Joel L. Schiff

# The Mathematical Universe

## From Pythagoras to Planck

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# Contents

*Not only is the universe stranger than we imagine, it is stranger than we can imagine...*

Sir Arthur Eddington

*This book is dedicated to the memory of Emmy Noether (1882–1935), who made many brilliant contributions to the world of Mathematics and whose eponymous theorem changed the world of Physics forever.*

Amalie Emmy Noether, date unknown but prior to entering Göttingen University in 1915. Renowned 20th century mathematician Norbert Wiener asserted: “Leaving all questions of sex aside, she is one of the ten or twelve leading mathematicians of the present generation in the entire world.” (Image in public domain.)

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# About the Author

**Joel L. Schiff** has a PhD in Mathematics from the University of California Los Angeles (UCLA). He has spent his career at the University of Auckland, Auckland, New Zealand and has written five books on mathematical and scientific subjects. With colleague Wayne Walker, he helped develop the *Arithmetic Fourier Transform* used in signal processing. He was also the founder publisher of the international journal *Meteorite* and in 1999, he and his wife discovered a new asteroid from their backyard observatory. They named it after the notable New Zealand meteorite scientist, Brian Mason.

# Foreword

On a sunny day, one million years ago, our world already looked much like it does now. There were mountains, forests, and deserts, and they were inhabited by plants and animals, much like it is today. But something very special was happening among some small groups of primates. They had grown to become more intelligent than any other animal, and they noticed all sorts of remarkable features in the world they lived in. In the weather, in the fields, and in the sky, they saw patterns, varying during the days and nights, and during the seasons.

They learned how to communicate among one another, how to foresee what was going to happen, and how to use their insights to improve their safety and to ensure the availability of food and shelter. This is how science began, and the primates could no longer be regarded as animals; they started to talk; they were becoming human.

Then, a few thousand years ago, they began to write and read. They learned to use new concepts of thought, such as numbers. They started to use their fingers to sort out the different numbers that one might have and soon discovered that we don't have enough fingers to account for all possible numbers. You can go all the way to 60, but then it gets harder. You have to combine numbers to describe larger numbers, and sometimes you need numbers that aren't integers to indicate how much water you have or how tall something is. This was the beginning of calculus. You can also use numbers to register time, like the hour of the day or the day of the year, and to get this accurately required some real thinking.

It sometimes happened that they thought they understood everything that was to be known, but time and again this turned out to be wrong. The art of using numbers to indicate the size of things became known as mathematics, and the science of describing forces and other features of the things you see became known as physics. Studying physics and mathematics became a specialty that only a few experts can understand. This led to the need for people who could bridge the gap between the experts and the rest of the population. It is not an easy job, but I think this book is a delightful endeavor in this regard.

Gerard 't Hooft (Nobel Prize in Physics, 1999).

# Prologue

## A BRIEF TALE

Early in the 20th century, a patent office clerk (class III) in Bern, Switzerland, was thinking about light. It was a topic he had been dwelling upon since he was 16. His patent office work was not very demanding, so he had time to think on his own. As a starting point, he stipulated two basic postulates: one about the constancy of the speed of light in empty space, the other about the invariance of the laws of Physics.

The clerk then considered the ramifications of these two principles. He toiled away, without doing any physical experiments of his own, but rather experiments inside his own head; that is, 'thought experiments'. Simply from his deliberations, a number of equations flowed from his pen in due course.

One of the conclusions drawn from the mathematical squiggles that the clerk wrote down was that a clock aboard a moving spaceship would tick more slowly than one back on Earth, and thus time would actually slow down for the space travelers. A year in space for them could even correspond to ten years back on Earth.

