100 essential things YOU DIDN'T KNOW you didn't know about MATHS & the ARTS

JOHN D. BARROW

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About the Book

What can maths tell us about art and design?

Professor John D. Barrow has all the answers. In *100 Essential Things You Didn’t Know You Didn’t Know About Maths and the Arts*, he shows us that mathematics and the arts are not so far removed from each other. He takes us on a 100-step tour, guiding us through art forms as various as sculpture, literature, architecture and dance, and reveals what maths can tell us about the mysteries of the worlds of art and design.

We find out why diamonds sparkle, how many words Shakespeare knew and why the shower is the best place to sing. We discover why an egg is egg-shaped, why Charles Dickens crusaded against maths and how a soprano can shatter a wine glass without touching it ...

Enlivening the everyday with a new way of looking at the world, this book will enrich your understanding of the maths and art that surround us in our day-to-day lives.

About the Author

John D. Barrow is Professor of Mathematical Sciences and Director of the Millennium Mathematics Project at Cambridge University, Fellow of Clare Hall, Cambridge, a Fellow of the Royal Society, and formerly Professor of both Geometry and Astronomy at Gresham College, London. His previous books include *The Book of Nothing*, *The Constants of Nature, The Infinite Book, Cosmic Imagery*, the bestselling *100 Essential Things You Didn’t Know You Didn’t Know, The Book of Universes*, and, most recently, *100 Essential Things You Didn’t Know You Didn’t Know About Sport*.

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Theories of Everything

The Left Hand of Creation
(with Joseph Silk)

L'Homme et le Cosmos
(with Frank J. Tipler)

The Anthropic Cosmological Principle
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The World Within the World

The Artful Universe

Pi in the Sky

Perché il mondo è matematico

Impossibility

The Origin of the Universe

Between Inner Space and Outer Space

The Universe that Discovered Itself

The Book of Nothing

The Constants of Nature: From Alpha to Omega

The Infinite Book: A Short Guide to the Boundless, Timeless and Endless

New Theories of Everything

Cosmic Imagery: Key Images in the History of Science

100 Essential Things You Didn't Know You Didn't Know

The Book of Universes

100 Essential Things You Didn’t Know You Didn’t Know About Sport

TO DARCEY AND GUY

who are still young enough to know everything

‘Art is I, science is we’

Claude Bernard

100 Essential Things You Didn't Know You Didn't Know About Maths and the Arts

John D. Barrow

THE BODLEY HEAD
London

Preface

Maths is all around us, underpinning situations that are not commonly thought of as ‘mathematical’ at all. This is a collection of mathematical bits and pieces – unusual applications of mathematics to our usual surroundings. The situations are taken from the world of ‘the arts’, a broadly defined discipline encompassing the large subcontinents of design and the humanities from which I have chosen a hundred examples across a wide landscape of possibilities. The selection can be read in any order: some chapters interconnect with others, but most stand alone and offer a new way of thinking about an aspect of the arts, including sculpture, the design of coins and stamps, pop music, auction strategies, forgery, doodling, diamond cutting, abstract art, printing, archaeology, the layout of medieval manuscripts and textual criticism. This is not a traditional ‘maths and art’ book, covering the same old ground of symmetries and perspective, but an invitation to rethink how you see the world around you.

The diverse spectrum of links between mathematics and all the arts is not unexpected. Mathematics is the catalogue of all possible patterns – this explains its utility and its ubiquity. I hope that this collection of examples looking at patterns in space and time will broaden your appreciation of how simple mathematics can shed new light on different aspects of human creativity.

