Matteo Ruggiero

Rigid Germs, the Valuative Tree, and Applications to Kato Varieties

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tesi di perfezionamento in Matematica sostenuta il 15 marzo 2011

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ISBN 978-88-7642-558-5
ISBN 978-88-7642-559-2 (eBook)

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Introduction

Holomorphic dynamics has several points of view: it can be discrete or continuous, and be studied locally or globally, but all these aspects are, sometimes surprisingly and in a very fascinating way, linked to one another.

The setting of global discrete holomorphic dynamics is the following: one has a complex space X of dimension d, and a holomorphic map $f : X \to X$, and wants to understand the behavior of the iterates $f^{\circ n}$ of f. For example one can check if the orbit of a point $x \in X$ (*i.e.*, the set $\{f^{\circ n}(x) \mid n \in \mathbb{N}\}$) changes regularly by moving the starting point x.

On the other hand, local discrete holomorphic dynamics still studies the behavior of a map f, but near a given fixed point p, and hence in coordinates one is interested into the behavior of a holomorphic germ $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ and its iterates, existence of basins of attractions, or the structure of the stable set (where all the iterates of f are defined in a neighborhood of 0).

One of the main techniques to study the dynamics of a family $\mathcal{F}$ of holomorphic germs is looking for *normal forms*. Roughly speaking, one looks for a (possibly small) family $\mathcal{G}$ of germs, whose dynamics is easier to study, and such that every $f \in \mathcal{F}$ can be reduced to a germ $g \in \mathcal{G}$ by changing coordinates.

Definition. Let $f, g : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be two holomorphic germs. We shall say that f and g are (*holomorphically*, *topologically*, *formally*) *conjugated* if there exists a (biholomorphism, homeomorphism, formal invertible map) $\phi : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ such that

$$\phi \circ f = g \circ \phi.$$

Depending on the regularity of the change of coordinates: holomorphic, homeomorphic, formal, we talk about holomorphic, topological or formal classification.

We can say that holomorphic dynamics was born in 1884, when Kœnigs (in [44]) proved a conjugacy result in local discrete dynamics in dimension $d = 1$.

Theorem (Kœnigs). *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ such that the multiplier $\lambda := f'(0)$ is such that $|\lambda| \neq 0, 1$. Then f is holomorphically conjugated to the linear part $z \mapsto \lambda z$.*

Twenty years later, Böttcher proved a result on the same lines for non-invertible germs (see [10]).

Theorem (Böttcher). *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ of the form*

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \ldots,$$

with $p \geq 2$ and $a_p \neq 0$. Then f is holomorphically conjugated to the map $z \mapsto z^p$.

Still in the beginning of the 20th century, Leau (see [46] and [47]) and Fatou (see [24]) proved a local conjugacy result for the parabolic case, *i.e.*, when $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is such that its multiplier $\lambda = f'(0)$ is a root of 1. Up to taking a suitable iterate of f, we can suppose that $\lambda = 1$.

Definition. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a holomorphic germ. It is called *tangent to the identity* if $df_0 = \mathrm{Id}$.

In dimension $d = 1$, write $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ in the form

$$f(z) = z(1 + a_k z^k + a_{k+1} z^{k+1} + \ldots),$$

with $k \geq 1$ and $a_k \neq 0$. Then $k + 1$ is called the *parabolic multiplicity* of f.

Definition. A *parabolic domain* in $\mathbb{C}$ is a simply connected open domain Δ such that $0 \in \partial\Delta$.

A parabolic domain Δ is said to be an *attracting petal* (resp., *repelling petal*) for a map f tangent to the identity if $f(\Delta) \subset \Delta$ and $f^{\circ n}(x) \to 0$ for every $x \in \Delta$ (resp., the same for f^{-1}).

Theorem (Leau, Fatou). *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a tangent to the identity germ with parabolic multiplicity $k+1$. Then there exist k attracting petals and k repelling petals such that in every petal f is holomorphically conjugated to $z \mapsto z+1$. Distinct attracting petals are disjoint, and the same holds for repelling petals. The union of attracting and repelling petals form a punctured neighborhood of 0.*

The formal and topological classifications for this kind of germs are not so difficult at least to state, but the holomorphic classification is surprisingly complicated: the moduli space is infinite-dimensional, and the final answer of this question was given almost 70 years later, independently by Écalle using resurgence theory (see [21, 22, 23]) and Voronin (see [61]).