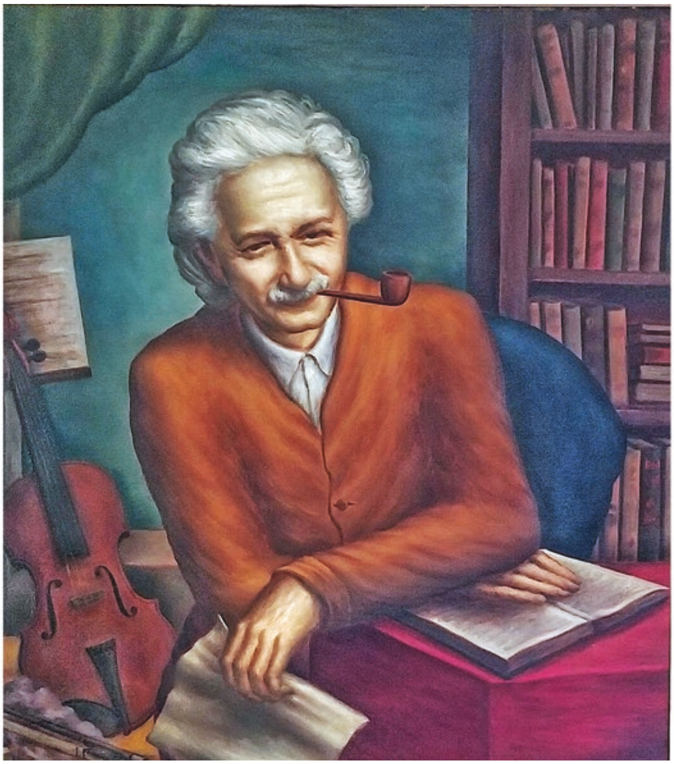
Yes, *time*, which had always been thought to be an immutable quantity, was no longer so. At least, that is what the equations said.

After the publication of the clerk's ideas, he turned his thoughts to gravity, another notion we are all familiar with, and the clerk eventually became a university professor. After some ten years of deliberation on gravity, his new squiggles showed that gravity could also slow down time, and could warp space that bent light beams which had no mass.

Many found this state of affairs very confusing and few could comprehend the meaning of the squiggles. This was time and space after all, things people were intimately familiar with from a lifetime of experience. The clerk's revelations simply defied common sense.

The above account has not been taken from some bizarre science fiction story. These considerations concerning light and gravity were spelled out in two landmark publications concerning the Theory of Relativity, in 1905 and 1916, respectively. The author's name was Albert Einstein. His squiggles on paper completely redefined the nature of time and space, and his theories have proven to be a profound platform for understanding the workings of the Universe. All the theoretical results of his work have been scientifically verified, often to many decimal places.

Another fantastical prediction to come out of Einstein's squiggles was the notion of black holes.



Albert Einstein, painted by American artist Paul Meltsner (1905–1966). This painting was responsible for the purchase of $US 1 million in war bonds in 1943. (Image courtesy of Rabbi Naomi Levy/Rob Eshman. Photo by Ted Levy.)

But do seemingly abstruse mathematical equations have anything to do with our everyday lives? The answer is yes. For without a complete understanding of how time can slow down and speed up aboard an orbiting satellite, there would be no Global Positioning System (GPS) and everyone would become hopelessly lost traveling from A to B. Well, at least some of us would and many of us did. Moreover, GPS is an integral part of aircraft navigation systems, as well as that of ships at sea.

Other squiggles that appeared in Einstein's first publication on Relativity said that mass and energy were equivalent, obeying the relation: $E = mc^2$. Now these three letters, plus one number and an equal sign, arranged just so, have had a most dramatic effect on the course of human history. Moreover, it is also operating every second of every day in our Sun, converting mass to energy and bringing us light. The light with which to explore the nature of reality all the way back to the origins of our Universe.

# Preface

*Nature imitates Mathematics...*
Mathematician Gian-Carlo Rota

This is a book about the mathematical nature of the Universe we live in. Perhaps in some other universe, if such exists, the Mathematics would be different. The question of whether our Mathematics is discovered or invented has exercised many great minds since the time of Plato (who claimed it was discovered). We will touch upon this issue since the author has his own view of the matter, as most mathematicians do. Regardless, there are certain fundamental features of our Universe, and even predictions that are beautifully described by the little squiggles on paper that we call Mathematics.

