MESSAGE PASSING CELLULAR AUTOMATA

Severino Fernández Galán

1945-2020
Marcombo 75 años

MESSAGE PASSING
CELLULAR AUTOMATA

Severino Fernández Galán
Artificial Intelligence Dept. at UNED

Marcombo
1945-2020 75 años

Message Passing Cellular Automata

Primera edición, 2020

E-mail address: seve@dia.uned.es
Personal web page: www.ia.uned.es/~seve/

www.marcombo.com

Corrección del inglés a cargo del autor.

Diseño de la cubierta: ENEDENÚ DISEÑO GRÁFICO

ISBN: 978-84-267-2997-2
D.L.: B-12196-2020

Impreso en Akoma
Printed in Spain

Dedicated with love to my wife, María, and my daughter, Daira

Contents

Preface

Cellular automata constitute a mathematical model that has been used in computer science since the 1940s. A cellular automaton consists of many identical simple processing units (or cells), arranged as a regular lattice, that interact with each other in a local way and in discrete time. Cellular automata have been applied to a wide variety of tasks such as modeling, simulation, visualization, or pseudo-random number generation, among many others. Due to the growing interest of the scientific community in cellular automata, a good number of relevant journals and conferences are dedicated to their study and development.

One of the main characteristics of cellular automata is their simplicity, which makes them relatively easy to program. Nonetheless, despite their simplicity, cellular automata are able to generate complex results in many domains, even in the context of artificial life. Another remarkable characteristic of cellular automata is that they are suitable for parallel implementation.

The present book deals with the fundamentals of cellular automata. Additionally, a new approach to extending cellular automata with the use of message passing is introduced. Each of the seven chapters of this book includes a number of figures, bibliographic references, and exercises of interest to the reader. The book offers students, practitioners and researchers a concise but broad coverage of the main aspects of cellular automata.

Chapter 1 "Introduction to Cellular Automata" includes a review of the history of cellular automata from their creation in the 1940s. Additionally, a definiton of "cellular automaton" is provided, along with a description of its main components. The novel idea of "probabilistic neighborhood" is introduced. This type of neighborhood generalizes the classical von Neumann and Moore neighborhoods. The chapter ends with a definiton of second-order cellular automata.

Chapter 2 "One-Dimensional Classical Cellular Automata" reviews cellular automata in one dimension defined over binary, ternary and continuous alphabets. Special attention is given to elementary cellular automata, which adopt binary states. The four classes of behavior identified by Stephen Wolfram in elementary CA are illustrated: homogeneous, periodic, chaotic, and complex.

Chapter 3 "One-Dimensional Message Passing Cellular Automata" introduces the novel concept of "message passing cellular automaton" and defines its four phases. Message passing cellular automata extend classical cellular automata by using message passing to communicate information between neighboring cells. One-dimensional message passing cellular automata are described in detail for binary, ternary, and continuous alphabets. Elementary message passing cellular automata are illustrated and several theorems are provided for them.

Chapter 4 "Two-Dimensional Classical Cellular Automata" explains some important classical cellular automata in two dimensions. Several relevant rules for defining two-dimensional cellular automata are studied such as majority, parity, and the Game of Life. Due to its impact on the field of cellular automata, the characteristics of the Game of Life are described in detail. Special attention is paid to the complex structures generated in the Game of Life, which make this cellular automaton so appealing.

Chapter 5 "Two-Dimensional Message Passing Cellular Automata" presents how message passing cellular automata can be implemented in two dimensions. The different two-dimensional message passing cellular automata are classified according to how the messages are calculated: fixed messages, sender-dependent messages, receiver dependent-messages, link-dependent messages, and neighborhood dependent messages. In each case, some interesting examples are illustrated from the huge set of possibilities that exist. Additionally, like in the previous chapter, several specific relevant rules are studied: rotation messages, reflection messages, majority messages, parity messages, and two extensions of the classical Game of Life.

Chapter 6 "Applications of Classical Cellular Automata" reviews a broad set of cellular automata applications in physics, excitable media, biology, social science, and mathematics. Each application is illustrated through simulated examples.

