Solutions Manual to Accompany

ORDINARY DIFFERENTIAL EQUATIONS

MICHAEL D. GREENBERG

WILEY

Contents

Solutions Manual to Accompany Ordinary Differential Equations

Michael D. Greenberg
Department of Mechanical Engineering
University of Delaware
Newark, DE

WILEY
A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.

Published simultaneously in Canada.

United States at (800) 762-2974, outside the United States at (317) 572-3993 or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print, however, may not be available in electronic formats. For more information about Wiley products, visit our web site at www.wiley.com.

Library of Congress Cataloging-in-Publication Data:

Greenberg, Michael D.

Ordinary differential equations / Michael D. Greenberg.

p. cm.

Includes bibliographical references and index.

ISBN 978-1-118-39899-9

1. Differential equations—Textbooks. 2. Differential equations, Partial—Textbooks. I. Title.

QA372.G725 2012

515'.352—dc23   2011042287

CHAPTER 1

First-Order Differential Equations

Section 1.1

This first section is simply to introduce you to **differential equations:** what they look like, some ideas as to how they arise in applications, and some important definitions. We see that the complete problem might be not just the differential equation, but also one or more "initial conditions." If such conditions are prescribed, the problem is called an **initial value problem, or IVP.** For instance, (6) [that is, equation (6) in the text] is an IVP because in addition to the DE (differential equation) there are two initial conditions, given by (6b), so that the solution of the IVP must satisfy not only the DE (6a), but also those two initial conditions.

Chapter 1 is about first-order equations; that is, equations in which the highest derivative is of first order. In that case, hence all through Chapter 1, there will be only one initial condition. In later chapters we will find that the "appropriate" number of initial conditions for a DE is the same as the order of the equation. For instance, (6a) is of *second* order and, sure enough, there are *two* initial conditions in (6b).

The distinction between linear and nonlinear differential equations will be of great importance, so it is necessary to be able to tell if a given equation is linear or nonlinear. Later, we will find that the key is whether or not a certain *linearity property* is satisfied, but for now it will suffice not to know about that property, but simply to say that an nth-order equation is linear if it is in, or can be put into, the form (14). What is the form of (14)? First, put all occurrences of the unknown, that is, the dependent variable such as y in (14), on the LHS (left-hand side of the equation); anything else goes on the right. If the LHS is a linear combination *of*

y, y',..., $y^{(n)}$, then the DE is linear. Actually, the "constants" that multiply y, y',..., $y^{(n)}$ in (14) are permitted to be functions of x; the point is that they don't depend on y or its derivatives.

EXAMPLES

Example 1. (Definitions) State the order of the

$$x^2y'' + xy' - 9y = 32x^5,$$

whether it is linear or nonlinear, homogeneous or nonhomogeneous, and determine whether or not the given functions are solutions, that is, whether or not they "satisfy" the DE:

$$y_1(x) = e^{2x}, y_2(x) = 4x^3 + 2x^5.$$

SOLUTION. The equation is of second order because the highest derivative present is of second order; it is linear because it is of the form (14), with $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = -9$, and $f(x) = 32x^5$; and it is nonhomogeneous because the RHS, $32x^5$ is not zero. The RHS does happen to be 0 at $x = 0$, but the equation is nonhomogeneous because the RHS is not *identically* zero on the interval under consideration. [Actually, we did not specify an x interval. The default interval is (-∞, ∞). Getting back to this example, surely x^2 is not identically zero on (-∞, ∞).]

Now test $y_1(x) = e^x$ to see if it is a solution of the DE. Simply substitute it into the equation and see if the equation is there by reduced to an identity, such as $6x + 3 \sin x = 6x + 3 \sin x$. Inserting $y_1(x)$, gives $x^2(4e^{2x}) + x(2e^{2x}) - 9e^2 = 32x^5$, or $(4x^2 + 2x - 9)\, e^{2x} = 32x^5$. Surely, the latter is not identically true. How do we *know* that?

Hopefully, we can just look at it and see that there is "no way" a quadratic in x times an exponential function of x can equal a multiple of a power of x. At the least we can use "brute force" and check the values of the LHS and RHS at one or more x's. For instance, a convenient point to use is $x = 0$, and there the LHS is -9 whereas the RHS is 0. Thus, $y_1(x)$ is not a solution of the DE.

Now test $y_2(x) = 4x^3 + 2x^5$. This time, putting the latter into the DE gives, after some canceling of terms, $32x^5 = 32x^5$ which is an identity. Thus, $y_2(x)$ is indeed a solution of the DE.