What the author aims to do in the pages that follow is to explain some of the mathematical aspects of the Universe we live in by using no more than basic high school Mathematics. What is explored through these chapters are some selected topics that are easily accessible to anyone who has once brushed up against the Pythagorean theorem and the symbol $\pi$, with a light dusting of algebra. But do not worry if you have forgotten the details as these will be provided. As a bonus, nothing involving even a smattering of Calculus is needed for any of our considerations.

We will make numerous stops along the way, visiting the sublime but bizarre world of subatomic particles, honey bees, the Theory of Relativity, galaxies, black holes, and of course infinity, as well as a universe of our own creation with its own set of rules called Cellular Automata. The point is, the whole Universe is linked by bits of Mathematics that most people already know, and what the reader does not already know, the author hopes that you will come to understand by the end of the book because you will find it compelling and accessible.

So, this is a Science book for the layperson that has a few equations, unlike many other such Science books that are purely descriptive. But simply knowing the equation for a straight line ($y = mx + b$) enables us to determine the age of our Solar System, or to measure the size of a supermassive black hole that lies at the heart of most galaxies, or even allows us to describe the expansion of the Universe itself. Moreover, we will describe how time speeds up and slows down and you will become a believer. Now all that is pretty extraordinary, and just a sample of what we can do with the most basic of squiggles.

We will only be able to touch on the highlights of our Mathematical Universe, since Mathematics is at the very core of so much scientific endeavor and the rigorous details of those endeavors require concepts that are beyond the scope of this book. Nevertheless, there is much we can explore with the simplest of mathematical tools.

The author cannot guarantee that you will have six-pack abs by the end of this book, but the mental exertion could possibly lead to a six-pack brain.

# 1

# The Mystery of Mathematics

*Pure Mathematics is religion...*
Philosopher Friedrich von Hardenberg

*It is impossible to be a mathematician without being a poet in soul…*
Mathematician Sofia Kovalevskaya

Everyone who has gone to school has learned some Mathematics, with the experience for many being a painful one. Whenever someone on a plane or at a party asks the author what he does and he replies that he is a mathematician, the conversation either halts immediately, as if he had said he was an undertaker or worked for the Internal Revenue Service, or they confess that they were never very good at math at school. So, he appreciates the phobia, dread, and forbidding nature regarding what he is about to say, but can assure you that it will be entirely painless.

In 1999, the author published a book with the title *Normal Families*. The title was somewhat misleading, and he suspects that some copies of the book were purchased thinking that it held some deep psychological insights into family life. However, it was entirely devoted to a very esoteric branch of Mathematics by the same name. Indeed, it included many beautiful theorems that had absolutely no bearing on the natural world. In fact, most of the material was as remote from reality as it could be. Yet some of the results could even be considered majestic, in the same way that Mahler's Fifth Symphony might be so considered. The structure of their proofs was just so elaborate, so rich and magnificent in their construction, and the final results so illuminating of other dark corners of the mathematical



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realm, that the author was often in awe of the genius that went into their creation. These were the results of others and he could in no way take any of the credit for the results themselves. He was merely the messenger.

But that begs the question, 'Messenger of what'? What really is Mathematics? Where does it reside? Is it the product merely of our imagination, or is it found in some netherworld outside of space and time. Is Mathematics a religion? Is it the language of God? Or even, is God Mathematics?

These are very deep questions in and of themselves and have been fretted over for millennia by mathematicians and philosophers alike. The author has pondered over them himself for the more than four decades that he has been a mathematician, since he has frequently wondered what in the world he has actually been doing all these years. It is hoped, however, that by the end of this book, the reader will have a better understanding of what Mathematics is, although there is no simple answer.

There is at least a 4,000-year glorious history of Mathematics and the sophistication of some of the earliest work is quite remarkable. For example, the Babylonians used a base 60 system of numbers. Indeed, we still do when measuring seconds and minutes of time, or in arcseconds and arcminutes of angle. The diagonal of the square in Fig. 1.1 (with the horizontal row of numbers) represents:

$$1 + 24/60 + 51/(60)^2 + 10/(60)^3 = 1.41421296,$$

which gives the value of $\sqrt{2}$ accurate to 6 parts in 10 million $\left(\sqrt{2} = 1.41421356\ldots\right)$.