Finally, **Chapter 7 "Applications of Message Passing Cellular Automata"** introduces some novel applications of message passing cellular automata in domains such as diffusion of innovations, Schelling segregation model, cellular evolutionary algorithms, graph layout, and fractals.

The author of this book is an associate professor in the Department of Artificial Intelligence at UNED (Spanish Open University). Since the middle 1990s, he has performed teaching and research activities within the field of artificial intelligence, mainly in the areas of Bayesian networks and evolutionary computation. The present book is the result of a long journey that started when the author became interested in NetLogo [Wilensky, 1999]. NetLogo is an agent-based programming environment well suited for modeling and inspecting complex systems developing over time.

Severino Fernández Galán

Madrid, May 2020

Chapter 1

Introduction to Cellular Automata

A *cellular automaton* (CA) [von Neumann, 1966, Toffoli & Margolus, 1987, Wolfram, 2002] is formed by a regular lattice of cells where each cell adopts one of a set of states. The cells are updated according to a transition function defined locally in the lattice.

The three essential characteristics of CA are that they consist of many identical simple processing cells, that interactions between cells take place in a small neighborhood compared to the lattice size, and that discrete time is used. The rest of characteristics can be extended in order to apply CA to a wide range of processes: (1) the alphabet for the cell states can be Boolean, integer, real, or symbolic, (2) the transition function can be deterministic or probabilistic, and (3) the updating scheme can be synchronous or asynchronous.

1.1 History of Cellular Automata

The concept of CA was originally defined by John von Neumann along with his friend Stanislaw Ulam while working at Los Alamos National Laboratory in the 1940s. At that time, Ulam was interested in the growth of crystals, and von Neumann was studying self-replicating machines. In order to define his self-replicating machine, von Neumann suggested the use of a CA. As explained in [von Neumann, 1966], von Neumann's self-replicating machine was embedded in a two-dimensional cellular lattice of around two hundred thousand cells in which each cell had twenty-nine states and a five-cell neighborhood (today called "von Neumann neighborhood"). Interestingly, this self-replicating machine was designed without employing a computer. After von Neumann's death in 1957, Ulam continued working on several simpler CA and the results were published during the early 1960s.

The next important event in CA history took place in 1970. A divulgation article authored by Martin Gardner and published in Scientific American [Gardner, 1970] helped to popularize John H. Conway's Game of Life. The Game of Life, created by the British mathematician John H. Conway, is a CA whose rules allow simple patterns to change and give rise to different structures. Despite its simple rules, the Game of Life exhibits a complex behavior that has attracted the attention of many researchers.

In the early 1980s, Stephen Wolfram developed the first serious and exhaustive study of elementary CA, which provided a number of iconic images. Stephen Wolfram's work on CA initially gave rise to several seminal manuscripts [Wolfram, 1983, Wolfram, 1986] and culminated in 2002 with the publication of his comprehensive milestone book in the field *A New Kind of Science* [Wolfram, 2002]. Wolfram established four basic classes into which CA can be qualitatively classified in terms of their behavior: homogeneous, periodic, chaotic, and complex.

An excellent pioneering work on CA developed by Tommaso Toffoli and Norman Margolus at MIT can be found in the book *Cellular Automata Machines: A New Environment for Modeling* [Toffoli & Margolus, 1987]. The book by Andrew Ilachinski entitled *Cellular Automata: A Discrete Universe* [Ilachinski, 2001] offers an advanced mathematical treatment of CA.

1.2 Components of a Cellular Automaton

The present section describes the main components of a CA. Specifically, the following four components are dealt with: (1) the regular lattice of cells, (2) the set of states that the cells can adopt, (3) the local transition function that updates the state of each cell, and (4) the scheme that determines the order in which the cells are updated.

1.2.1 Lattice of Cells

The cells of a CA, denoted as C, are locally interconnected and arranged as a regular d-dimensional *lattice*. As shown in Figure 1.1a, in the one-dimensional case each cell is usually represented as a square and connected to its left and right neighbors. For two-dimensional CA, the most important lattice types are the square, the hexagonal, and the triangular (see Figures 1.1b, 1.1c, and 1.1d respectively).