Now suppose we append to the DE these initial conditions at $x = 0$: $y(0) = 0$, $y'(0) = 0$. $y_2(x)$ does satisfy these conditions, so it is a solution of the IVP consisting of the DE and the two given initial conditions. If the initial conditions were $y(0) = 0$ and $y'(0) = 3$, say, instead, then $y_2(x)$ would not be a solution of the IVP, because although it satisfies the first initial condition, it does not satisfy the second.

Let's also bring *Maple* usage along, as we proceed. Here, let's use it to see if y_1 and y_2 are solutions of ("satisfy") the DE.

MAPLE:

y1 := exp(2·*x*) :

#(The # permits us to enter a "comment".) The foregoing line simply defines the function $y_1(x)$. x^2·*diff*(*y1*, *x*, *x*) + *x*·*diff*(*y1*, *x*) -9·*y1*(*x*)

(1) $4\,x^2\,e^{2\,x} + 2\,x\,e^{2\,x} - 9\,e^{2\,x}(x)$

The latter is not equal to $32x^5$ so $y_1(x)$ is not a solution of the DE. Now try $y_2(x)$.

y2 := 4·x^3 + 2·x^5 :

$x^2 \cdot diff(y2, x, x) + x \cdot diff(y2,x) - 9 \cdot y2$

(2) $x^2\left(24x+40x^3\right)+x\left(12x^2+10x^4\right)-36x^3-18x^5$

simplify(%)

(3) $32x^5$

Thus, $y_2(x)$ is a solution.

Example 2. (Is it a solution?) Is $y(x) = e^{-x^2}\left(\int_0^x e^{t^2}\,dt + A\right)$, in which A is any constant, a solution of the DE

$$y' + 2xy = 1?$$

SOLUTION. It is not surprising to find integrals within solutions to DEs; after all, integration is the inverse operation of the differentiations present in the DE. In most cases that occur in this book, such integrals can be evaluated in terms of the familiar elementary functions, but this integral cannot. Actually, it can be evaluated in terms of "nonelementary" functions, but let's not get into that; let's just leave it as it is. To see if the given $y(x)$ is a solution, differentiate it to obtain y':

$$y'(x) = (-2x)e^{-x^2}\left(\int_0^x e^{t^2}\,dt + A\right) + e^{-x^2}\left(e^{x^2} + 0\right),$$

and if we put that, and $y(x)$ into the DE we obtain

$$(-2x)e^{-x^2}\left(\int_0^x e^{t^2}\,dt + A\right) + e^{-x^2}\left(e^{x^2} + 0\right) + 2x\,e^{-x^2}\left(\int_0^x e^{t^2}\,dt + A\right) = 1,$$

which is seen, after cancelations, to be an identity. Thus, the given $y(x)$ is indeed a solution, for any value of the constant A. Of course, we could have used Maple, as we did in Example 1.

NOTE: To differentiate e^{-x^2}, we used chain differentiation:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg}\,\frac{dg}{dx},$$

with $f(g) = e^g$, and $g(x) = -x^2$. And to differentiate the integral we used the calculus formula

$$\frac{d}{dx}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

Example 3. (Classification) Classify the DE

$$\frac{y'' - 2}{y' + 3} = x.$$

SOLUTION. It is a linear second-order equation because it can be re-arranged as $y'' - 2 = x(y' + 3)$, or $y'' - xy' = 3x + 2$. That is, it is of the form (14), with $n = 2$, $a_0(x) = 1$, $a_1(x) = -x$, $a_2(x) = 0$**,** and $f(x) = 3x + 2$. And it is nonhomogeneous because $f(x)$ is not identically zero.

Example 4. Is the DE

$$\frac{y'' - 2}{y' + 3} = xy$$

linear or nonlinear?

SOLUTION. It is nonlinear, because when we try to rearrange it in the form (14) the best we can do is $y'' - xyy' - 3xy = 2$. The presence of the product yy' makes the equation nonlinear.

Example 5. (Seeking exponential solutions) A powerful and simple solution method that we will develop is that of seeking a solution in a certain form. For instance, see whether you can find any solutions of

$$y'' + 4y' = 0$$

in the exponential form $y(x) = e^{rx}$, in which r is a yet-to-be-determined constant.

SOLUTION. Just put the latter into the DE and see if any r 's can be found so that e^{rx} is a solution. That step gives $r^2 e^{rx} + 4re^{rx} = (r^2 + 4r)e^{rx} = 0$. Now, e^{rx} is not zero for any values of *x.* In fact, even if it were zero for certain values of *x* that wouldn't suffice, for we need substitution to reduce the DE to an identity, that is, for *all x.* Thus, we can cancel the e^{rx} and obtain $r^2 + 4r = 0$**.** That is merely a quadratic equation for *r,* and it gives the two values $r = 0$ and $r = -4$**.** Thus, we have been successful in finding solutions of the DE in the assumed exponential form, namely, both $y(x) = e^{0x} = 1$ and $y(x) = e^{-4x}$. These solutions are readily verified by substitution into the DE.