I would like to thank many people who encouraged me to write this book, or helped to gather illustrative material and bring it into its final form. In particular, I would like to thank Katherine Ailes, Will Sulkin and his successor Stuart Williams at Bodley Head. Thanks for their contributions are also due to Richard Bright, Owen Byrne, Pino Donghi, Ross Duffin, Ludovico Einaudi, Marianne Freiberger, Geoffrey Grimmett, Tony Hooley, Scott Kim, Nick Mee, Yutaka Nishiyama, Richard Taylor, Rachel Thomas, and Roger Walker. I would also like to thank Elizabeth and our growing family generations for noticing occasionally that this book was in progress. I just hope they notice when it comes out.

John D Barrow
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1

The Art of Mathematics

WHY ARE MATHS and art so often linked? We don't find books and exhibitions about art and rheology or art and entomology, but art and maths are frequent bedfellows. There is a simple reason that we can trace back to the very definition of mathematics.

Whereas historians, engineers and geographers will have little difficulty telling what their subjects are, mathematicians may not be so sure. There have long been two different views of what mathematics *is*. Some believe it is discovered, while others maintain that it is invented. The first opinion sees mathematics as a set of eternal truths that already 'exist' in some real sense and are found by mathematicians. This view is sometimes called mathematical Platonism. The second contrasting view sees mathematics as an infinitely large game with rules, like chess, which we invent and whose consequences we then pursue. Often, we set the rules after seeing patterns in Nature or in order to solve some practical problem. In any case, it is claimed, mathematics is just the outworking of these sets of rules: it has no meaning, only possible applications. It is a human invention.

These alternative philosophies of discovery or invention are not unique to the nature of mathematics. They are a pair of alternatives that go back to the dawn of philosophical thinking in early Greece. We can imagine exactly the same dichotomy applied to music, or art, or the laws of physics.

The odd thing about mathematics is that almost all mathematicians act as though they are Platonists, exploring and discovering things in a mentally accessible world of mathematical truths. However, very few of them would defend this view of mathematics if pressed for an opinion about its ultimate nature.

The situation is muddied somewhat by those, like me, who question the sharpness of the distinction between the two views. After all, if some mathematics is discovered, why can't you use it to invent some more mathematics? Why does everything we call 'mathematics' have to be either invented *or* discovered?

There is another view of mathematics which is weaker in some sense, in that it includes other activities like knitting or music within its definition, but I think it is more helpful for non-mathematicians. It also clarifies why we find mathematics to be so useful in understanding the physical world. On this third view mathematics is the catalogue of all possible patterns. This catalogue is infinite. Some of the patterns exist in space and decorate our floors and walls; others are sequences in time, symmetries, or patterns of logic or of cause and effect. Some are appealing and interesting to us but others are not. The former we study further, the latter we don't.

The utility of mathematics, that surprises many people, is on this view not a mystery. Patterns must exist in the universe or no form of conscious life could exist. Mathematics is just the study of those patterns. This is why it seems to be so ubiquitous in our study of the natural world. Yet there remains a mystery: why have such a small number of simple patterns revealed so much about the structure of the universe and all that it contains? It might also be noted that mathematics is remarkably effective in the simpler physical sciences but surprisingly ineffective when it comes to understanding many of the complex sciences of human behaviour.

This view of mathematics as the collection of all possible patterns also shows why art and mathematics so often come together. There can always be an identification of patterns in artworks. In sculpture there will be patterns in space; in drama there will also be patterns in time. All these patterns can be described by the language of mathematics. However, despite this possibility, there is no guarantee that the mathematical description will be interesting or fruitful, in the sense of leading to new patterns or deeper understanding. We can label human emotions by numbers or letters, and we can list them, but that does not mean that they will obey the patterns followed by numbers or by English grammar. Other, subtle patterns, like those found in music, clearly fall within this structural view of mathematics. This doesn’t mean that the purpose or meaning of music is mathematical, just that its symmetries and patterns comprise a little part of the great catalogue of possibilities that mathematics seeks to explore.