Fatou, Julia, Cremer, Siegel, Brjuno, Sullivan, Douady, Hubbard, Yoccoz and many others gave their contribution to the study of holomorphic dynamics in dimension 1, and right now most of the main issues for both local and global holomorphic dynamics in dimension 1 are solved.

In higher dimensions, a very fruitful theory has been developed for the global setting, by Bedford, Sibony, Fornaess, Smillie and many others, whereas in the local setting only a few simpler cases are well understood, such as the invertible attracting case, while the more complicated ones are still subject of study, even in dimension $d = 2$.

Definition. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ, and let us denote by $\mathrm{Spec}(df_0) = \{\lambda_1, \lambda_2\}$ the set of eigenvalues of df_0. Then f is said:

- *attracting* if $|\lambda_i| < 1$ for $i = 1, 2$;
- *superattracting* if $df_0 = 0$;
- *nilpotent* if df_0 is nilpotent (*i.e.*, $df_0^2 = 0$; in particular, superattracting germs are nilpotent germs);
- *semi-superattracting* if $\mathrm{Spec}(df_0) = \{0, \lambda\}$, with $\lambda \neq 0$;
- *of type* $(0, D)$ if $\mathrm{Spec}(df_0) = \{0, \lambda\}$ and $\lambda \in D$, with $D \subset \mathbb{C}$ a subset of the complex plane.

In particular the semi-superattracting germs are the ones of type $(0, \mathbb{C}^*)$.

We shall always consider only *dominant* holomorphic germs, *i.e.*, holomorphic germs f such that $\det df_z \not\equiv 0$. For non-dominant holomorphic germs, the dynamics is essentially 1-dimensional.

Favre in 2000 gave the holomorphic classification of a special type of germs, namely the (attracting) *rigid germs* (see [25]).

Definition. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a holomorphic germ. We denote by $\mathcal{C}(f) = \{z \mid \det(df_z) = 0\}$ the *critical set* of f, and by $\mathcal{C}(f^\infty) = \bigcup_{n \in \mathbb{N}} f^{-n}\mathcal{C}(f)$ the *generalized critical set* of f. Then a (dominant) holomorphic germ f is *rigid* if:

(i) $\mathcal{C}(f^\infty)$ (is empty or) has simple normal crossings (SNC) at the origin; and
(ii) $\mathcal{C}(f^\infty)$ is forward f-invariant.

Another very interesting class of holomorphic germs is given by strict germs, that (in dimension 2, but not in higher dimensions) are a subset of rigid germs.

Definition. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a (dominant) holomorphic germ. Then f is a **strict germ** if there exist a SNC divisor with support C and a neighborhood U of 0 such that $f|_{U \setminus C}$ is a biholomorphism with its image.

Besides giving interesting classes of examples of local dynamics in higher dimensions, in the 2-dimensional case rigid and strict germs are very important for at least two reasons: first, every (dominant) holomorphic germ is birationally conjugated to a rigid germ; second, every strict germ gives rise to a compact complex non-Kähler surface (Kato surface).

Valuative tree

A very useful tool for the study of holomorphic dynamics (in dimension 2 or higher) has been borrowed from algebraic geometry: *blow-ups*. Roughly speaking, a blow-up of a point p in $\mathbb{C}^d$ consists in replacing p by the set $\mathbb{P}(T_p X)$, *i.e.*, the set of "directions" through p.

If then one wants to study holomorphic dynamics locally at $0 \in \mathbb{C}^2$, one can look for a suitable *modification* over 0, (*i.e.*, a sequence of blow-ups, the first one over 0), to get a simpler dynamical situation on the blown-up space.

These techniques were first used for studying foliations by Seidenberg, Camacho and Sad, and many others (see, *e.g.*, [55] and [13]), and then transferred, by Hakim, Abate and others, to the tangent to the identity case in local dynamics in $\mathbb{C}^2$ (see [34] and [2]).

To study local (and global) holomorphic dynamics, Favre and Jonsson in [26] developed a tool, the *valuative tree*, that roughly speaking is a way to look at all possible modifications over the origin.

Using the valuative tree, and the action induced on it by a germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, they proved that up to modifications you can suppose that a *super-attracting* germ is actually rigid. Let us be more precise.

