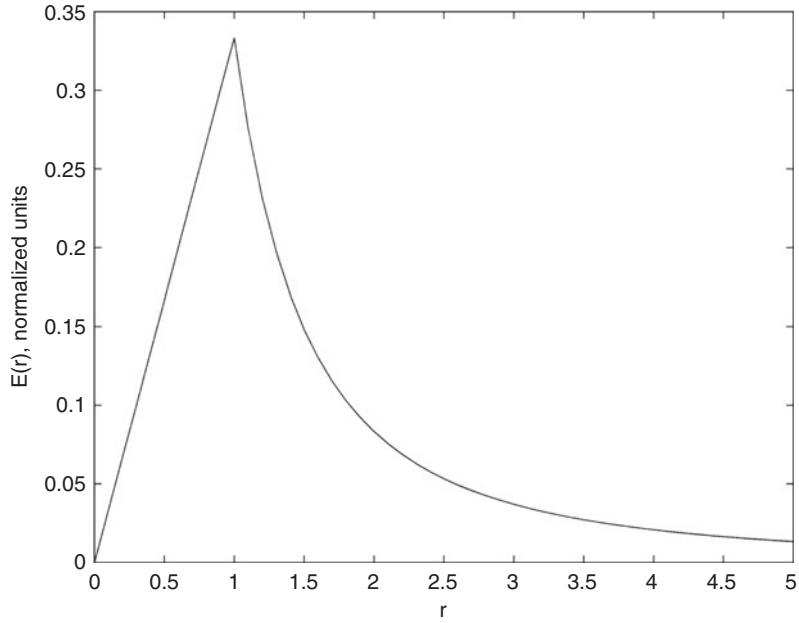

FIGURE 1.5 $E(r)$ for a sphere of radius a, charge density ρ.

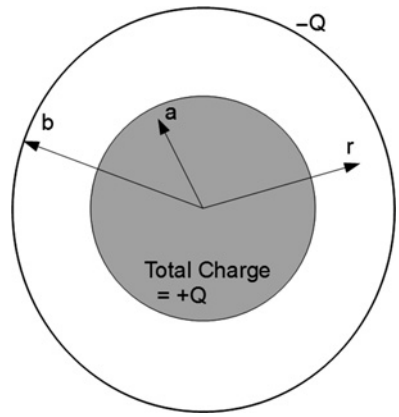

FIGURE 1.6 Two concentric spherical shells.

(Chapters 3, 4, etc.). Right now we will consider only situations with geometries whose symmetries require that charge distributions be uniform.

Consider Figure 1.6. A metal sphere has been placed at $r = a$, and a spherical metal shell has been placed at $r = b$. A charge $-Q$ equal to the total charge enclosed by the inner shell ($+Q$) has been placed on the outer shell so that the entire system is now charge-neutral. The symmetry of the structure implies that charge must be uniform in terms of angle. The charges on the inner sphere repel each other and are attracted to the charges on the outer sphere. This means that the charges on the inner sphere will all move to the outer surface of the inner sphere, which, in turn, means that there is no electric field inside the inner sphere.