The square root of 2 was known mathematically as the ratio of the diagonal of a square to a side of length 1. In this particular ancient school exercise, the value 1 is replaced by the value '30' at the top left for the length of a side, so that the diagonal would have a length 30 times greater, namely: $30 \times \sqrt{2} = 42 + 25/60 + 35/(60)^2$, represented by the bottom numbers. This is sophisticated mathematics beyond the capability of any measuring device at the time, and would (using base 60) be a challenging problem for a high school student of today. Try it.

However, to do the historical side of Mathematics justice would require a completely separate volume to this one. Nevertheless, the names of many famous individuals who made important contributions to our understanding of the mathematical and physical worlds are sprinkled throughout this text.

In the remainder of the text, we shall explore the mysterious relation between Mathematics and the Universe, for without Mathematics we would have little left to explore. We could not even count sheep at night to go to sleep. But first we need to consider some of the basic elements of mathematical logic in order to make this exploration possible.

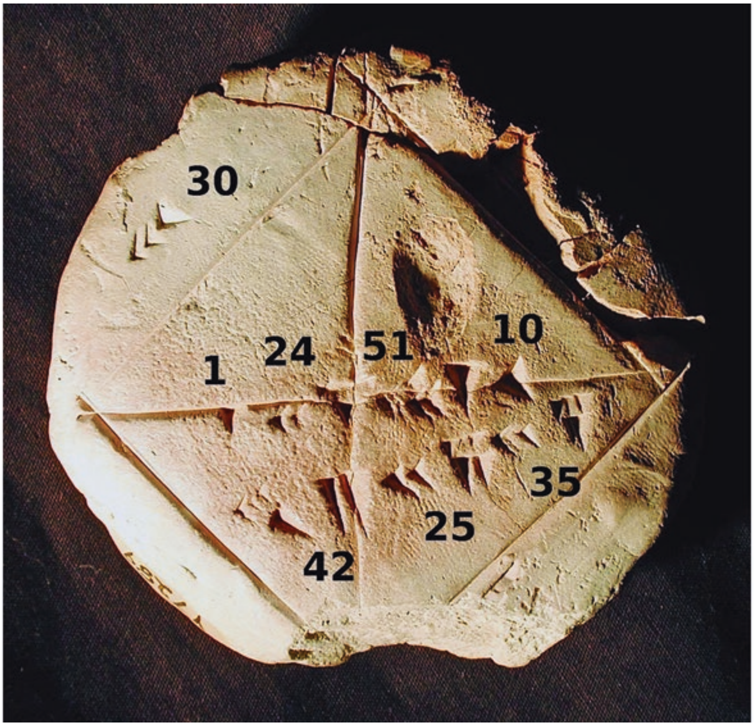


**Figure 1.1:** They were not just counting goats and sheep in Babylonian times. This is the cuneiform Babylonian school tablet YBC 7289 from 1600–1800 B.C. See text for explanation. (Image courtesy of Bill Casselman (https://www.math.ubc.ca/~cass/Euclid/ybc/ybc.html) and Yale Babylonian Collection.)

## LET US BE REASONABLE

*Logic: The art of thinking and reasoning in strict accordance with the limitations and incapacities of the human misunderstanding...*
Ambrose Bierce, *The Devil's Dictionary*

Mathematics at its heart relies on the power of reasoning in a rigorous fashion. Such systematic reasoning, known as *symbolic logic*, is a mode of thought that was initiated by Aristotle, developed by the Stoics, and further expounded upon in a more mathematical setting beginning in the 19th century by George Boole, Augustus De Morgan, and Charles Sanders Peirce, among others. It is an attempt to make reasoning – and in particular mathematical reasoning – highly rigorous.