In a two-dimensional square lattice, the *von Neumann neighborhood* is formed by a cell and its vertical and horizontal neighbors (see Figure 1.2a), whereas the *Moore neighborhood* incorporates the diagonal neighbors (see Figure 1.2b). The *probabilistic neighborhood* of a cell $c \in C$, denoted as $\mathcal{N}_\theta(c)$ with $\theta \in [0,1]$, was introduced in [Galán, 2019]. A cell $c' \in C$ is included in $\mathcal{N}_\theta(c)$ with a probability defined by the following expression:

$$P(c' \in \mathcal{N}_\theta(c)) = \begin{cases} 0 & \text{if } c' \notin \mathcal{N}_\mathrm{M}(c) \\ 1 & \text{if } c' \in \mathcal{N}_\mathrm{N}(c) \\ \theta & \text{if } c' \in \mathcal{N}_\mathrm{M}(c) - \mathcal{N}_\mathrm{N}(c) \end{cases},$$

where $\mathcal{N}_\mathrm{M}(c)$ and $\mathcal{N}_\mathrm{N}(c)$ represent the Moore and von Neumann neighborhoods of c respectively. Note that $\mathcal{N}_\theta$ is a generalization of $\mathcal{N}_\mathrm{N}$ and $\mathcal{N}_\mathrm{M}$, since $\mathcal{N}_\theta \equiv \mathcal{N}_\mathrm{N}$ if $\theta = 0$

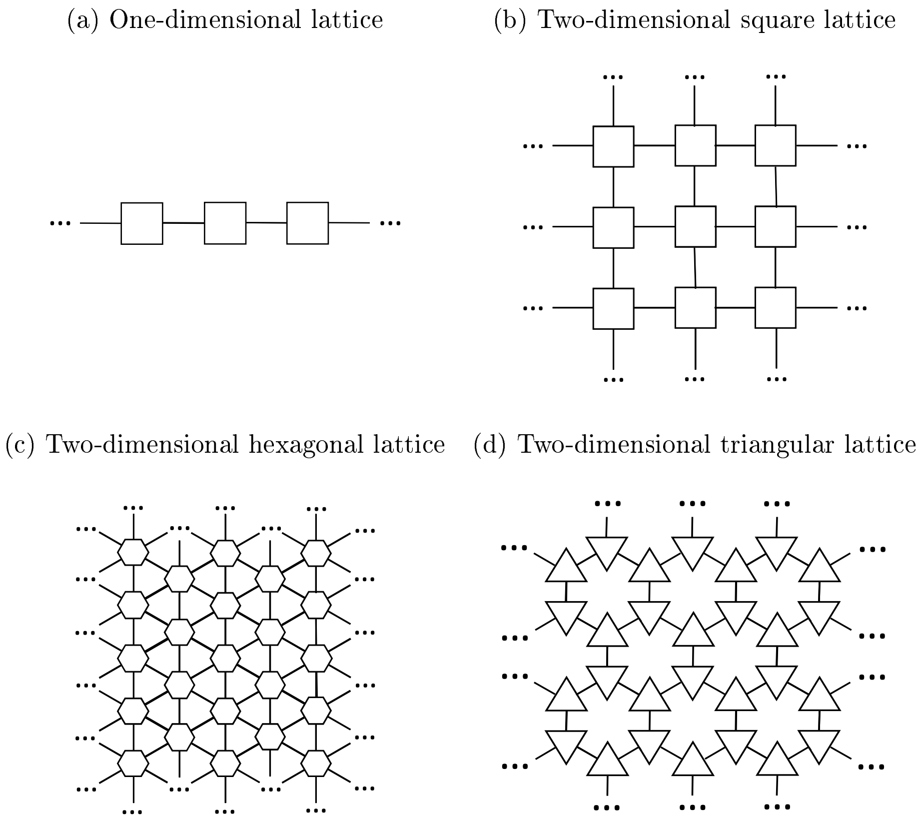

Figure 1.1 Examples of a one-dimensional lattice (a), a two-dimensional square lattice (b), a two-dimensional hexagonal lattice (c), and a two-dimensional triangular lattice (d).