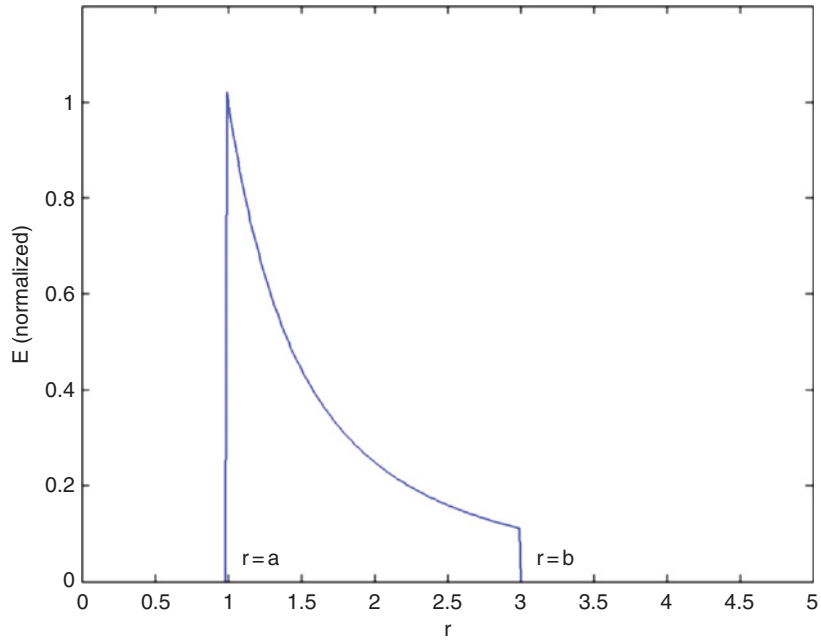

FIGURE 1.7 Electric field between two concentric opposite-charge conductive shells.

A further conclusion is that, in terms of the electric field between the outer shell and the inner sphere, the inner sphere can be either a solid conductor or simply a thin conductor shell at $r = a$.

This latter characteristic of electrostatic systems is put to very good use in ultra-high-voltage systems such as the Van de Graaff generator.[5] The safest place for people to be is inside one of the large metal spheres used in the device, as it is a field-free region.

Returning to Figure 1.6, in the region $a \le r \le b$, charge $+Q$ is enclosed, and

$$\vec{E} = E_r = \frac{Q}{4\pi\varepsilon r^2} \tag{1.14}$$

When $r \ge b$, the sum of the charge on both the inner shell and the outer shell is zero, so that there is no net charge enclosed and E abruptly drops to zero (Figure 1.7).

Next, consider the structure shown in Figure 1.8. Two large parallel conductor plates have surface charge densities $+\sigma$ and $-\sigma$ [expressed in coulombs per square meter (C/m^2)]. The plates are separated by a distance d.

Near the center of these plates, far from the edges, the charge density on both plates is uniform. The only possible electric field distribution in this region is uniform, directed from the positively charged plate toward the negatively charged plate. The figure shows a virtual right circular cylinder extending from the bottom plate up to some point between the plates. The actual shape of the virtual structure is insignificant as long as its walls are directed normal to the plates' surfaces (i.e., parallel to the electric field lines).

If the area of the top and bottom surfaces of the virtual structure is A, the charge enclosed by the structure, as long as the top surface is somewhere between the surfaces, is σA. Because the sidewalls of the structure are parallel to the electric field lines, no lines cross

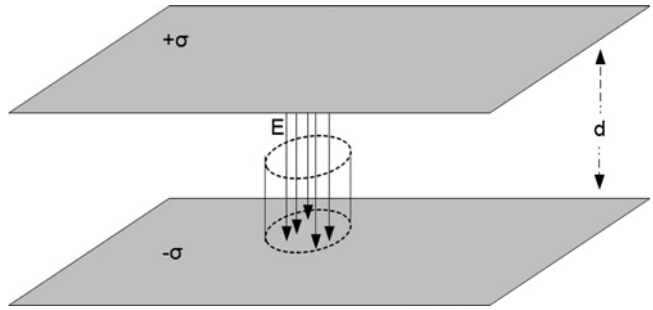

FIGURE 1.8 Electric field between two large parallel plates, near the center.

the surfaces, and therefore the only contribution to the right-handside of equation (1.6) is the top surface. Thus Gauss's law tells us that

$$\sigma A = \varepsilon EA \tag{1.15}$$

or

$$E = \frac{\sigma}{\varepsilon} \tag{1.16}$$

Gauss's law can also be expressed in differential, or point, form as[3]

$$\nabla \cdot \overrightarrow{D} = \rho \tag{1.17}$$

where $\nabla \cdot$ is the divergence operator. In rectangular coordinates this is

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{1.18}$$

where $\rho = \rho(x,y,z)$ is the charge density, that is

$$q = \iiint_V \cdot \rho \, dV \tag{1.19}$$

where V is the total volume enclosed by s.