We use basic forms of logical reasoning all the time in our daily lives without even realizing it:

*If it is 2:30, it is time to go to the dentist*[1].
*It is 2:30.*
*Therefore, it is time to go to the dentist.*

This sort of reasoning, or *inference rule*, has a specific name: *modus ponens.* Another basic inference rule is known as *modus tollens*, as in:

*If my grandmother had wheels she would be a trolley car*[2].
*My grandmother does not have wheels.*
*Therefore, she is not a trolley car.*

Both of these forms of logical inference have their origins in the mists of antiquity. In classical logical reasoning, any logical statement (proposition) *P* in the form: 'It is time to go to the dentist' (or 'my grandmother is a trolley car') is considered to be either true or false. The *negation* of *P* is the statement: 'It is not time to go to the dentist' (or 'my grandmother is not a trolley car') and is referred to as the statement '*not P*', as in: 'it is *not* the case that *P* is true'.

Another form of logical reasoning going back to Aristotle is that:

*Either a statement, P, is true, or its negation, not P, is true*.

Thus, 'it is 2:30' or 'it is not 2:30'; either 'my grandmother is a trolley car' or 'my grandmother is not a trolley car'. There is no middle ground and that is why this mode of thinking is called the *law of the excluded middle*.

This is a cornerstone of mathematical reasoning, whereby any given mathematical statement is either true or false. Either 17 + 32 – 6 = 43, or it does not; either 10,357 is a prime number or it is not[3]. If we have the statements (a) 10,357 is a prime number and (b) 10,357 is not a prime number, then one or the other must be true.

Just in case you were wondering, 10,357 *is* a prime number, so that it is only divisible by the number 1 and itself.

---

[1] To make the time more explicit, it should have been given as 14:30, but there is an implied child's joke here and dentists generally do not work at 2:30 am anyway.

[2] Trolley car is the American term for tram car. The expression, which the author heard many times as a young child, is a response to someone who makes a wistful statement of pure speculation, such as, "If only my parents had made me persist with my violin lessons, I could have become a great violinist."

[3] Recall that a prime number is a positive integer (counting number) greater than 1 that cannot be divided (with no remainder) by any other positive integer except for the number 1 and itself. Thus, the first prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, …

Based on the law of the excluded middle, here is an argument that many of us have no doubt encountered on the school playground. Suppose our proposition $P$ is:

$P$: There is no largest number.

The negation of $P$ is therefore simply the statement:

*not* $P$: There *is* a largest number.

If we assume for a moment that *not* $P$ is the true statement, let us call the largest number $N$. But then $N + 1$ is still larger, so that the statement, *not* $P$, cannot be true. Since one of the statements $P$ or *not* $P$ must be true according to the law of the excluded middle, it follows that our proposition $P$: 'there is no largest number', is the true statement.

Indeed, with this playground example we have actually proved a genuine mathematical theorem, namely that *there is no largest number*. If the reader at some stage has enunciated some form of the above proof, it likely would have been done without ever realizing you were using sophisticated forms of logical reasoning and doing something mathematicians do every day, which is to prove theorems.

As seemingly obvious as the *law of the excluded middle* appears, it came under attack from a 20th century Dutch mathematician named L.E.J. (Luitzen Egbertus Jan) Brouwer (1881–1966), who rejected it on philosophical grounds when dealing with infinite sets. For Brouwer, an infinite set is something that is incomplete, as for example, the natural numbers, 0, 1, 2, 3, … which cannot be thought of in their entirety no matter how smart you are. That is why we need the three dots (ellipsis), which means *and so forth in the same vein*. See Appendix I for further discussion on this matter, where we give a proof by contradiction, as well as a 'constructive' proof of the proposition:

$P$: There is an infinite number of primes.

In 1946, the famous mathematician Hermann Weyl wrote, "Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers... The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation but is not a closed realm of things existing in themselves."

Indeed, there is a world of difference between the realm of finite entities and infinite ones and just how we treat the infinite is the subject of Chapter 2.

On the other hand, the law of the excluded middle leads to a method of proof known as 'proof by contradiction', which sounds more impressive in Latin: *reductio ad absurdum.* This is just the line of reasoning we employed to prove our little theorem that there is no largest number. We assumed the negation of the theorem to be true, i.e. that there is a largest number, $N$, and derived a contradiction since the number $N + 1$ is obviously larger. Therefore, the negation of the theorem cannot be true and it follows that our theorem must be true after all: There is no largest number. This is actually a very powerful technique when it is not possible to find a direct proof of a theorem. In Appendix III, there appears a famous proof by contradiction.

