Solutions Manual to Accompany

BEGINNING PARTIAL DIFFERENTIAL EQUATIONS

Third Edition

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WILEY

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Solutions Manual to Accompany Beginning Partial Differential Equations

Third Edition

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey

Published simultaneously in Canada

Library of Congress Cataloging in Publication Data is available.

ISBN 978-1-118-63009-9

Preface

This manual contains solutions for many of the problems in Beginning Partial Differential Equations, third edition.

Because solutions for many odd-numbered problems are included in Chapter Nine of the book, most of the problems included here are even-numbered. However, particularly in the case of problems exploring ideas beyond the text discussion, some odd-numbered solutions are also included.

Chapter 1
First Ideas

1.1 Two Partial Differential Equations

2. Verifying that the function is a solution of the heat equation is a straightforward exercise in differentiation. One way to show that $u(x,t)$ is unbounded is to observe that if $t > 0$ and $x = 2\sqrt{kt}$, then

$$u(x,t) = \frac{1}{e} t^{-3/2}$$

and this can be made as large as we like by choosing t sufficiently close to zero.

4. By the chain rule,

$$\begin{aligned} u_x &= \frac{1}{2}(f'(x-ct) + f'(x+ct)), \\ u_{xx} &= \frac{1}{2}(f''(x-ct) + f''(x+ct)), \\ u_t &= \frac{1}{2}(f'(x-ct)(-c) + f'(x+ct)(c)), \text{ and} \\ u_{tt} &= \frac{1}{2}(f''(x-ct)(-c)^2 + f''(x+ct)(c)^2). \end{aligned}$$

It is routine to verify that $u_{tt} = c^2 u_{xx}$.

7. One way to show that the transformation is one to one is to evaluate the Jacobian

$$\begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} = b - a \neq 0.$$

Finally, solve $\xi = a + at$, $\eta = x + bt$ for x and t to obtain the inverse transformation

$$x = \frac{1}{b-a}(b\xi - a\eta), t = \frac{1}{b-a}(\eta - \xi).$$

8. With $V(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta))$, chain rule differentiations yield:

$$\begin{aligned} u_x &= V_\xi \xi_x + V_\eta \eta_x = V_\xi + V_\eta, \\ u_t &= V_\xi \xi_t + V_\eta \eta_t = aV_\xi + bV_\eta, \end{aligned}$$

and, by continuing these chain rule differentiations and using the product rule,

$$\begin{aligned} u_{xx} &= V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta}, \\ u_{tt} &= a^2 V_{\xi\xi} + 2abV_{\xi\eta} + b^2 V_{\eta\eta}, \text{ and} \\ u_{xt} &= aV_{\xi\xi} + (a+b)V_{\xi\eta} + bV_{\eta\eta}. \end{aligned}$$

Now collect terms to obtain

$$\begin{aligned} &Au_{xx} + Bu_{xt} + Cu_{tt} = \\ &\quad (A + aB + a^2C)V_{\xi\xi} + (2A + (a+b)B + 2abC)V_{\xi\eta} + (A + bB + b^2C)V_{\eta\eta}. \end{aligned}$$

This, coupled with the fact that $H(x,t,u,u_x,u_t)$ transforms to some function $K(\xi,\eta,V,V_\xi,V_\eta)$, yields the conclusion.

9. From the solution of problem 8, the transformed equation is hyperbolic if $C \neq 0$ because in that case we can choose a and b to make the coefficients of $V_{\xi\xi}$ and $V_{\eta\eta}$ vanish. This is done by choosing a and b to be the distinct roots of

$$A + Ba + Ca^2 = 0 \text{ and } A + Bb + Cb^2$$

which are the same quadratic equation. For example, we could choose

$$a = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \text{ and } b = \frac{-B - \sqrt{B^2 - 4AC}}{2C}.$$

If $C = 0$, use the transformation

$$\xi = t,\ \eta = -\frac{B}{A}x + t.$$

Now chain rule differentiations yield

$$u_x = -\frac{B}{A}V_\eta,\ u_t = V_\xi + V_\eta,$$
$$u_{xx} = \frac{B^2}{A^2}V_{\eta\eta},\ u_{xt} = -\frac{B}{A}V_{\xi\eta} - \frac{B}{A}V_{\eta\eta}.$$

We do not need u_{tt}, because $C = 0$ in this case. Now we obtain

$$Au_{xx} + Bu_{xt} + Cu_{tt} = -\frac{B^2}{A}V_{\xi\eta},$$

yielding a hyperbolic canonical form

$$V_{\xi\eta} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

of the given partial differential equation.

10. In this case suppose $B^2 - 4AC = 0$. Now let

$$\xi = x,\ \eta = x - \frac{B}{2C}t.$$

Now

$$u_x = V_\xi + V_\eta,\ u_t = -\frac{B}{2C}V_\eta,$$
$$u_{xx} = V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta},\ u_{tt} = \frac{B^2}{4C^2}V_{\eta\eta},\ \text{and}$$
$$u_{xt} = -\frac{B}{2C}V_{\xi\eta} - \frac{B}{2C}V_{\eta\eta}.$$

Then

$$
\begin{aligned}
&Au_{xx} + Bu_{xt} + Cu_{tt} \\
&= A(V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta}) - \frac{B^2}{2C}(V_{\xi\eta} + V_{\eta\eta}) + \frac{B^2}{4C}V_{\eta\eta} \\
&= AV_{\xi\xi} + V_{\xi\eta}\left(2A - \frac{B^2}{2C}\right) + V_{\eta\eta}\left(A - \frac{B^2}{2C} + \frac{B^2}{4C}\right) \\
&= AV_{\xi\xi},
\end{aligned}
$$

with two terms on the next to last line vanishing because $B^2 - 4AC = 0$. This gives the canonical form

$$V_{\xi\xi} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

for the original partial differential equation when $B^2 - 4AC = 0$.