Definition. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a (dominant) holomorphic germ. Let $\pi : X \to (\mathbb{C}^2, 0)$ be a modification and $p \in \pi^{-1}(0)$ a point in the exceptional divisor of π. Then we shall call the triple $(\pi, p, \hat{f})$ a *rigidification* for f if the lift $\hat{f} = \pi^{-1} \circ f \circ \pi$ is a holomorphic rigid germ with fixed point $p = \hat{f}(p)$.

Finding a rigidification is, a priori, extremely hard, since if we have a germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, a modification $\pi : X \to (\mathbb{C}^2, 0)$ and a point $p \in \pi^{-1}(0)$, the lift $\hat{f} = \pi^{-1} \circ f \circ \pi$ in general is just a rational map, and it is already not easy to have $\hat{f}$ to be a holomorphic germ in a fixed point p.

In this thesis we extend Favre's and Jonsson's result (see [27, Theorem 5.1]) to germs with non-invertible differential in 0 (for invertible germs, the result is trivial, being the map already rigid), getting

Theorem. *Every (dominant) holomorphic germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ admits a rigidification.*

Then we shall study more carefully the case of *semi-superattracting* germs, getting a sort of uniqueness of the rigidification process, and a result on the existence (and uniqueness) of invariant curves.

Theorem. *Let f be a (dominant) semi-superattracting holomorphic germ of type $(0, \lambda)$. Then there exist two curves C and D such that the following holds:*

- *C is a (possibly formal) curve through 0, with multiplicity equal to 1 and tangent to the λ-eigenspace of df_0, such that $f(C) = C$.*
- *D is a (holomorphic) curve through 0, with multiplicity equal to 1 and tangent to the 0-eigenspace of df_0, such that either $f(D) = D$ or $f(D) = 0$.*
- *There are no other invariant or contracted (not even formal) curves for f besides C and D.*

Thanks to this result, the formal classification of semi-superattracting rigid germs can be found. We shall quote here only a consequence of this classification (see Section 2.5 for the precise statement):

Corollary. *The moduli space of semi-superattracting germs in $\mathbb{C}^2$ up to formal conjugacy is infinite-dimensional.*

This result shows how difficult (if not impossible) is to give an explicit classification of semi-superattracting germs up to holomorphic conjugacy. As a matter of fact, one has always to consider the complexity of the generalized critical set, that generally has an infinite number of irreducible components.

With the rigidification result and the last remark in mind, we can then focus on better understanding the dynamics of rigid germs. For semi-superattracting rigid germs of type $(0, \mathbb{D})$ Favre's result gives the holomorphic classification; in this thesis we focus our attention on a sort of limit case, germs of type $(0, 1)$. Hakim proved (see [33]) the following result on the existence of basins of attraction.

Definition. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ of type $(0, 1)$. Let C be the f-invariant (formal) curve associated to the

1-eigenspace of the differential df_0 at 0, parametrized by a suitable (formal) map $\gamma : \mathbb{C}[[t]] \to \mathbb{C}[[z, w]]$. Then we shall call *parabolic multiplicity* of f the parabolic multiplicity of $\gamma^{-1} \circ f|_C \circ \gamma$.

Theorem (Hakim). *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ of type $(0, 1)$, with parabolic multiplicity $k + 1$, and let us denote by $\mathbb{D}_\rho$ the open disc in $\mathbb{C}$ centered at 0 and with radius $\rho > 0$. Then there exist k (disjoint) parabolic domains $\Delta_0, \ldots, \Delta_{k-1} \subset \mathbb{C}$, such that, for ρ small enough,*

$$W_j := \Delta_j \times \mathbb{D}_\rho$$

are basins of attraction for f and there exist holomorphic submersions

$$\phi_j : W_j \to \mathbb{C}$$

that satisfy the following functional equation:

$$\phi_j\big(f(p)\big) = \phi_j(p) + 1.$$

Notice that, even if the basins of attraction are a product of a parabolic domain Δ_j and a disc $\mathbb{D}_\rho$, the parabolic domain $\Delta_j \times \{0\}$ is not necessarily f-invariant (as happens for germs tangent to the identity, see [34]), and f might not admit parabolic curves.