Are these the *only* solutions of the DE? If not, what are the others? We cannot answer these important questions yet, but we will in Chapter 2.

Section 1.2

As one begins with $y = mx + b$ when studying functions, the analogous starting point in solving differential equations is the first-order linear equation

$$y' + p(x)y = q(x),$$

in which $p(x)$ and $q(x)$ are known and $y(x)$ is the unknown. We see in this section that there is actually an infinite number of solutions of the latter since the "general solution," the "all-encompassing" solution, contains an arbitrary constant, usually called A (or C). Each different choice of A gives a solution.

EXAMPLES

Example 1. (Homogeneous equations) Find the particular solution of the IVP

$$y' = 6x^2 y, \; y(3) = 1,$$

and give its interval of existence.

SOLUTION. The DE is of the form (6), with $p(x) = -6x^2$, so its general solution is given by (8) as

$$y(x) = Ae^{-\int p(x)\,dx} = Ae^{\int 6x^2\,dx} = Ae^{2x^3}.$$

Then, the initial condition gives $y(3) = 1 = Ae^{54}$, so $A = e^{-54}$. Thus the desired solution is

$$y(x) = e^{-54}e^{2x^3} = e^{2x^3 - 54}.$$

For its interval of existence, we can use Theorem 1.2.1: $p(x) = -6x^2$ is continuous on $-\infty < x < \infty$, so the theorem assures us that the foregoing solution exists on $-\infty < x < \infty$.In this example we used the off-the-shelf solution formula (8). More generally, **in working the text exercises you can use whatever formulas are available in the text-**

unless the problem statement or your instructor asks for a specific line of approach.

Example 2. (This time using separation of variable to get the general solution) Derive the general solution of

$$xy' + 3y = 0,$$

this time not by using the solution formula (8), but by using the method of separation of variables.

SOLUTION. Divide by *x* and *y*, assuming that *y* ≠ 0, to separate the variables, then integrate:

$$\int \frac{dy}{y} + \int \frac{3\,dx}{x} = 0, \quad \ln|y| + 3\ln|x| = A, \quad \ln|x^3y| = A,$$

$$|x^3y| = e^A, \quad x^3y = \pm e^A \equiv C, \quad y(x) = C/x^3. \text{ (A)}$$

Now, - ∞ < *A* < ∞, so C= ± e^A is arbitrary, but nonzero (because e^A is nonzero for all A). Now check the possibility *y* = 0 that we disallowed when we divided the DE by *y*: We see that *y*(*x*) = 0 does happen to satisfy the DE, because its substitution gives 0 = 0, so we can bring that solution under the umbrella of (A) by now allowing C to be zero as well. Thus, the general solution of $xy' + 3y = 0$ is $y(x) = C/x^3$, with Can arbitrary constant.

Example 3. (Nonhomogeneous equations) Find the general solution of the DE

$$x^2y' + 3xy = 4.$$

Then find the particular solution corresponding to the initial condition *y*(2) = 0.

SOLUTION. The simplest way to get these solutions is to use (27) and (37), respectively, but, instead, let's begin by using the integrating factor method to find the general solution: Multiply the DE by a yet-to-be-determined function σ(*x*), so σ x^2y' + 3σ xy = 4σ, and require that $3\sigma x = \frac{d}{dx}\left(\sigma x^2\right),$ that is, 3σ x=2xσ +x^2σ'. The latter is separable, giving $\frac{d\sigma}{\sigma} = \frac{dx}{x}.$

Integrating (and not bothering to include an integration constant because all we need is *an* integrating factor, not the most general one), we obtain ln σ = ln x, so σ = x. Thus, our DE becomes $x^3y' + 3x^2y = 4x$. Now the coefficient *of y* is indeed the derivative of the coefficient *of y'*, $\frac{d}{dx}\left(x^3y\right) = 4x,$ which can be solved merely by integrating. That step gives $x^3y = 2x^2 + A$, so the general solution of the DE is

$$y(x) = \frac{2}{x} + \frac{A}{x^3},$$

in which A is an arbitrary constant. To evaluate A, apply the initial condition: $y(2) = 0 = 1 + A/8$, which gives A =-8. Thus, the particular solution satisfying $y(2) = 1$ is

(A) $$y(x) = \frac{2}{x} - \frac{8}{x^3}.$$

It is simpler to use (27) for the general solution, or (37) if we want the particular solution (but less helpful in achieving understanding):