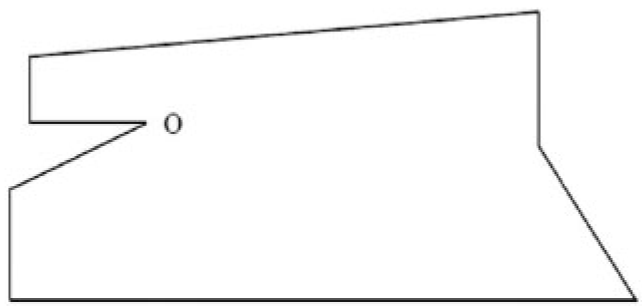
2

How Many Guards Does an Art Gallery Need?

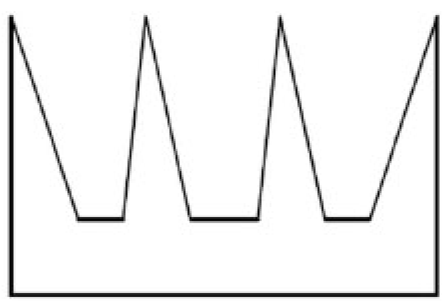
IMAGINE YOU ARE head of security at a large art gallery. You have many valuable paintings covering the gallery walls. They are hung quite low so that they can be viewed at eye level and therefore they are also vulnerable to theft or vandalism. The gallery is a collection of rooms of different shapes and sizes. How are you going to make sure that each one of the pictures is being watched by your attendants all of the time? The solution is simple if you have unlimited money: just have one guard standing next to every picture. But art galleries are rarely awash with money and wealthy donors don't tend to earmark their gifts for the provision of guards and their chairs. So, in practice, you have a problem, a mathematical problem: what is the smallest number of guards that you need to hire and how should you position them so that all the walls of the gallery are visible at eye level?

We need to know the minimum number of guards (or surveillance cameras) required to watch all of the walls. We will assume that the walls are straight and that a guard at a corner where two walls meet will be able to see everything on both those walls. We will also assume that a guard's view is never obstructed and can swivel around 360 degrees. A triangular gallery can obviously be watched by just one guard placed anywhere inside it. In fact, if the gallery floor is shaped like any polygon with straight walls whose corners all point outwards (a ‘convex’ polygon, like any triangle, for example) then one guard will always suffice.

Things get more interesting when the corners don’t all point outwards. Here is a gallery like that with eight walls which can still be watched by just one guard located at the corner O (although not if the guard is moved to the top or bottom left-hand corner):



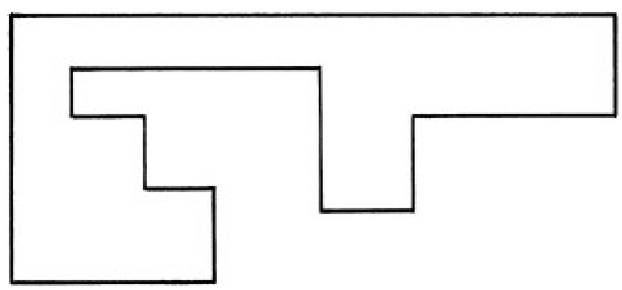
So, this is still a rather economical gallery to run. Here is another ‘kinkier’ twelve-walled gallery that is not so efficient. It needs four guards to keep an eye on all the walls:



To solve the problem in general we just look at how we can divide the gallery up into triangles that don’t overlap.[1] This can always be done. Since a triangle is one of those convex polygons (the three-sided one) that only need a single guard, we know that if the gallery can be completely covered by, say, T non-overlapping triangles then it can always be watched by T guards. It might, of course, be watched by fewer. For instance, we can always divide a square into two triangles by joining opposite diagonals but we don’t need two guards to watch all the walls – one will do. In general, the maximum number of guards that might be necessary to guard a gallery with W walls is the whole number part[2] of W/3. For our twelve-sided comb-shaped gallery this maximum is 12/3 = 4, whereas for an eight-sided gallery it is two.