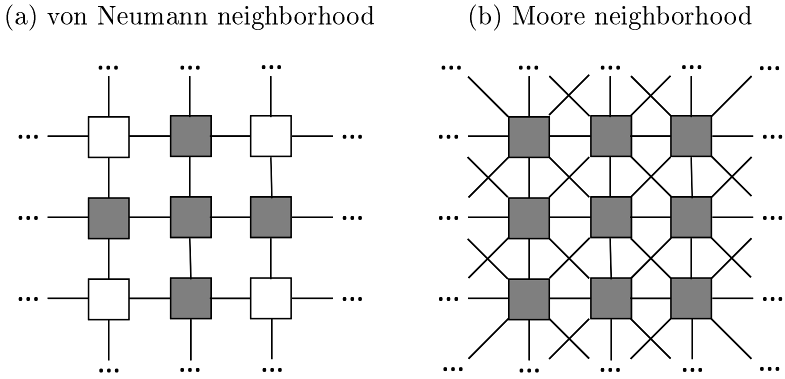

Figure 1.2 Examples in two dimensions of a von Neumann neighborhood (a) and a Moore neighborhood (b).

and $\mathcal{N}_\theta \equiv \mathcal{N}_{\mathrm{M}}$ if $\theta = 1$. Figure 1.3 contains several examples of wavefront generated by applying $\mathcal{N}_\theta$ to an initial cell for one hundred iterations and $\theta \in \{0, 0.1, \ldots, 0.9, 1\}$.

In the formal definition of a CA, the lattice of cells is usually considered as infinite. However, boundary conditions have to be established when simulating a CA lattice in the limited memory of a computer. As an example, Figure 1.4a shows the boundary cells of a finite two-dimensional CA. In order for the boundary cells to operate, the states of their neighboring cells need to be set up through any of the following types of boundary conditions:

1. *Periodic*: Opposite boundary cells are connected along each axis direction (see Figure 1.4b).

2. *Reflecting*: The boundary acts like a mirror that reflects the states of the boundary cells (see Figure 1.4c).

3. *Fixed*: The same arbitrary state is assigned to all the boundary cells (see Figure 1.4d).

The most widespread are the periodic boundary conditions, which simulate an infinite lattice by using a finite one. The periodic boundary conditions are implemented as a ring in one dimension and as a torus in two dimensions.

In the rest of this book, both the one-dimensional and the two-dimensional square lattices will be extensively used. The cells in these two lattices will be referred to as c_i and c_{ij} respectively, where $i, j \in \mathbb{Z}$. Additionally, unless otherwise specified, periodic boundary conditions will be employed for the cells.

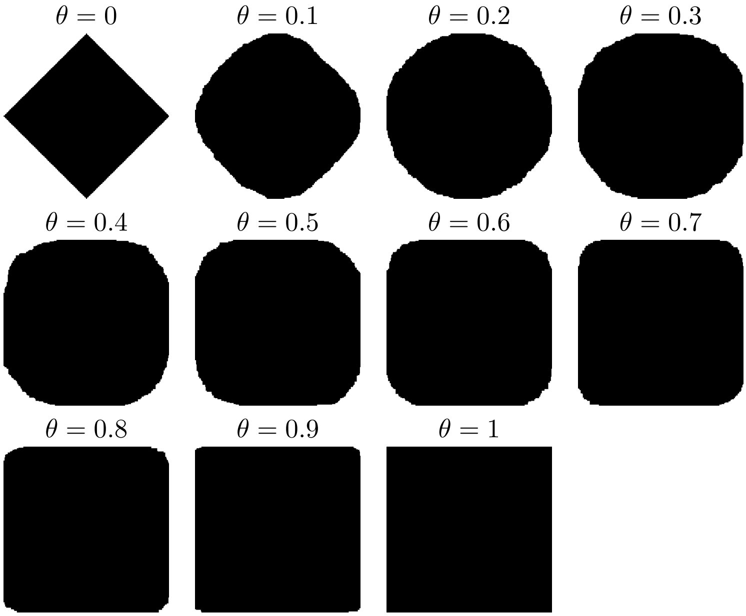

Figure 1.3 Examples of wavefront generated through $\mathcal{N}_\theta$ for $\theta \in \{0, 0.1, \ldots, 0.9, 1\}$. Each wavefront corresponds to one hundred iterations using a 201x201 grid.