General solution by (27): First, write the DE in the standard form as $y' + (3/x)y = 4/x^2$ so $p(x) = 3/x$ and $q(x) = 4/x^2$. Then, (25) gives σ(x) $= e^{\int (3/x)\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$ and (27) gives $y(x) = \frac{1}{x^3}\left(\int x^3 \frac{4}{x^2}\,dx + C\right) = \frac{2}{x} + \frac{C}{x^3},$ as found above. Having that general solution in hand, we can find the particular solution by applying the initial condition to that general solution, to find C. Instead, let's suppose we don't have the general solution, and let us get the desired particular solution directly from (37), with *a* chosen as the initial point, 2, and *b* = 0: Evaluating $\sigma(x) = e^{\int (3/x)dx} = x^3$, as above, then (37) gives

$$y(x) = \frac{1}{x^3}\left(\int_2^x s^3 \frac{4}{s^2}\, ds + (0)\left(2^3\right)\right) = \frac{1}{x^3}\left(2x^2 - 8\right) = \frac{2}{x} - \frac{8}{x^3},$$

as found above.

What is the interval of existence of the solution (A)? It is well-behaved (that is, continuous and even differentiable) on the two separate intervals (- ∞, 0) and (0, ∞). Of these, we must choose the latter since it is the one that contains the initial point *x* = 2. Thus, the interval of existence of (A) is (0, ∞).

Example 4. (Interchange of variables) Solve

$$y' = \frac{y}{4y - x}; \quad y(2) = 1.$$

SOLUTION. This DE cannot be put into first-order linear form (try it), so it is nonlinear. Hence, the methods of this section don't apply. However, try interchanging the roles of the independent and dependent variables, now letting *x, y* be the dependent and independent variables, respectively, so we seek *x*(*y*). Setting the *y*' equal to $\frac{1}{dx/dy}$, the DE becomes $\frac{dx}{dy} = \frac{4y - x}{y} = 4 - \frac{x}{y}$, or $\frac{dx}{dy} + \left(\frac{1}{y}\right)x = 4.$ Then, $\sigma(y) = e^{\int (1/y)\, dy} = e^{\ln y} = y,$ so the DE becomes *yx*' + *x* = 4*y*, or $\frac{d}{dy}(yx) = 4y.$ Thus, $yx = \frac{4y^2}{2} + A$ so $x(y) = 2y + \frac{A}{y}$. We could write the latter as $2y^2 - xy + A = 0$, then solve the latter by the quadratic formula for y, and then apply the initial condition to find A, but it is much simpler **to** apply the initial condition to the solution in the form $yx = 2y^2 + A$, given above: That is, set $x = 2$ and $y = 1$, so $2 = 2 + A$, so $A = 0$. *Thus,* $yx = 2y^2$, so $y = \frac{x}{2}$ is the desired solution to the IVP.

Example 5. (Direction field and straight-line solution)

(a) Find any straight-line solutions of the DE

$$y' + 3y = 9x.$$

(b) Then, obtain the direction field for that DE, on the box $-1 < x < 6$, $-15 < y < 15$.

(c) Obtain the direction field again, but this time including the solution curves through the initial points $(x, y) = (2, 1)$ [that is, $y(2) = 1$] and $(0,-1)$.

SOLUTION.

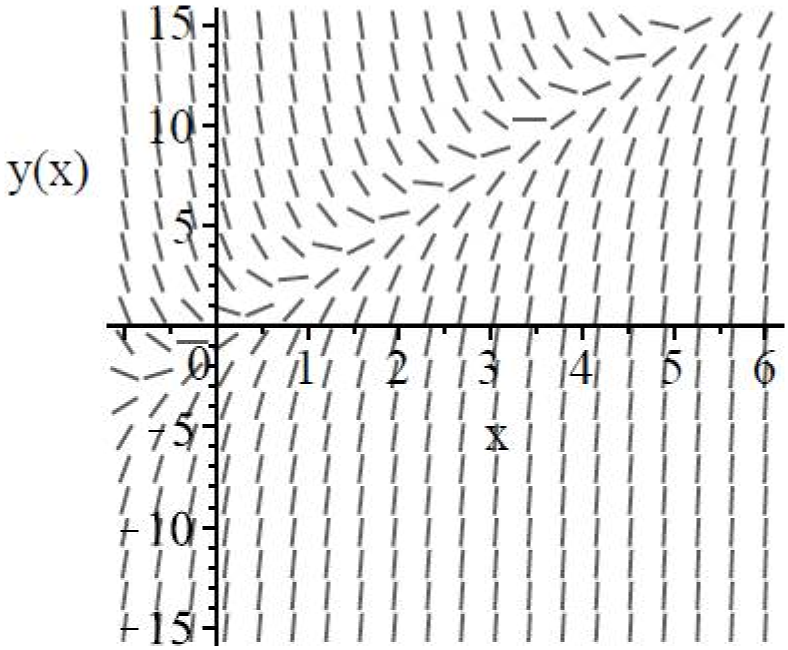
(a) That is, seek $y(x)$ in the form $y = mx + b$. Putting the latter into the DE gives

