Mathematical Physics Studies

Kasia Rejzner

Perturbative Algebraic Quantum Field Theory

An Introduction for Mathematicians

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York, UK Kasia Rejzner

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Chapter 1
Introduction

Quantum field theory is an important research area in theoretical physics, with a wide range of applications and an impressive agreement with experiment. Despite this success, the mathematical foundations of this theory are still under investigation and many fundamental questions remain open. The rapid development of the field makes it difficult to find textbooks which are up to date with all the recent advances, especially if one looks for a mathematically rigorous approach. It is a common misconception that working with QFT necessarily implies doing something "not-well defined", while in fact most of the formal manipulations presented in the physics literature can be made completely rigorous.

For me quantum field theory is a beautiful bizarre world full of wonders suspended somewhere in-between mathematics and physics. It charms physicists by providing results that agree with experiments with incredible precision. It lures mathematicians seeking to explore the land of QFT and get a closer look at the beautiful mathematical structures that inhabit it. And yet, after more than 50 years of research, we do not fully understand what QFT really is and what wonders it is hiding from us deep in its conceptual roots.

As both a physicist and a mathematician, I am fascinated by the richness of structures that one can encounter in QFT land, and from my first visit I have decided that I do not want to leave it ever again. So what is this book about? Well, maybe first I should explain what it isn't about…It is far from being a complete account of what has been done in QFT research (this would have taken multiple volumes!). It also doesn't touch the problem of non-perturbative construction of models of interacting quantum field theories, which at the moment remains open.

You can think of this book as a mathematician's diary from a journey into an exotic land. As opposed to some other textbooks on the subject, I will not use the excuse that "physicists often do something that is not well defined", so as mathematicians we don't need to bother and just turn around for a while, until it's over. Instead, I will jump straight into the lion's den and will try to make mathematical sense of perturbative QFT all the way from the initial definition of the model to the interpretation of the

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results. This is not always easy and sometimes I will have to bring into the story results from several fields of mathematics at once. I hope this will not discourage you from exploration of the QFT wonderland. After all, its beauty lies in the fact that it is so diverse and full of surprises… So, come along! Our journey starts here.

Chapter 2
Algebraic Approach to Quantum Theory

2.1 Algebraic Quantum Mechanics

Before entering the realm of the quantum theory of fields, let's have a look at something simpler and better understood, namely *quantum mechanics* (QM). To prepare the ground for what follows, we will present an abstract formulation of QM and discuss how it relates to the more standard Dirac–von Neumann axioms [Dir30, vN32]. The exposition presented in this chapter is based on [BF09b, Mor13, Fre13, Str08].

2.1.1 Functional Analytic Preliminaries

Let us start by recalling some basic definitions from functional analysis. For more information see [Rud91, RS80, BR87, BR97, Kad83]. Readers familiar with basic functional analysis can skip this subsection.

Definition 2.1 An *algebra* $\mathfrak{A}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a $\mathbb{K}$-vector space with an operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ called the *product* with the following properties:

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C), \quad \forall A, B, C \in \mathfrak{A}$ (associativity),
2. $A \cdot (B + C) = A \cdot B + A \cdot C$, $(B + C) \cdot A = B \cdot A + C \cdot A$,
   $\alpha(A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B)$, for all $A, B, C \in \mathfrak{A}$, $\alpha \in \mathbb{K}$ (distributivity).

We will usually denote the algebra product $\cdot$ simply by juxtaposition, i.e. $A \cdot B \equiv AB$.

Definition 2.2 An algebra $\mathfrak{A}$ is said to have a unit (i.e. $\mathfrak{A}$ is unital) if there exists an element $\mathbb{1} \in \mathfrak{A}$ such that $\mathbb{1}A = A\mathbb{1} = A$, for all $A \in \mathfrak{A}$.

Definition 2.3 An *involutive complex algebra* (a $*$-*algebra*) $\mathfrak{A}$ is an algebra over the field of complex numbers, together with a map, $^* : \mathfrak{A} \to \mathfrak{A}$, called an *involution*. The image of an element A of $\mathfrak{A}$ under the involution is written A^*. Involution is required to have the following properties:

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1. for all $A, B \in \mathfrak{A}$: $(A+B)^* = A^* + B^*$, $(AB)^* = B^* A^*$,
2. for every $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^* = \overline{\lambda} A^*$,
3. for all $A \in \mathfrak{A}$: $(A^*)^* = A$.

Definition 2.4 A $*$-morphism is a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ between $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, which is an algebra morphism compatible with the involution, i.e.:

1. $\varphi(AB) = \varphi(A)\varphi(B)$, for all $A, B \in \mathfrak{A}$,
2. $\varphi(\lambda A + B) = \lambda\varphi(A) + \varphi(B)$, for all $A, B \in \mathfrak{A}$, $\lambda \in \mathbb{C}$,
3. $\varphi(A^*) = \varphi(A)^*$ for every $A \in \mathfrak{A}$.