Definition. A *parabolic curve* for a germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ at the origin is a injective holomorphic map $\varphi : \Delta \to \mathbb{C}^d$ satisfying the following properties:

(i) Δ is a parabolic domain in $\mathbb{C}$;
(ii) φ is continuous at the origin, and $\varphi(0) = 0$;
(iii) $\varphi(\Delta)$ is invariant under f, and $f^{\circ n}(z) \to 0$ for every $z \in \varphi(\Delta)$.

Roughly speaking, Hakim's result tells us the behavior of "one coordinate" of f in a basin of attraction. We can focus on understanding the behavior of f also with respect to the "other coordinate", to get a complete description of the dynamics f in the basins of attraction. For the reasons we anticipated above, we shall consider rigid germs and prove the following result.

Theorem. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic rigid germ of type $(0, 1)$ of parabolic multiplicity $k + 1$. Let*

$$W_j := \Delta_j \times \mathbb{D}_\rho$$

for $j = 0, \dots, k-1$ *be basins of attraction for* f *as above. If there is a parabolic curve in* W_j*, then there exists a holomorphic conjugation* $\Phi_j : W_j \to \widetilde{W}_j$ *between* $f|_{W_j}$ *and the map*

$$\widetilde{f}(z, w) = \left(\frac{z}{\sqrt[k]{1+z^k}}, z^c w^d \big(1 + \widetilde{h}(z)\big) \right),$$

where $\widetilde{W}_j$ *is a suitable parabolic domain. Moreover, if* $d \geq 2$ *then we can get* $\widetilde{h} \equiv 0$.

In particular, the action of such germ in the second coordinate is either monomial or linear with respect to w.

The assumption of the existence of parabolic curves is quite technical; the feeling on the matter is that, if parabolic curves do not exist, then f should be conjugated (in each basin of attraction) to a map that in the second coordinate is affine with respect to w.

Kato Varieties

Coming back to local dynamics in the attracting case, given an attracting germ $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$, it is natural to look at its basin of attraction to 0, and its fundamental domain, *i.e.*, (any dense open subset of) the basin of attraction modulo the action of f itself.

Starting from the 70's, Kato, Inoue, Dloussky, Oeljeklaus, Toma and others proved that one can find some compactifications of these fundamental domains when $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is also strict, getting interesting examples of compact complex non-Kähler surfaces, called *Kato surfaces*.

Kato surfaces are of great interest also for the Kodaira-Enriques classification of compact complex surfaces. Indeed, they all belong to the so called *Class* VII.

Definition. A surface X is called of *class* VII if it has Kodaira dimension $\mathrm{kod}\,(X) = -\infty$ and first Betti number $b_1 = 1$. If moreover X is a minimal model, it is called of *class* VII_0.

We have to explain now what is the Kodaira dimension of a compact complex manifold, and what is a minimal model. We shall start from the second.

Definition. A compact complex surface X is called *minimal model* if it does not exist a compact complex surface Y and a modification $\pi : X \to Y$.

This definition can seem difficult to check directly, but thanks to the Castelnuovo-Enriques criterion (see [31, p. 476]), it is equivalent to asking that the surface X has no *exceptional curves*, *i.e.*, rational curves with self-intersection -1.

Theorem (Castelnuovo-Enriques Criterion). *Let X be a 2-manifold, and $D \subset X$ a curve in X. Then there exists a 2-manifold Y so that $\pi : X \to Y$ is the blow-up of a point $p \in Y$ with $D = \pi^{-1}(p)$ if and only if D is a rational curve of self-intersection -1.*

So, up to modifications (and hence up to birationally equivalent models), a compact complex surface can be supposed to be a minimal model.

If reducing to minimal models can be considered as the first step for the classification of compact complex surfaces, the second step would be sorting surfaces with respect to the *Kodaira dimension*.

Definition. Let X be a compact complex n-manifold. For every $m \in \mathbb{N}^*$, we shall call the m-th *plurigenera* the dimension

$$P_m := h^0\big(X, \mathcal{O}\,(m\mathcal{K}_X)\big)$$

of the space of holomorphic sections of the line bundle $m\mathcal{K}_X$, where $\mathcal{K}_X = \bigwedge^n T^*X$ is the *canonical bundle* of X (here T^*X denotes the holomorphic cotangent bundle of X).