$m + 3(mx + b) = 9x$. The latter is of the form $ax + b = cx + d$, where a, b, c, d are constants. For the latter to be an identity we must "match coefficients": $a = c$, $b = d$. Thus, $3m = 9$ and $m + 3b = 0$, which give $m = 3$, $b = -1$**,** so we do find one straight-line solution, namely $y = 3x - 1$.

(b) Using *Maple,* with the arrows = line option, for instance:

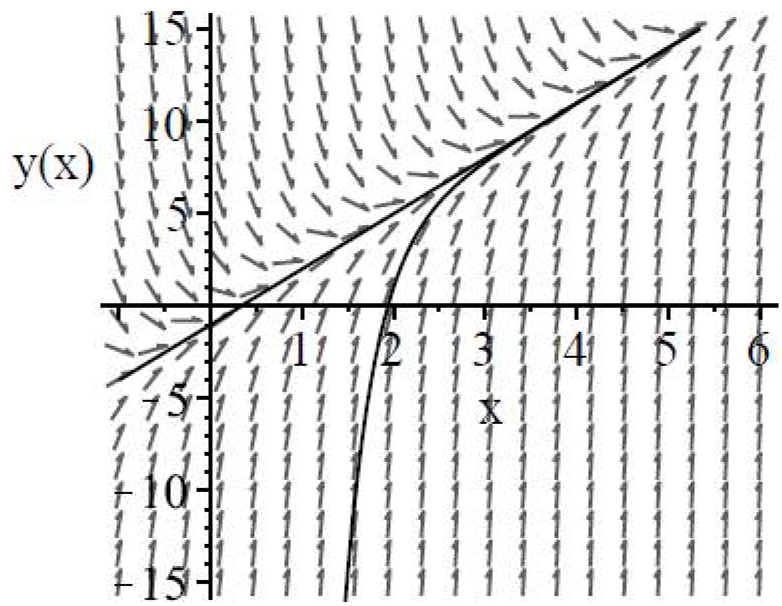
with (*DEtools*) :

dfieldplot(*diff*(*y*(*x*), *x*) + 3· *y*(*x*) = 9·*x*, *y*(*x*), *x* =-1 ..6, *y* =-15 ..15, *arrows = line*)



(c) To include solution curves we cannot use dfieldplot; instead use phaseportrait:

phaseportrait(*diff*(*y*(*x*),*x*) +3·*y*(*x*) = 9·*x*,*y*(*x*), *x* = -1 ..6, [[*y*(2) = 1], [*y*(0) =-1]], *stepsize* = 0.05, *y* =-15..15, *linecolor* = *black*, *thickness*= 1)



Note that the initial point *y*(0) = -1 gives the straight-line solution that we found in part (a).

Example 6. (Working backwards) If possible, find a first-order linear DE that hasy $y_1(x) = 1$ and $y_2(x) = x$ among its solutions.

SOLUTION. We'll just give a hint. Putting each of the two given solutions into $y' + p(x)y = q(x)$ will give equations that can be solved for $p(x)$ and $q(x)$.

Example 7. (Bernoulli's equation.) Bernoulli's equation will be covered in Section 1.8.1, so it will be simplest to refer you to that section and to Example 1 given therein.

Section 1.3

As its title indicates, this is an applications section. The only new mathematics is the material in Section 1.3.4 on the phase line, equilibrium points, and stability (of those equilibrium points), for autonomous equations. That subsection is a prerequisite for Chapter 7, which covers the phase plane for systems of *two* autonomous differential equations.

EXAMPLES

Example 1. (Exponential popnlation model) If a population governed by the exponential model has 4500 members after five years and 6230 after ten years, what is its growth rate? Its initial population?

SOLUTION. $N(t) = N(0)e^{kt}$, so $4500 = N(0)e^{5k}$ and $6230 = N(0)\ e^{10k}$. Dividing the latter two equations gives $e^{5k} = 1.384$, so the growth rate is $k = (\ln 1.384)/\ 5 = 0.065$. Putting that result into $6230 = N(0)e^{10k}$ gives the initial population $N(0) = 6230\ e^{-10(0.065)} = 3250$.