The subject of logical reasoning is a fascinating one but takes us too far afield, so the interested reader is directed to the excellent book by R.L. Epstein in the Bibliography.

Interestingly, some of the very basics of symbolic logic find their way into the design of electrical switches, known as logic gates, that are at the heart of electronic computers. See Appendix XIII, where the three most fundamental logic gates are discussed in relation to symbolic logic.

It should also be mentioned that Brouwer is perhaps more famous for his 'fixed-point theorem.' A simple example would be to take a circular disk and rotate the disk $30^\circ$ about the center. At the end of the rotation, which is a smooth continuous action, all the points in the disk will have moved, except for one point, the center, which remains fixed. Brouwer's theorem says that any continuous action that transforms a suitably closed region to itself will always leave at least one point in its fixed position. Ironically, Brouwer's proof of his fixed-point theorem is *not* constructive; it merely 'proves' that there exists a fixed point without actually providing a means to determine (construct) it explicitly[4].

There are now numerous fixed-point theorems and, while they may seem only of mathematical interest, they have important applications in many branches of Science, such as in Economics. Indeed, a fixed-point theorem was at the heart of the game theoretic work of American mathematician John Nash, which earned him the 1994 Nobel Prize in Economics. The 'Nash equilibrium' point is a fixed point of a particular continuous function[5], and to prove it, one can use the Brouwer fixed-point theorem, although Nash originally used an alternative fixed-point theorem attributed to Shizuo Kakutani.

A fine biographical film about John Nash, *A Beautiful Mind*, starring Russell Crowe and Jennifer Connelly and directed by Ron Howard, came out in 2001. Like John Nash himself, the film won numerous awards.

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[4] Thanks to Douglas Bridges for pointing out the irony of Brouwer's non-constructive proof.

[5] A *function* in Mathematics is a relationship between the elements of two sets (discussed in the subsequent section of the text) that assigns to each element of the first set (the domain) a unique element of the second set (the range). An example of a function would be that given by the expression: $y = x^2$, which assigns, to each element $x$ in the domain of real numbers, the unique real number $y$, determined by the relationship: $y = x^2$, where $y$ is again a real number. The domain often depends on context but is generally taken to be as large as possible.

## ALL SET

Another issue that arose in the early 20th century was the discovery of some cracks in the very structure of the mathematical edifice of its day. This had to do with the subject of *sets*, which are simply collections of distinct objects, with the objects themselves being the *elements (members)* of the set. For example, the set of letters of the English alphabet can be written as:

$$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\},$$

and consists of 26 elements. Note also the conventional curly bracket notation, $\{\bullet\}$, for denoting a set. For convenient further reference, we can give the set a name, usually by a capital letter. So, one can write,

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\},$$

but the specific letter designation is somewhat arbitrary.

Furthermore, it would seem that we can create sets of almost anything we can think of, such as specific sets of whole numbers like $\{1,2,3\}$, the set of all U.S. States, the set of birthdays of the author (a large set indeed), or even a set which has no elements at all, known as the *empty set*[6].

The theory of sets is an important branch of Mathematics and much of its development stems from the seminal work of the German mathematician Georg Cantor (1845−1918). The above examples of sets are all *finite* sets; that is, one can count up the finite number of elements in any of the sets. Working with finite sets is somewhat mundane and they can even be taught in Primary School.

Cantor, on the other hand, was particularly interested in the exotic realm of *infinite* sets, and in dealing with them we are forced to leave everyday common sense behind. It took a genius like Cantor to figure how to proceed in this infinite realm, as will be seen Chapter 2, although he suffered for his efforts.

Now let us consider one of the bumps in the road of the Mathematical Universe involving sets, namely the Russell Paradox (1901) formulated by the English philosopher Bertrand Russell. He questioned the status of the set,

$S = \{$all sets that are not members of themselves$\}$.

This sounds like a reasonable set to consider. Or is it?

Let us now consider the following two statements:

(i) $S$ is a member of itself.
(ii) $S$ is not a member of itself.

---

[6] The set of observed aliens from outer space would be an example of an empty set, although some who claim to have been abducted by aliens might argue otherwise.