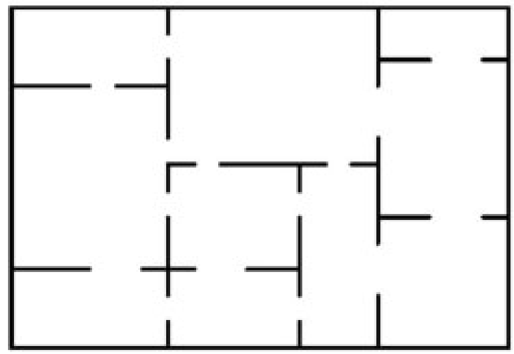
Unfortunately, determining whether you need to use the maximum is not so easy and is a so-called ‘hard’ computer problem for which the computing time can double each time you add another wall to the problem.[3] In practice, this will only be a worry if W is a very large number.

Most of the galleries you visit today will not have kinky, jagged wall-plans like these examples. They will have walls which are all at right angles like this:



If there are many corners in a right-angled gallery like this, then it can be divided up into rectangles each of which requires no more than one guard to watch its walls.[4] Now, the number of guards located at the corners that might be necessary and is always sufficient to guard the gallery is the whole number part of ¼ × Number of Corners: for the fourteen-cornered gallery shown here this is three. Clearly, it is much more economical on wages (or cameras) to have a gallery design like this, especially when it grows large. If you have a hundred and fifty walls then the non-right-angled design might need fifty guards; the right-angled set-up will need at most thirty-seven.

Another traditional type of right-angled gallery will be divided into rooms. Here is a ten-roomed example:



In these cases you can always divide the gallery up into a collection of non-overlapping rectangles. This is expedient because if you place a guard at the opening connecting two different rooms then both are watched at the same time. Yet no guard can watch three or more rooms at once. So now the number of guards that is sufficient, and occasionally necessary, to keep a complete watch on the gallery is the next whole number bigger than or equal to ½ × Number of Rooms, or five for the ten-roomed gallery drawn here. This is a more economical use of resources. All manner of more realistic scenarios have been studied by mathematicians, some in which the guards move, others in which they have limited fields of view or where there are mirrors to help them see around corners. There are also studies of the optimum routes for art thieves to take through a gallery watched by cameras or moving guards so as to avoid all of them! Next time you plan to steal the *Mona Lisa* you will have a head start.

3

Aspects of Aspect Ratios

AN ALARMINGLY LARGE fraction of the population spends a good deal of the waking day watching a TV or a computer screen. In fifty years' time there will no doubt be articles in learned journals that reveal effects on our eyesight over the period of the computer revolution that 'health and safety' was blind to.

The screens used in the computer industry have evolved towards particular shapes and sizes over the past twenty years. The 'size', as we had from the start with TV screens, is labelled by the length of the diagonal between opposite top and bottom corners of the monitor screen. The shape is defined by an 'aspect ratio' which gives the ratio of the width to the height of the screen. There have been three or four common aspect ratios used in the computer industry. Before 2003, most computer monitors had an aspect ratio of 4 to 3. So, if they were 4 units wide and 3 units high, then Pythagoras' theorem tells us that the length of the diagonal squared would be 4-squared (16) plus 3-squared (9), which equals 25, or 5-squared, and so the diagonal would be of length 5 units. Screens of this almost-square shape became the old TV-industry standard for desktop computers. Occasionally, you would see a monitor with a 5 to 4 aspect but 4 to 3 was the most common until 2003.

From 2003 until 2006 the industry moved towards an office standard of 16 to 10, that was less square and more ‘landscape’ in dimension. This ratio is almost equal to the famous ‘golden ratio’ of 1.618, which is presumably no accident. It has often been claimed by architects and artists to be aesthetically pleasing to the eye and has been widely incorporated into art and design for hundreds of years. Mathematicians have been aware of its special status since the days of Euclid. We shall meet it again in later chapters but for now we just need to know that two quantities, A and B, are said to be in the golden ratio, R, if:

$$A/B = (A + B)/A = R.$$

Multiplying across, we see that $R = 1 + B/A = 1 + 1/R$, so:

$$R^2 - R - 1 = 0.$$

The solution of this quadratic equation is the irrational number $R = \frac{1}{2}(1 + \sqrt{5}) = 1.618$.