Example 2. (Exponential popnlation model) The world population is increasing at approximately 1.3% per year. If that growth rate remains constant, how many years will it take for its population to triple?

SOLUTION. It follows from (3) and the problem statement that $k = 0.013$, so $N(t) = N(0)e^{0.013t}$. For it to triple after T years, $N(T) = 3N(0) = N(0)e^{0.013T}$. Canceling $N(0)$'s and solving gives $T = (\ln 3)/0.013 = 84.5$ years.

Example 3. (E. coli population) A colony of E. coli is grown in a culture having a growth rate $k = 0.2$ per hour. (From $N' = kN$ it follows that k has dimensions of 1/time.) At the end of 5 hours the culture conditions are modified by increasing the nutrient concentration in the medium, such that the new growth rate is $k = 0.5$ per hour. If the initial population is $N(0) = 500$, evaluate $N(20)$, that is, after 20 hours.

SOLUTION. For $0 \leq t \leq 5$, $N(t) = N(0)e^{0.2t} = 500e^{0.2t}$ so $N(5) = 500e^{0.2(5)} = 500e$. Letting this time, $t = 5$, be the new initial time, we obtain $N(15) = (500\ e)e^{0.5(15)} = 2,$ 457, 000 as the population at the end of 20 hours.

Example 4. (Radioactive decay) (a) A seashell contains 90% as much C-14 as a living shell of the same size (that is, ofthe same weight). How old is it? NOTE: The half-life of C-14 is 5,570 years. (b) How many years did it take for its C-14 content to diminish from its initial value to 99% of that value?

SOLUTION. (a) It is more convenient to use (12) than (11) because we know T in (12), but would first need to evaluate k in (11) (from the known half-life): $m(t) = m_0\ 2^{-t/T}$ gives $0.9m_0 = m_0 2^{-t/5570}$, solution of which gives $t = 847$ years.

(b) $0.99m_0 = m_0\ 2^{-t/5570}$ gives $t = 81$ years.

Example 5. (Radioactive decay) If 20% of a radioactive substance disappears in 70 days, what is its half-life?

SOLUTION. $m^{(t)} = m_0 e^{-kt}$, so $0.8m_0 = m_0 e^{-70k}$, which gives $-70k = \ln 0.8$ and $k = 0.00319$. Thus $T = (\ln 2)/k =$ 217 days.

Example 6. (Mixing tank) For the mixing tank shown in the text Fig.3, let the initial concentration be $c(0) = 0$. At time T, the inflow concentration is changed from c_i to 0.

(a) Solve for $c(t)$, both *for* $t < T$ *and for* $t > T$.

(b) Taking $c_i = Q = v = T = 1$, for simplicity, sketch the graph of $c(t)$.

SOLUTION. (a) The problem is this, $c' + (Q/v)c = c_i\, Q/v$ *for* $t < T$, and $c' + (Q/v)c = 0$ for $t > T$ or,

$$c' + \left(\frac{Q}{v}\right)c = \begin{cases} c_i\, Q/v, & t < T \\ 0, & t > T \end{cases} \quad \text{with } c(0) = 0.$$

In Chapter 5 we will learn how to treat this as a single problem, using the Laplace transform, but here we will proceed by breaking the problem into two sequential problems. The first is $c' + (Q/v)c = c_i\, Q/v$, with $c(0) = 0$, the solution of which is

(A) $c(t) = c_i\left(1 - e^{-Qt/v}\right)$ for $t < T$.

Use the final value from the first time interval, $c(T)$, as the initial value for the second time interval. Thus, for the second time interval the problem is $c' + (Q/v)c = 0$ with initial condition $c(T) = c_i(1-e^{-QT/v})$. Its solution is

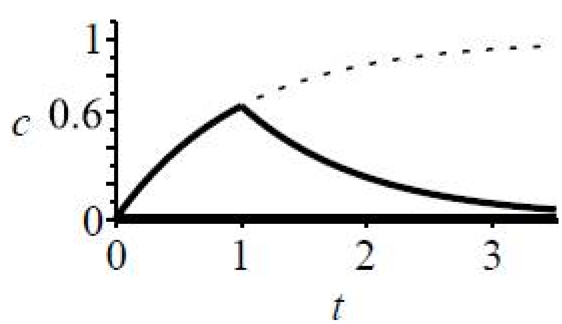
$c(t) = c_i\left(1 - e^{-QT/v}\right) e^{-(Q/v)(t-T)}$ for $t > T$. (B)

Don't let c_i, Q, v, T confuse you; they are simply constants.