Suppose that $S$ *is* a member of itself, which is statement (i). Then according to the definition of $S$, it is one of those sets that *is not* a member of itself, which is statement (ii). Okay, then suppose that $S$ *is not* a member of itself (statement (ii)). By the very definition of the set $S$, it *is* a member of itself (statement (i)). Therefore, assuming the truth of either (i) or its negation (ii) we arrive at a contradiction. Maybe Brouwer had a point after all.

The preceding dilemma arises from allowing the existence of sets such as $S$ in the first place. This has to be more carefully managed, and has been in the current axioms of Set Theory developed by Ernst Zermelo and Abraham Fraenkel in the early 20th century. This will be discussed in Chapter 2, since the most interesting sets are infinite.

Other variations on the Russell Paradox abound:

> *The barber of Seville shaves all and only those men who do not shave themselves.*

If the barber does not shave himself, then according to the statement, he does shave himself. But if the barber does shave himself, then the statement says that he does not shave himself. A way out of this dilemma is to say that no such barber exists.

All this serves to remind us that when we wish to make meaningful statements about the world, we need to proceed with caution. Even the ideal world of Mathematics has some pitfalls to be wary of.

**Figure 1.2:** The central figures of the famous painting, *Socrates in Athens speaking with Plato,* painted by Raphael. It resides in the Vatican. (Image in public domain.)

## WHERE IS MATHEMATICS?

*God gave us the integers, all else is the work of man...*
German mathematician Leopold Kronecker (1823−1891)

We have had an inkling of mathematical reasoning, but what of Mathematics itself? What exactly is it? Some investigators, like neuropsychologist Brian Butterworth, have argued on evolutionary grounds that the human brain is hardwired for numeracy (see the Bibliography for his very interesting account).

The author would argue that the human genome – the full set of genes that make us what we are – contains instructions for building specialized circuits of the brain, which he calls the Number Module. The job of the Number Module is to categorize the world in terms of numerosities – the number of things in a collection…

Our Mathematical Brain, then, contains these two elements: a Number Module and our ability to use the mathematical tools supplied by our culture.

Those tools would include counting on one's fingers for starters, as well as an abacus, hand calculator, and all the other calculating devices that history has provided us with. Of course, trade and commerce made numeracy an essential ingredient and hastened the development of arithmetic.

For many people, Mathematics is just some form of glorified arithmetic. Or if they studied some algebra or geometry in high school, people often confess that they were never very good at it. Humorist Fran Lebowitz nicely sums up a prevailing view when she states that, "In real life, I assure you, there is no such thing as algebra[7]." This is not only a good joke, but it contains an element of truth in it, in so far as it makes the point that algebra is actually a mathematical abstraction. You will not see it anywhere on the streets of New York, Fran.

On the other hand, the concept of number is also an abstraction, so we are forced to conclude that we are going to have to deal with abstractions if we are to talk about Mathematics at all.

Indeed, besides the abstract notion of number, Mathematics inhabits a world of perfect circles, straight lines, triangles with exactly 180 degrees and so forth. Yet this is only an idealization and the world we live in can never contain a perfect circle or perfectly straight line. All lines will have some variations from absolute perfection, although these may be exceedingly small. They will also have some

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[7] Because many people who have encountered algebra in school have found it so arcane, it has attracted a considerable body of humor:

"*When you are dissatisfied and would like to go back to youth, think of algebra*" … Will Rogers

"*What is algebra exactly; is it those three-cornered things?*" James M. Barrie

"*As long as algebra is taught in school, there will be prayer in school*" … Cokie Roberts width, whereas a line in Mathematics has none. Numbers also go on forever, yet nothing in our experience seems to go on forever, except perhaps boring lectures about Mathematics. Even the Universe itself is expected to come to an end – more on the end of the world later.

Yet Mathematics is the most powerful tool we have that allows us to describe the world. Why should this even be so? In a certain sense, subscribed to by the author, Mathematics lies outside this world, in an ‘ideal world’ postulated by Plato (Fig. 1.2). Thus, mathematical concepts seem to occupy some transcendent realm (a view called *Platonism*) existing outside time and space.