Up to now all the properties we have considered are purely algebraic. In order to quantify the notion of distance between the elements of the algebra we need some topology.

Let us start with some basic definitions and notation.

Definition 2.5 *A topological space* $\mathfrak{X}$ is a pair (X, τ), where X is a set X and τ is a collection of subsets of X (called open sets), with the following properties:

- $X \in \tau$
- $\varnothing \in \tau$
- the intersection of any two open sets is open: $U \cap V \in \tau$ for $U, V \in \tau$
- the union of every collection of open sets is open: $\bigcup_{\alpha \in A} U_\alpha \in \tau$ for $U_\alpha \in \tau \; \forall \alpha \in A$, where A is some index set.

Consider mappings between topological spaces. A topology tells us something about the regularity of those mappings, since it contains already a notion of "being close to something" and we can ask ourselves to what extend a given map preserves this notion.

Definition 2.6 A function $f : \mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are topological spaces, is *continuous* if and only if for every open set $V \subseteq Y$, the inverse image:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \tag{2.1}$$

is open.

Given a collection of topological spaces, one can define a new topological space by taking their *Cartesian product*. This is a very commonly used operation, so we recall here the definition of a natural topology on such product.

Definition 2.7 Let X be a set such that

$$X = \prod_{i \in I} X_i$$

is the *Cartesian product of topological spaces* X_i, indexed by i in some set I. Let $p_i : X \to X_i$ be the canonical projections. The product topology on X is defined as the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous.

In our applications the topology will not be enough to capture all the structure we need. In the physics context it is common that we want to add certain quantities and scale them. This leads in a natural way to a vector space structure. We want this structure to be compatible also with the topology.

Definition 2.8 A *Topological vector space* (TVS) over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (with their standard topologies) is a pair $(X, \tau) \equiv \mathcal{X}$, where τ is a topology such that:

- every point of X is a closed set (i.e. its complement is an open set),
- vector addition $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and scalar multiplication $\mathbb{K} \times \mathcal{X} \to \mathcal{X}$ are continuous functions with respect to the product topology on the respective domains.

Definition 2.9 Let $\mathcal{X}$, $\mathcal{Y}$ be topological vector spaces over the field $\mathbb{K}$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ to $\mathcal{Y}$ and by $\mathcal{X}'$ the topological dual of $\mathcal{X}$, i.e. the space of continuous linear maps from $\mathcal{X}$ to $\mathbb{K}$.

A topology can be introduced for example by means of a norm. This leads to the concept of a normed space.

Definition 2.10 A complex normed space is a vector space $\mathcal{X}$ over $\mathbb{C}$, equipped with a map $\|.\| : \mathcal{X} \to \mathbb{R}$, which satisfies:

1. $\|\lambda A\| = |\lambda|\|A\|$ (scaling),
2. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality also called subadditivity),
3. If $\|A\| = 0$, then A is the zero vector (separates points).

One of the nice features of normed spaces is that the continuity of maps between such spaces can be probed by convergent sequences. Recall that in general:

Definition 2.11 A point x of the topological space $\mathcal{X}$ is the limit of the sequence (x_n) in $\mathcal{X}$ if, for every neighbourhood U of x, there is an N such that, for every $n \geq N$, $x_n \in U$.

In particular, for normed spaces:

Definition 2.12 A point x of a normed space $(X, \|.\|)$ is the limit of the sequence (x_n) if, for all $\varepsilon > 0$, there is an N such that, for every $n \geq N$, $\|x_n - x\| < \varepsilon$. A sequence that has a limit is called convergent.

Definition 2.13 Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces. Then a function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *sequentially continuous* if for every convergent sequence (x_n) in $\mathcal{X}$ with the limit x we have $f(x_n) \to f(x)$ in $\mathcal{Y}$.

An elementary result from analysis states that if $\mathcal{X}$, $\mathcal{Y}$ are normed spaces equipped with topologies induced by the respective norms then $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it is sequentially continuous. However, in Sect. 2.4.1 we will consider spaces where these two notions do not coincide.

Having defined the notion of convergence of sequences, we are now ready to introduce the notion of *completeness*. First we define:

Definition 2.14 A sequence (x_n) in a normed space $\mathcal{X}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers m, n such that $m, n > N$ we have $\|x_n - x_m\| < \varepsilon$.

Definition 2.15 A normed space $\mathcal{X}$ in which every Cauchy sequence converges to an element of $\mathcal{X}$ is called *complete*.