The *Kodaira dimension* of X is

$$\operatorname{kod}(X) = \min\{k \mid P_m = O(m^k) \text{ for } m \to +\infty\}.$$

The Kodaira dimension somehow tells us how positive is the canonical bundle.

When $P_m = 0$ for every $m \geq 1$, we shall say that the Kodaira dimension is $-\infty$.

For a (compact complex) n-manifold, the Kodaira dimension can take values in $\{-\infty, 0, \dots, n\}$. The case $\operatorname{kod}(X) = n$ is said to be of *general type*.

In dimension 2, surfaces of general type are not completely understood. There are results on the structure of the moduli spaces, but it seems not easy to compute them for all cases.

However for Kodaira dimension 1 and 0 the classification is done and classical, while for Kodaira dimension $-\infty$, only one case is still not completely understood: class VII surfaces.

When the second Betti number $b_2(X) = 0$, these surfaces have been completely classified, thanks to the work of Kodaira ([42, 43]), Inoue ([38]), Bogomolov ([8]), Li, Yau and Zheng ([48]), Teleman ([60]).

For $b_2 > 0$, the classification is not completed yet. Before describing the known results, we need a definition.

Definition. Let X be a compact complex n-manifold. A *spherical shell* is a holomorphic embedding $i : V \hookrightarrow X$, where V is a neighborhood of $\mathbb{S}^{2n-1} = \partial\mathbb{B}^{2n} \subset \mathbb{C}^n$. A spherical shell is said *global* (or *GSS*) if $X \setminus i(V)$ is connected.

Kato introduced a construction method for surfaces of class VII_0 with $b_2 > 0$, called *Kato surfaces* (see [40]), that starts from a Kato data.

Definition. Let $B = \overline{\mathcal{B}_\varepsilon}$ be a closed ball in $\mathbb{C}^n$ of center 0 and radius $\varepsilon > 0$, and $\pi : \widetilde{B} \to B$ a modification over 0. Let $\sigma : B \to \widetilde{B}$ be a biholomorphism with its image such that $\sigma(0)$ is a point of the exceptional divisor of π. The couple (π, σ) is called a *Kato data*.

Kato datas and (rigid, strict) germs are strictly related, as the following definition shows.

Definition. Let (π, σ) be a Kato data. Then we can consider $f_0 = \pi \circ \sigma : B \to B$, that turns out to be a holomorphic rigid and strict germ, with a fixed point in 0 the center of B. We shall call this germ the *base germ* associated to the given Kato data.

On the other hand, given a (rigid and strict) holomorphic germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, we shall call *resolution* for f_0 a decomposition $f_0 = \pi \circ \sigma$, with π a modification over 0 and σ a (germ) biholomorphism that sends 0 into a point of the exceptional divisor of π.

Then, roughly speaking, a Kato variety is constructed as follows.

Definition. Let (π, σ) be a Kato data. Let $B = \mathcal{B}_\varepsilon$ be a ball in $\mathbb{C}^n$ of center 0 and radius $\varepsilon > 0$. The *Kato variety* associated to the given Kato data is the quotient of $\widetilde{X} = \pi^{-1}(\overline{B}) \setminus \sigma(B)$ by the action of $\sigma \circ \pi : \pi^{-1}(\partial B) \to \sigma(\partial B)$, that is a biholomorphism on a suitable neighborhood of $\pi^{-1}(\partial B)$ for ε small enough.

Dloussky in his PhD thesis [15] studied deeply this construction and properties of Kato surfaces. Among these properties, we shall underline the following (see [40] and [20]).

Theorem (Kato, Dloussky-Oeljeklaus-Toma). *Let X be a surface of class VII_0 with $b_2 = b_2(X) > 0$. Then X admits at most b_2 rational curves. Moreover, X admits a GSS if and only if X has exactly b_2 rational curves.*

There are no known examples of surfaces X of class VII_0 that do not admit global spherical shells, and a big conjecture (called the GSS Conjecture) claims that there are none.

Dloussky and Oeljeklaus (see [17]) studied the case when the germ f_0 arises from the action at infinity of an automorphism of $\mathbb{C}^2$.

Definition. An automorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ is said to be a *Hénon map* if it is of the form

$$f(x, y) = \big(p(x) - ay, x\big),$$

with p a polynomial of degree $d = \deg p \geq 2$.

