Lecture Notes in Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} \, dx$$

David R. Adams

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ISSN 2296-5009 ISSN 2296-5017 (electronic)
Lecture Notes in Applied and Numerical Harmonic Analysis
ISBN 978-3-319-26679-4 ISBN 978-3-319-26681-7 (eBook)
DOI 10.1007/978-3-319-26681-7

Library of Congress Control Number: 2015954629

Mathematics Subject Classification (2010): 31A15, 31B15, 43A15, 46E35

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

Dedicated to my wife and daughter,
Jeanie Artis Adams
and
Sissy B. Meredith, Ph.D.

LN-ANHA Series Preface

The *Lecture Notes in Applied and Numerical Harmonic Analysis (LN-ANHA)* book series is a subseries of the widely known *Applied and Numerical Harmonic Analysis (ANHA)* series. The Lecture Notes series publishes paperback volumes, ranging from 80-200 pages in harmonic analysis as well as in engineering and scientific subjects having a significant harmonic analysis component. *LN-ANHA* provides a means of distributing brief-yet-rigorous works on similar subjects as the *ANHA* series in a timely fashion, reflecting the most current research in this rapidly evolving field.

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative crossfertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, bio-medical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in applicable topics such as the following, where harmonic analysis plays a substantial role:

Bio-mathematics, Bio-engineering, and Bio-medical signal processing;
Communications and RADAR;
Compressive sensing (sampling) and sparse representations;
Data science, Data mining, and Dimension reduction;
Fast algorithms;
Frame theory and noise reduction;
Image processing and super-resolution;
Machine learning;
Phaseless reconstruction;
Quantum informatics;
Remote sensing;
Sampling theory;
Spectral estimation;
Time-frequency and Time-scale analysis – Gabor theory and Wavelet theory.

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of “function.” Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor’s set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener’s Tauberian theorem in

Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequencyscale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'ˆetre of the *ANHA* series!

University of Maryland
College Park

John J. Benedetto
Series Editor

Preface

While preparing this volume, I was encouraged to present the details of three new claims that I have made concerning Morrey Spaces and the action of Riesz potential operators on these spaces. The claims all follow from a new formulation of the predual to a Morrey Space (2012): $H^{p',\lambda}(\mathbb{R}^n)$; see Chapter 5, a version that generally meets all the desired requirements for such in Harmonic Analysis. The full theory is presented here for the first time. These include:

(1) Determine the integrability classes of the trace of a Riesz potential of an arbitrary Morrey function, in the unbounded case. The key idea here is the existence of certain Wolff potentials; see Chapter 10.
(2) Determine the capacity of the singular sets (sets of discontinuities) of weak solutions of a $2m$th-order quasilinear elliptic systems, with mth-order derivatives belonging to L^p, e.g., the celebrated Meyers-Elcrat system. Here the mth-order derivatives can be upgraded to belong to a Morrey class; see Chapters 15 and 16.
(3) Are there any "full" interpolation results for linear operators between Morrey Spaces in the light of the now known counterexamples in the literature? I claim that there are - with important restrictions. For this we refer the reader to Chapter 11. This all depends on an atomic decomposition of a Morrey Space and a full use of our duality of Chapter 5.

Lexington, KY, USA David R. Adams
July 2014

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