1.4 POTENTIAL, GRADIENT, AND CAPACITANCE

Since there is a force on a charged body in an electric field, moving that body through the field must require work. (If energy is transferred to the body, we'll consider it negative work done.) This is analogous to the work done lifting a mass in a gravitational field. As in the case of work done in a gravitational field, we can define a potential difference as the work done in moving the body, where $\overrightarrow{dl}$ is a differential length element along the path from p to q:

$$\phi_q - \phi_p = -\int_p^q \overrightarrow{E} \cdot \overrightarrow{dl} \tag{1.20}$$

The electrical potential φ is also called the *voltage V*, so equation (1.20) can equivalently be written

$$V_q - V_p = -\int_p^q \overrightarrow{E} \cdot \overrightarrow{dl} \tag{1.21}$$

As the preceding equations show, only a voltage difference between two points is defined. Strictly speaking, the voltage at a point has no meaning. It is common, however, to define the voltage at some point as zero, often called the *ground* or *reference* voltage or potential. It is then possible to refer to the voltage at any point using a single number—the implied meaning is that we are talking about the voltage difference between that point and the reference point.

Returning to the example of the concentric spheres (Figure 1.6), we can easily find the voltage difference (commonly called the *voltage*) between the two spheres by integrating equation (1.14):

$$V(r) = \frac{-Q}{4\pi\varepsilon}\int_a^r \frac{d\bar{r}}{\bar{r}^2} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{r}\right) \tag{1.22}$$

Here, we have chosen $V(a) = 0$ as the voltage reference.

The voltage between the two metal shells is then

$$V_b = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{1.23}$$

From a circuital perspective, we are often more interested in voltages (and fields) at different places in terms of the applied voltage. We obtain this result by dividing equation (1.22) by equation (1.23):

$$V(r) = V_b \frac{1/a - 1/r}{1/a - 1/b} \tag{1.24}$$

For the parallel plate structure (Figure 1.8), taking $z = 0$ as the bottom plate and $z = d$ as the top plate, with the bottom plate at ground and the top plate at V_0, integrating equation (1.16), and repeating the same procedure as above, we obtain

$$V(z) = \frac{\sigma}{\varepsilon} z = V_0 \frac{z}{d} \tag{1.25}$$

Again analogous to the mass in a gravitational field, the potential difference between two points is path independent, it is inconsequential which path the integral takes from point p to point q. This implies that the electrostatic field is conservative—any path leading from point p back to point p will yield a zero-voltage difference. In other words, electrostatic

energy is neither gained nor lost going around a closed path. An important point to make here, even though it is beyond the purview of this book, is that this is not a general electromagnetic system property—it is valid only in the electrostatic case.

Restating equation (1.21) to yield the field in terms of the voltage difference, in rectangular coordinates, we have

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}\right] \tag{1.26}$$

The operator ∇ is called the *gradient operator*. This equation shows clearly why an arbitrary reference voltage choice has no effect on the electric field.

Suppose that there is a charge q at the origin of our coordinate system. If q is the only charge present, then no work was required to bring q from anywhere else to the origin. Now, let us bring a test charge from infinity (where the field due to q is zero) to some radius a. Using equation (1.21), we obtain

$$V_a = -\int_{\infty}^{a} \vec{E}\cdot\vec{dl} \tag{1.27}$$

and using equation (1.2), the potential at a is

$$V_a = -\int_{\infty}^{a} \frac{q}{4\pi\varepsilon r^2}dr = \frac{q}{4\pi\varepsilon a} \tag{1.28}$$

Equation (1.28) is a scalar equation, which is almost always easier to work with than is a vector equation. Also, once the voltage is known, it is a straightforward job to calculate the field. Consequently, we will concentrate on finding voltages and then (if necessary) finding the field, not the other way around.

For the single-point charge of equation (1.28), we already know that the field lines point radially outward (from the charge), going to infinity. Figure 1.9. shows surfaces of constant potential, known as equipotential surfaces or more commonly equipotentials. These surfaces cross the field lines normally and in this situation are spheres.

If, instead of a single charge q, we have a collection of (discrete) charges, we must replace equation (1.28) by the sum of the contributions of all the charges, and a is replaced by the distances from each of the charges (x_i,y_i,z_i) to the measurement point $p=(x_p,y_p,z_p)$. In other words,

$$r_{ip} = \sqrt{(x_i-x_p)^2 + (y_i-y_p)^2 + (z_i-z_p)^2} \tag{1.29}$$

and then

$$V_p = \sum_i \frac{q_i}{4\pi\varepsilon r_{i,p}} \tag{1.30}$$

The gradient [equation (1.26)] operating on V produces en electric field vector whose direction is the same as that of the maximum change in V. Since the direction of maximum