Polynomial automorphisms of $\mathbb{C}^2$ can be subdivided into two classes, *elementary automorphisms*, whose dynamics is easier to study, and compositions of Hénon maps; see [29].

The idea is then to consider the extension $F : \mathbb{P}^2 \to \mathbb{P}^2$, that has an indeterminacy point at $[0 : 1 : 0]$, and an indeterminacy point for the inverse at $[1 : 0 : 0]$.

Looking at the action of F on the line at infinity, one finds that there is a fixed point $p = [1 : 0 : 0]$, and the germ $F_p := f_0$ is strict and admits a resolution, and hence an associated Kato surface.

This new approach gives a connection between the dynamics of f and the Kato surface X associated to f_0: in particular, if we denote by U the basin of attraction of f to p, then X turns out to be a compactification of the fundamental domain V of U.

From this dynamical interpretation of Kato surfaces, some questions arise:

- Can we add a point "at infinity" to V and get a (possibly singular) compact complex manifold? Or equivalently, is the Alexandroff one-point compactification of V a (possibly singular) complex manifold?
- Can we lift objects that are invariant for f_0 to X, obtaining some additional structure on X (such as the existence of subvarieties, foliations, vector fields)?

The first property is actually equivalent to contracting all the rational curves to a point, that was a result already known for Kato surfaces (see [15]).

The second phenomenon has been studied for example by Dloussky, Oeljeklaus and Toma, starting from [15]; see, *e.g.*, [16, 18] and [19].

Favre, in his PhD thesis (see [25]), used his classification of attracting rigid germs in $\mathbb{C}^2$ and studied the construction of a holomorphic foliation on X, and the computation of the first fundamental group of the basin U, already obtained using other techniques by Hubbard and Oberste-Vorth (see [37]), that turns out to be very complicated.

In this work, we shall study an example of these phenomena in dimension 3. Several aspects become more complicated. First of all, while the structure of polynomial automorphisms in $\mathbb{C}^2$ is pretty clear, in higher dimensions it is still a subject of research, and we have no "Hénon maps" that we can use. Indeed only a few cases have been classified: for instance, the automorphisms of degree 2 in $\mathbb{C}^3$ (see [30] and [49]).

However, Sibony and others identified a special property of Hénon maps, *regularity*, and studied maps with this property in higher dimensions, getting a very fruitful theory on global holomorphic dynamics, using currents and pluripotential theory.

Definition. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial automorphism, $F : \mathbb{P}^n \to \mathbb{P}^n$ be its extension to $\mathbb{P}^n$, and denote by I^+ and I^- the indeterminacy sets for F and F^{-1} respectively. Then f is said to be *regular* (in the sense of Sibony) if $I^+ \cap I^- = \emptyset$.

Furthermore, the structure of birational maps is more complicated in dimension ≥ 3: one can blow-up not only points, but also curves and varieties of higher dimension, and not every birational map is obtained as a composition of blow-ups followed by a composition of blow-downs (the inverse of blow-ups, see [9]). Moreover, the problem of finding an equivalent of minimal models in dimensions higher than 2 is still open (the project to solve this problem is called "Minimal Model Program", based on Mori theory).

So not so much is known about Kato varieties in higher dimensions. We shall then study the case of a specific regular quadratic polynomial automorphism $f : \mathbb{C}^3 \to \mathbb{C}^3$ in normal form with respect to [30], namely

$$f(x, y, z) = (x^2 + cy^2 + z, y^2 + x, y),$$

where $c \in \mathbb{C}$. The example we chose is essentially the only example of regular polynomial automorphism in $\mathbb{C}^3$ of degree 2. Indeed by direct computation one gets that $I^+ = [0 : 0 : 1 : 0]$ is a point, while $I^- = \{z = t = 0\}$ is a line at infinity.

In this thesis we shall construct a Kato variety associated to f^{-1} and its basin of attraction to $[0 : 0 : 1 : 0]$.

This polynomial automorphism was already considered by Oeljeklaus and Renaud in 2006 (see [52]), who constructed a different kind of 3-fold (called of Class L, see [41]) associated to the basin of attraction of f to I^-.

We shall then prove the following properties.

Theorem. *Let $f : \mathbb{C}^3 \to \mathbb{C}^3$ be the regular polynomial automorphism given by*

$$f(x, y, z) = (x^2 + cy^2 + z, y^2 + x, y),$$