Given a normed space $\mathcal{X}$ that is not complete one can always construct its *completion*,[1] i.e. a complete normed space that contains $\mathcal{X}$ as a dense subspace.

Let us now come back to our algebras. If an algebra $\mathfrak{A}$ is equipped with a norm, we can ask for the continuity of the algebraic relations with respect to the norm topology and for some notion of completeness. This leads to the following definitions.

Definition 2.16 A *normed algebra* $\mathfrak{A}$ is a normed vector space whose norm $\|.\|$ satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

If $\mathfrak{A}$ is unital, then it is a normed unital algebra if in addition $\|\mathbb{1}\| = 1$.

Definition 2.17 A *Banach space* is a normed vector space equipped with the norm-induced topology that is complete with respect to this topology. A Banach (unital) algebra is a Banach space and a normed (unital) algebra with respect to the same norm.

A particularly important class of Banach algebras with involution is distinguished by the C^*-property. We will see in this chapter that such algebras can be used to describe spaces of observables in quantum systems.

Definition 2.18 A *C^*-algebra* is a Banach involutive algebra (Banach algebra with involution satisfying $\|A^*\| = \|A\|$), such that the norm has the C^*-property:

$$\|A^*A\| = \|A\|\|A^*\|, \quad \forall A \in \mathfrak{A}.$$

2.1.2 Observables and States

In this section we will see how the structures introduced in the previous section are used in quantum physics. First note that in order to describe a physical system we need to specify a collection of physical quantities, which we want to measure (we call them *observables*) and a collection of *states* in which the system can be prepared. Now we want to deduce what kind of mathematical structure is suitable to describe observable and states. Operationally, each observable corresponds to some measurement apparatus, which measures given properties of the system. An example of such an apparatus is a particle detector localized in some region of space.

---

[1]The completion of $\mathcal{X}$ can be constructed as a set of equivalence classes of Cauchy sequences in $\mathcal{X}$.

Next, one considers operations that can be performed on observables. Scaling of the measurement apparatus means multiplying the corresponding observable A by a real number. One can also consider other functions of the observables, which can be operationally realized as "repainting the scale". The simplest examples are monomials A^n, interpreted as measuring the observable A and taking the nth power of the result.

Now we discuss the notion of states. We need to assume that we are able to repeat experiments, so that we can measure a given observable repeatedly in the same state (i.e. for the same preparation of the system). This statistical interpretation presupposes that each experiment comes with a protocol that allows us to obtain the same initial condition each time it is repeated. Under this assumption, a state ω associates to an observable A a real number $\omega(A)$ obtained by averaging the results of measurements of A for the system prepared to be in the state ω. It is natural to assume that $\omega(\lambda A) = \lambda\omega(A)$ for $\lambda \in \mathbb{R}_+$ (scaling). Let $\mathbb{1}$ be the observable, which always takes value 1. For this observable we require that $\omega(\mathbb{1}) = 1$. One can also deduce the positivity of states from the fact that the average of positive numbers is positive, so $\omega(A^2) \geq 0$.

If we assume that physical properties of observables can be measured only by looking at expectation values in various states of the system, it is natural to identify the observables that give the same expectation values in all the states. Now let $\mathcal{A}$ be the space of equivalence classes of observables, where $A \sim B$ if $\omega(A) = \omega(B)$ for all states ω of the system. A notion of a norm can be introduced by assigning to each observable $A \in \mathcal{A}$ a finite positive number defined by

$$\|A\| \doteq \sup_{\omega} |\omega(A)|$$

The operational properties of states imply that $\|\lambda A\| = |\lambda|\|A\|$ for $\lambda \in \mathbb{R}$ and $\|A\| = 0$ implies that $A = 0$ (states separate observables). What is still missing is the linear structure on $\mathcal{A}$ and the product. Let us start with the linear structure. We want to be able to construct measuring devices that measure the sum of any two observables A and B, i.e. we need the operation "$A + B$". This operation has to satisfy

$$\omega(A + B) = \omega(A) + \omega(B),$$

for all states of the system. It is, however, not clear if an element "$A + B$" exists in $\mathcal{A}$, so one needs to embed the initial space of observables in a larger structure in such a way that states will remain positive linear functionals on this enlarged space. Further considerations (see for example [Str08]) lead to the notion of Jordan algebras [Jor33, JvNW34] and finally, by bringing in a complex structure, to C^*-algebras, introduced in [Gel43] and discussed in [Seg47a, Seg47b] in the context of quantum mechanics.

We can summarize the basic axioms in the algebraic approach to QM as follows:

1. A physical system is defined by its unital C^*-algebra $\mathfrak{A}$.
2. States are identified with positive, normalized linear functionals on $\mathfrak{A}$, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and $\omega(\mathbb{1}) = 1$.