The golden ratio aspect ratio, R, was used for the first-generation laptops, and then for stand-alone monitors that could be attached to any desktop. However, by 2010, things had undergone another evolutionary change, or perhaps it was just an arbitrary change, to a ratio of 16 to 9 aspect. These numbers – the squares of 4 and 3 – have a nice Pythagorean air to them and a screen that is 16 units wide and 9 high would have a diagonal whose length is the square root of 256 + 81 = 337, which is approximately 18.36 – not quite so round a number. Between 2008 and 2010 computer screens were almost all in the ratio 16 to 10 or 16 to 9 but by 2010 most had moved away from the golden ratio to the 16 to 9 standard, which is the best compromise for watching movies on computer screens. However, the user seems to be a loser again, because if you take two screens with the same diagonal size, then the old 4 to 3 aspect ratio results in a larger screen area than the newer 16 to 9 ratio: a 4 to 3 aspect 28-inch screen has a viewing area of 250 sq. inches, whereas its 16 to 9 aspect 28-inch counterpart has only 226 sq. inches of display.[1] Of course, the manufacturers and retailers who are seeking constantly to get you to upgrade your screen size will not tell you these things. An upgrade could well be a downgrade.

4

Vickrey Auctions

AUCTIONS OF WORKS of art or houses are open in the sense that participants hear the bids being made by their fellow bidders or their agents. The sale is made to the highest bidder, at the price of the top bid. This is a 'pay what you bid' auction.

The sellers of small items like stamps, coins or documents have made extensive use of another type of auction which can be operated by post or Internet as a 'mail sale' and is cheaper to operate because it doesn't need to be run by a licensed auctioneer. Participants send in sealed bids for a sale item by a specified date. The highest bidder wins the auction but pays the price bid by the second-highest bidder. This type of sealed-bid auction is called a Vickrey auction after the American economist, William Vickrey, who in 1961 studied its dynamics, along with those of other types of auction.[1] Vickrey certainly didn't invent this style of auction. It was first used to sell postage stamps to collectors and dealers in 1893 when auctions began to attract interest from bidders on both sides of the Atlantic and it was not practical for them to travel to the auction in person. Nowadays it is how Internet auctions like eBay work (although eBay requires the next bid to beat the previous highest by a minimum amount).

The usual 'pay what you bid' style of sealed-bid auction that is so popular with house sales has problems. If everyone putting in a sealed bid thinks that only he or she knows the real value of the item being sold, then each bid is likely to be less than the item's true value and the seller will be sold short. A buyer bidding for something like a house, whose value is less well defined, feels driven to overbid and can end up paying far more than should have been necessary to win in an open sale. Some buyers also feel nervous about putting in high bids to a sealed-bid auctioneer because they are giving information to the seller. If you see one item in a mixed auction lot that is very valuable, then by bidding high for it you signal something to the seller who may suddenly realise what you have seen and withdraw the item from sale.

Overall, the 'pay what you bid' sealed-bid style of auction seems to discourage people from buying and selling items for what they are worth. The Vickrey auction does much better. The optimal strategy to adopt in a Vickrey auction is to bid what you think the value of the item is. To see why, imagine that your bid is B and you judge the item's value to be V, while the largest bid from all the other bidders is L. If L is bigger than V then you should make your bid less than or equal to V so that you don't end up buying it for more than it is worth. However, if L is smaller than V then you should bid an amount equal to V. If you bid less you won't get the item more cheaply (it will still cost you L, the price of the second-highest bid) and you may lose it to another bidder. Your optimal strategy is therefore to bid an amount equal to the item's value, V.