11. Suppose now that $B^2 - 4AC < 0$. Let the roots of $Ca^2 + Ba + A = 0$ be $p \pm iq$. Let

$$\xi = x + pt,\ \eta = qt.$$

Proceeding as in the preceding two problems, we find that

$$
\begin{aligned}
&Au_{xx} + Bu_{xt} + Cu_{tt} \\
&= (A + Bp + Cp^2)V_{\xi\xi} + (qB + 2pqC)V_{\xi\eta} + q^2V_{\eta\eta}.
\end{aligned}
$$

Now we need some information about p and q. Because of the way $p + iq$ was chosen,

$$C(p + iq)^2 + B(p + iq) + A = 0.$$

This gives us

$$Cp^2 - Cq^2 + Bp + A + (2Cpq + Bq)i = 0.$$

Then

$$Cp^2 - Cq^2 + Bp = 0 \text{ and } 2Cpq + Bq = 0.$$

In this case,

$$Au_{xx} + Bu_{xt} + Cu_{tt} = q^2(V_{\xi\xi} + V_{\eta\eta})$$

and we obtain the canonical form

$$V_{\xi\xi} + V_{\eta\eta} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

for this case.

12. The diffusion equation is parabolic and the wave equation is hyperbolic.

14. $B^2 - 4AC = 33 > 0$, so the equation is hyperbolic. With

$$a = \frac{1+\sqrt{33}}{8} \text{ and } b = \frac{1-\sqrt{33}}{8}$$

the canonical form is

$$V_{\xi\eta} - \frac{16}{49\sqrt{33}}\left(\frac{-7-\sqrt{33}}{8}\xi + \frac{7-\sqrt{33}}{8}\eta\right).$$

16. With $A = 1, B = 0$, and $C = 0$, $B^2 - 4AC = -36 < 9$, so the equation is elliptic. Solve $9a^2 + 1 = 0$ to get $a = \pm i/3$. Thus use the transformation

$$\xi = x,\, \eta = \frac{1}{3}t$$

to obtain the canonical form

$$V_{\xi\xi} + V_{\eta\eta} + \xi^2 - 3\eta V = 0.$$

1.2 Fourier Series

2. $\cos(3x)$ is the Fourier series of $\cos(3x)$ on $[-\pi, \pi]$. This converges to $\cos(3x)$ for $-\pi \le x \le \pi$.

4. The Fourier series of $f(x)$ on $[-2, 2]$ is

$$\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2}\cos(n\pi x/2),$$

converging to $1-|x|$ for $-2 \leq x \leq 2$. Figure 1.1 compares a graph of $f(x)$ with the fifth partial sum of the series.

6. The Fourier series is

$$\frac{2}{\pi}+\frac{4}{3\pi}\cos(x)-\sin(x)$$
$$+\sum_{n=2}^{\infty}\frac{4(-1)^{n+1}}{\pi(4n^2-1)}\cos(nx).$$

Figure 1.2 compares a graph of the function with the fifth partial sum of the series.

8. The Fourier series converges to

$$\begin{cases} \cos(x) & \text{for } -2<x<1/2, \\ \sin(x) & \text{for } 1/2<x<2, \\ (\cos(2)+\sin(2))/2 & \text{for } x=\pm 2. \end{cases}$$

10. The series converges to

$$\begin{cases} 1 & \text{for } -2<x<0, \\ -1 & \text{for } 0<x,1/2, \\ x^2 & \text{for } 1/2<x<2, \\ 0 & \text{at } x=0, \\ -3/8 & \text{at } x=1/2, \\ 5/2 & \text{at } x=\pm 2. \end{cases}$$

12. The series converges to

$$\begin{cases} 1-x & \text{for } -3<x<-1/2, \\ 2+x & \text{for } -1/2<x<1, \\ 4-x^2 & \text{for } 1<x<2, \\ 1-x-x^2 & \text{for } 2<x<3, \\ 3/2 & \text{at } x=-1/2, \\ 3 & \text{at } x=1, \\ -5/2 & \text{at } x=2, \\ -7/2 & \text{at } x=\pm 3. \end{cases}$$

14. Multiply by $f(x)$ to obtain

$$(f(x))^2 = \frac{1}{2}a_0 f(x) + \sum_{n=1}^{\infty}\left(a_n f(x)\cos(n\pi x/L) + b_n f(x)\sin(n\pi x/L)\right).$$

Integrate term by term:

$$\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{2}a_0\int_{-L}^{L}f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}f(x)\cos(n\pi x/L)\,dx + b_n\int_{-L}^{L}f(x)\sin(n\pi x/L)\,dx\right).$$

Then

$$\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{2}a_0(La_0) + \sum_{n=1}^{\infty}L(a_n^2 + b_n^2).$$

Upon division by L, this yields Parseval's equation.

16. The cosine series is

$$\sum_{n=1}^{\infty}\frac{4\sin(n\pi/2)}{n\pi}\cos(n\pi x/2),$$

converging to 1 for $0 \le x < 1$, to -1 for $1 < x \le 2$, and to 0 at $x = 1$. Figure 1.3 compares the function to the 100th partial sum of this cosine expansion. The sine series is

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}(-4\cos(n\pi/2) + 2(1 + (-1)^n))\sin(n\pi x/2),$$

converging to 0 at the end points and at 1, and to the function for $0 < x < 1$ and $1 < x < 2$. Figure 1.4 is the 100th partial sum of this sine series.