Note that on a unital C^*-algebra a positive, normalized linear functional is automatically continuous with respect to the topology induced by the C^*-norm. More generally, we can define states also on involutive topological algebras.

Definition 2.19 A *state* on an involutive algebra $\mathfrak{A}$ is a linear functional ω, such that:

$$\omega(A^*A) \geq 0,\ \omega(\mathbb{1}) = 1.$$

Observables are self-adjoint elements of $\mathfrak{A}$ and possible measurement results for an observable A are characterized by its spectrum $\sigma(A)$. Recall that an element A of a C^*-algebra is called self-adjoint if $A^* = A$.

Definition 2.20 The *spectrum* $\mathrm{spec}(A)$ of $A \in \mathfrak{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda\mathbb{1}$ has no inverse in $\mathfrak{A}$.

A standard result from functional analysis states that a spectrum of self-adjoint element is a subset of the real line and this agrees with the physical intuition, as outcomes of measurements have to be real.

2.1.3 Hilbert Space Representations

Having defined the abstract setup we can proceed to a more concrete description that provides a way to recover the Dirac–von Neumann axioms. The crucial observation is that abstract elements of an involutive algebra $\mathfrak{A}$ can be realized as operators on some Hilbert space by a choice of a *representation*. Definitions introduced in this section follow closely [Mor13, RS80]. First let us recall the definition of a Hilbert space.

Definition 2.21 Let $\mathcal{H}$ be a complex vector space. A map $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a *Hermitian inner product* if

1. $\langle u, v \rangle = \overline{\langle u, v \rangle}, \forall u, v \in \mathcal{H}$,
2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$ (linear in the second argument),
3. $\langle v, v \rangle \geq 0$ where the case of equality holds precisely when $v = 0$ (positive definite).

Properties 1 and 2 imply that $\langle .,. \rangle$ is antilinear in the first argument. One can define a norm on $\mathcal{H}$ by setting

$$\|v\| \doteq \sqrt{\langle v, v \rangle}.$$

Definition 2.22 A *Hilbert space* $\mathcal{H}$ is a complex vector space with a Hermitian inner product $\langle ., . \rangle$ such that the norm induced by this product makes $\mathcal{H}$ into a Banach space.

In physics *separable Hilbert spaces* play an important role.

Definition 2.23 A Hilbert space $\mathcal{H}$ is called *separable* if it admits a countable subset whose linear span is dense in $\mathcal{H}$. In fact a Hilbert space is separable if it is either finite dimensional or has a countable basis.

We are ready to define the notion of linear operators on Hilbert spaces, which is important in the context of C^*-algebras and physical observables.

Definition 2.24 An *operator* A on a Hilbert space $\mathcal{H}$ is a linear map from a subspace $D \subset \mathcal{H}$ into $\mathcal{H}$. In particular, if $D = \mathcal{H}$ and A satisfies $||A|| \doteq \sup_{||x||=1}\{||Ax||\} < \infty$, it is called *bounded*.

We will always assume that D is dense in $\mathcal{H}$ (i.e. A is *denesly defined*).

Definition 2.25 Let A be a densly defined linear operator on a Hilbert space $\mathcal{H}$. Let $D(A^*)$ be the set of all $v \in \mathcal{H}$ such that there exists $u \in \mathcal{H}$ with

$$\langle Aw, v \rangle = \langle w, u \rangle, \quad \forall w \in D(A).$$

For each such $v \in D(A^*)$ we define $A^* v = u$. A^* is called the adjoint of A.

An important class of bounded operators is provided by the unitary ones.

Definition 2.26 A bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called a *unitary operator* if it satisfies $U^* U = U U^* = \mathbb{1}$.

Note that the space $\mathscr{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ forms a C^*-algebra. We will see later on that one can argue the other way and realize any abstract C^*-algebra as the algebra of bounded operators on some $\mathcal{H}$. If A is a bounded operator on a Hilbert space then the self-adjointness is the same as hermiticity, i.e. is the condition that $A^* = A$. In general this is not sufficient.

Definition 2.27 An operator A on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$ is called *symmetric* if for any vectors $u, v \in D(A)$ we have $\langle u, Av \rangle = \langle Au, v \rangle$. This implies that $D(A) \subseteq D(A^*)$. A symmetric operator A is *self-adjoint* if in addition $D(A^*) \subset D(A)$.

Definition 2.28 Let A be an operator on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$. A self-adjoint operator A' is called a *self-adjoint extension* of A if $D(A) \subseteq D(A')$ and if $A'v = Av$ for any $v \in D(A)$.

A is called *essentially self-adjoint* if it admits a unique self-adjoint extension.

Abstract elements of an involutive algebra $\mathfrak{A}$ are realized as operators on some Hilbert space by a choice of a *representation*.