5

How to Sing in Tune

THE PERFECT PITCH and note-hitting of pop singers often sounds suspicious, particularly when they are amateur competitors in talent shows. Listen to old music shows and there is nowhere near the same degree of perfection. Our suspicions are justified. Some mathematical tricks are being played which clean up and enhance a singer's performance so that out-of-tune voices sound precise and pitch perfect.

In 1996, Andy Hildebrand was using his signal processing skills to prospect for oil. He would study the rebounds of seismic signals sent below the earth to map out the underground distribution of rock and (he hoped) oil. Next, he decided to use his acoustic expertise to study correlations between different musical sounds and devise an automatic intervention system to remove or correct sounds that were off-key or in some other way discordant. Apparently it all started when he decided to retire from his oil prospecting and wondered what to do next. A dinner guest challenged him to find a way to make her sing in tune. He did.

Hildebrand's Auto-Tune program was first used by only a few studios but gradually became an industry standard that can in effect be attached to a singer's microphone for instantaneous recognition and correction of wrong notes and poor pitch. It automatically retunes the output to sound perfect regardless of the quality of the input. Hildebrand was very surprised by these developments. He expected his program to be used to fix occasional discordant notes, not to process entire productions. Singers have come to expect that their recordings will be processed through the Auto-Tune box. Of course, this has a homogenising effect on recordings, particularly those of the same song by different artists. At first this software was expensive, but cheap versions soon became available for home use or karaoke performers, and its influence has now become all-pervasive.

The first that most listeners not involved with the music business heard about all this was when a fuss blew up because contestants were having their voices improved by Auto-Tune when singing on the popular *X Factor* TV talent show. Following an outcry, the use of this device was banned on the show and the singers now found it far more challenging to sing live.

The Auto-Tune program doesn't just correct frequencies of the notes sung by the singer to the nearest semitone (keys on a piano keyboard). The frequency of a sound wave equals its speed divided by its wavelength so a frequency change would alter its speed and duration. This would make the music sound as if it was being continually slowed down or speeded up. Hildebrand's trick was to digitise the music into discontinuous sequences of sound signals and change the wave durations so as to keep it sounding right after the frequencies are corrected and the cleaned musical signal is reconstructed.

The process is complicated and relies on the mathematical method know as Fourier analysis. This shows how to split up any signal into the sum of different sinusoidal waves. It is as if these simple waves are the basic building blocks out of which any complicated signal can be constructed. The split of the complex musical signal into a sum of building-block waves with different frequencies and amplitudes enables the pitch correction and timing compensation to be effected very quickly, so that the listener doesn't even know it's being done. That is, of course, unless he or she suspects that the singer’s output is a little too perfect.

6

The *Grand Jeté*

BALLERINAS CAN SEEM to defy gravity and 'hang' in the air when they jump. They can't actually defy gravity, of course, so is all this talk of 'hanging in the air' mere hyperbole, created by overenthusiastic fans and commentators?

The sceptic points out that when a projectile, in this case the human body, is launched from the ground (and air resistance can be neglected) then its centre of mass[1] will follow a parabolic trajectory: nothing the projectile can do will change that. However, there is some fine print to the laws of mechanics: it is only the *centre of mass* of the projectile that must follow a parabolic trajectory. If you move your arms around, or tuck your knees into your chest, you can change the location of parts of your body relative to your centre of mass. Throw an asymmetrical object, like a tennis racket, through the air and you will see that one end of the racket may follow a rather complicated backward looping path through the air. The centre of mass of the racket, nevertheless, still follows a parabolic trajectory.

Now we can begin to see what the expert ballerina can do. Her centre of mass follows a parabolic trajectory but her head doesn't need to. She can change her body shape so that the trajectory followed by her head stays at one height for a noticeable period. When we see her jumping we only notice what her head is doing and don't watch the centre of mass. The ballerina's head really does follow a horizontal