(b) Setting $c_i = Q = v = T = 1$, the solution is

$$c(t) = \begin{cases} 1 - e^{-t}, & t < 1, \\ 0.632\, e^{-(t-1)}, & t > 1. \end{cases}$$

The graph looks like the solid curve



Example 7. (Phase line) Develop the phase line for the autonomous DE $x' = x^2 - x$, identify any equilibrium points, and classify each as stable or unstable.

SOLUTION. Sketch the graph of $f(x) = x^2 - x$ versus x. It is a parabola with a minimum at $x = 1/2$ and is zero at $x = 0, 1$, which are the two equilibrium points. Now draw the phase line, which is the x axis: $f > 0$ for $x < 0$, so for $x < 0$ the flow arrow is to the right; similarly for $x > 1$ the flow arrow is to the right; and $f < 0$ for $0 < x < 1$, so there the flow is to the left. Since the flow approaches the equilibrium point at $x = 0$, that equilibrium point is stable; and since the flow is away from the equilibrium point at $x = 1$, that one is unstable.

Example 8. (Light extinction; Lambert's law) Consider window glass subjected to light rays normal to its surface, and let x be a coordinate normal to that surface, with $x = 0$ at the incident face. It is found that the light intensity I in the glass is not a constant, but diminishes with x according to Lambert's law, which says that $\frac{dI}{dx} = -kI$. If 80% of the light penetrates a 1-inch-thick slab of this glass, how thin must the glass be to let 95% penetrate?

SOLUTION. The solution of the DE, with initial condition $I(0) = I_0$, is $I(x) = I_0 e^{-kx}$. From the data given, $0.8\, I_0 = I_0 e^{-k\,(1)}$, which yields $k = 0.223$. Thus, $I(x) = I_0 e^{-0.223\,x}$. Then, $0.95\, I_0 = I_0 e^{-0.223\,x}$ gives $x = 0.230$ inches as the thickness of the slab.

Example 9. (Cooling of coffee) Newton's law of cooling states that a body that is hotter than its environment will cool at a rate that is proportional to the difference of the temperatures $u(t)$ of the body, and U of the environment, so that

(A) $\frac{du}{dt} = k(U - u)$,

in which k is the constant of proportionality - which can be determined empirically. The equation (A) is a linear equation $u' + ku = kU$, with general solution

(B) $u(t) = U + Ae^{-kt}$.

in which A is an arbitrary constant. Here is the problem: A cup of coffee in a room that is at 70° F is at 200°F when it is poured. After 10 minutes it has cooled to 180°F.

(a) How long will it take to cool to 100°F?

(b) What will be its temperature three hours after it was poured?

SOLUTION. If we take into account that the coffee temperature is not spatially uniform within the cup then the problem is MUCH more difficult, so let us assume that it is indeed spatially uniform, and hence a function only of t, which seems not such a bad assumption since the cooling process is so slow that the temperature within the cup has the opportunity to remain spatially equilibrated.

(a) Now, $U = 70$, so (A) becomes $u(t)=70+Ae^{-kt}$, Next, $u(0) = 200 = 70+Ae^{0}$ gives $A = 130$, so $u(t) = 70 + 130e^{-kt}$. Next, the data that $u(10) = 180$ enables us to evaluate k: $180 = 70 + 130e^{-10k}$, which gives $k = 0.01671$, so $u(t) = 70 + 130e^{-0.016711t}$. Finally, $u(T) = 100 = 70 + 130\ e^{-0.01671T}$ gives $T = 87.8$ minutes.

(b) And after 3 hours (180 minutes), $u(180) = 70 + 130e^{-0.01671(180)} = 76.4$°F.

Section 1.4

First-order linear equations $y' + p(x)y = q(x)$ are an "open and shut case," in the sense that the general solution is known, and we even know that the particular solution satisfying any initial value $y(a) = b$ exists, and is unique on the broadest open x interval, containing the initial point a, on which both $p(x)$ and $q(x)$ are continuous. As we tum now to equations $y' = f(x,y)$ that are nonlinear, we find that we can obtain solutions only in special cases, by methods that are specialized to those cases. Further, even when we do find a solution, it may be in the less convenient implicit, rather than explicit, form. In this section we begin with the first of those cases, the very important case in which the equation happens to be separable: that is, in which $f(x,y)$ can be factored as a function ofxtimes a function of y. As Section 1.3 covers applications of first-order linear equations, Section 1.6 will cover applications of nonlinear equations.