How this transcendent ideal world interacts with the real world is the subject of this book and when we get to Quantum Mechanics, the real world will also take on the appearance of something quite other worldly as well.

Let us take one historical example that most people are familiar with: the 2,500-year-old great theorem of Pythagoras, which tells us that in any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides (Fig. 1.3).

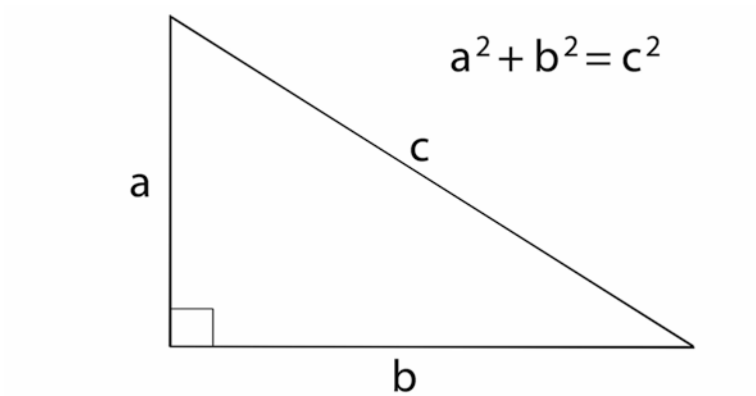


**Figure 1.3:** The famous theorem is attributed to Pythagoras (ca. 570–495 B.C.) although the evidence for a proof by him is sketchy and the result had been known since earlier times. (Illustration courtesy of Katy Metcalf.)

But again, this is only an idealization that is valid in the Mathematical Universe, and any actual right triangle that we can produce will have some minor discrepancy from what the theorem states. But the more accurately we draw a right triangle, the more precise the result will become.

On the other hand, this ancient result named after Pythagoras, perhaps initially drawn in the sand, has had countless applications to our understanding of the real world. It even makes a crucial appearance in Einstein’s Theory of Relativity regarding how time in motion slows down, which is at the heart of the Global Positioning System (GPS)[8].

There is also a converse of the theorem; that is, for any triangle having sides of length $a$, $b$, $c$, that satisfy the relationship

$$c^2 = a^2 + b^2,$$

then the angle formed by the sides of lengths $a$ and $b$ is a right angle ( = 90°). Proofs of both theorems appear in Euclid's *Elements*, discussed later in this chapter.

Mathematical theorems, like that of Pythagoras, are seemingly 'discovered'. A sprawling labyrinth of mathematical discovery arose from the concept of the 'imaginary number' which represents the square root of −1, and which is the subject of Chapter 3. It is a beautiful body of work, known as *complex analysis*, that is the basis for a lot of 'real' Science from Quantum Mechanics, to electrical circuitry, the Theory of Relativity, or the design of airplane wings... Yet the mathematics of complex analysis is not something concrete. It exists outside our physical world, dare we say in a universe of its own.

Roger Penrose, certainly one of the greatest living scientific minds on the planet, says, "I have been arguing that such 'God-given' mathematical ideas should have some kind of timeless existence, independent of our earthly selves." The author will not venture a guess as to whether this is a theistic statement or not.

The eccentric, itinerant (and atheist) Hungarian mathematical genius, Paul Erdös, often mentioned a book in which God had recorded all the most elegant and beautiful mathematical proofs. This sentiment reflects the notion, felt by many mathematicians, that Mathematics is fundamental to the very nature of the Universe.

Similarly, Godfrey Harold (G.H.) Hardy, one of the 20th century's finest mathematicians, held a similar view: "I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards…[9]"

On the other hand, many have argued against the Platonist perspective and take the view that Mathematics is a game played by mathematicians based on a set of logical rules and a set of axioms. No set of axioms is to be preferred over any other. This is a view more readily maintained by a non-mathematician and is held by a number of philosophers.

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[8] See the section Time in Motion of Chapter 7, as well as Appendix XV where it is explicitly utilized.

[9] *A Mathematician's Apology*, G.H. Hardy, 1940, p. 123.