EXAMPLES

In Examples 1 and 2, the given IVP is solvable by separation of variables. Solve, use computer software to get a graph of the solution, and state its interval of existence. In Examples 3 - 6 the solution will be in implicit form, and we will need to deal with that extra "wrinkle." I think these examples will be challenging in terms of the graphs, especially for the cases in which the solution is obtained only in implicit form, so you are urged to pay special attention to that aspect. In some cases one can look at the functions and successfully develop a hand sketch of the relevant graph, but in general you will probably need some computer graphics support; we'll use *Maple,* but you may use any other CAS that you prefer. Also, we might add that there is certainly a pattern to the solutions that follow, but you will need to think your way through the steps, rather than following a step-bystep "procedure."

Example 1. (Solution by separation of variables) Solve the IVP

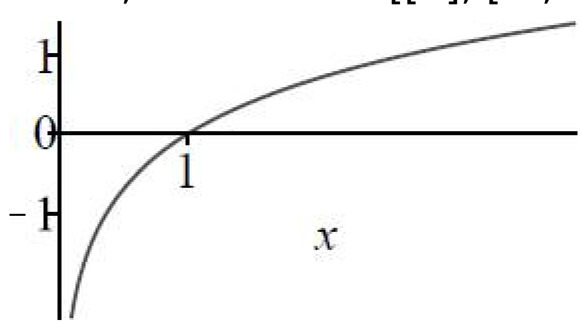
$$y' - 3x^2e^{-y} = 0; \quad y(0) = 0.$$

SOLUTION. Divide by e^{-y} (or, multiply by e^y), multiply by dx, and integrate: $\int e^y dy - \int 3x^2 dx = 0,$ $e^y - x^3 = C$. Rather than solve the latter for $y(x)$ and then applying the initial condition, to evaluate C, it is more convenient to apply the initial condition first: $e^0 - 0 = C$, which gives $C = 1$. Thus, $e^y = x^3 + 1$. The latter is the desired solution, but in implicit form. We happen to be able to solve for y, and we obtain the solution in explicit form as

(A) $\mathbf{y(x) = \ln(x^3 + 1).}$

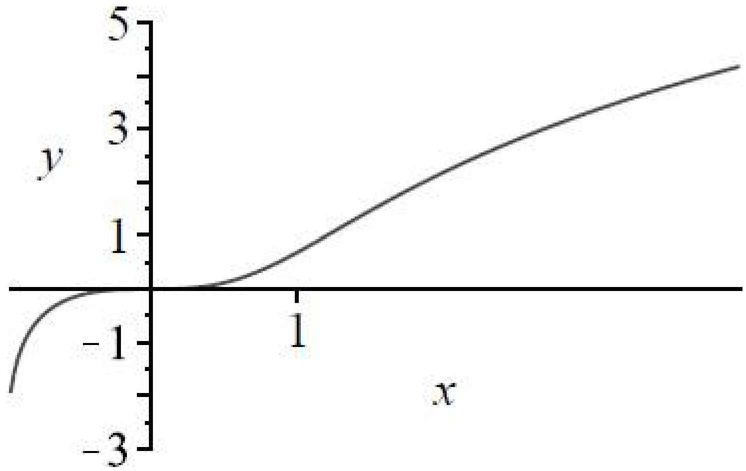
Remember that ln x tends to -∞ as $x \to 0$ (from the right), is 0 at $x = 1$, and then increases monotonely without bound asx increases. Including the *Maple* plot command, here is the graph of ln x, to keep in mind:

with (*plots*):
plot(ln(x), x = 0.1..4, *tickmarks*= [[1], [-1, 0, 1]])



Thus, the point(s) to watch out for, in (A), are those at which the argument $x^3 + 1 = 0$, namely, at $x = -1$. The graph of the solution (A) is this:

plot(ln(x^3 + 1), *x* =-.95..4, *y* =-3..5, *tickmarh* = [[1], *default*])



Note the approach to -∞ *as x* → -1, that we expected. The interval of existence of the solution (A) is (-1, ∞).

NOTE: The plot command doesn't give us the option of putting a heavy dot at the initial point [(0,0) in this case]. Ifwe really want to to that, we can do two plots: One would be a point plot, just plotting that single point, and the second would be the plot given above, and then we would use the display command to plot them together. We won't do that here, but that sequence is discussed in the *Maple* tutorial section.

Example 2. (Separation of variables) Solve

$$y' = (y+1)^2,$$

subject to each ofthese initial conditions: *y*(0) = -3, *y*(0) = -1, *y*(0) = 3.

SOLUTION. First, $\int (y+1)^{-2} dy = \int dx, \ -\frac{1}{y+1} = x + C.$ Apply each initial condition, in turn. $\boldsymbol{y(0) = -3:} \ -\frac{1}{-3+1} = C$ gives $C = \frac{1}{2}$ so $-\frac{1}{y+1} = x + \frac{1}{2}$. Which gives