1.2.2 States of Cells

Each cell of a CA adopts a *state* σ belonging to its local state space Σ. In this way, the whole CA has a global state space formed by the Cartesian product of the local state spaces. Usually, all the cells have the same local state space, which consists of a finite number of states $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k\}$ with $k \geq 2$.

CA can be extended so that the valid states that a cell can take on are defined in a continuous interval. In this book, the state of cell $c \in C$ at time t will be denoted as $c(t)$ unless otherwise specified.

1.2.3 Transition Function

CA use discrete time $t \in \{0, 1, 2 \ldots\}$ such that the initial time instant $t = 0$ corresponds to the situation where no change of the cells' states has taken place yet. A local *transition function* f governs how each cell alters its own state from the present instant t to the next instant $t + 1$. This function takes as arguments the cell's current state and the current states of its neighboring cells. The transition function of a CA can operate either in a deterministic or in a probabilistic way.

Among all the possible transition functions that can be defined in CA, an interesting subset is formed by the so-called *totalistic transition functions*, which are defined as a function of the sum of their arguments. Another interesting subset is composed by the

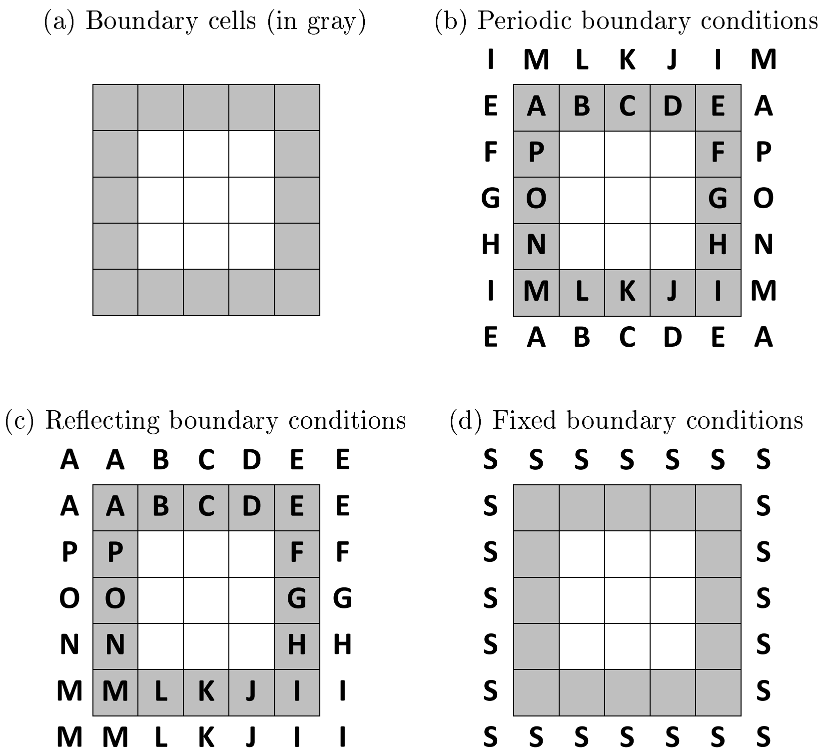

Figure 1.4 Boundary cells of a 5x5 square lattice (a) and three different boundary conditions for this lattice: periodic (b), reflecting (c), and fixed (d).

outer totalistic transition functions, in which the value of the central cell at the next time step depends on its current value and the sum of the values of its neighbors. The Game of Life (see Section 4.4) is a famous example of CA applying an outer totalistic transition function.

1.2.4 Updating Scheme

In a CA, the *updating scheme* of the cells' states can be performed in the following two ways:

1. *Synchronously*: All the cells update their states simultaneously at an externally provided clock step. This method is the most widely used in CA applications.
2. *Asynchronously*: The cells update their states sequentially in a random order. This process is repeated for each discrete time instant t.

A particular assignment of states to the cells of a CA is named *configuration*. Among all the possible configurations, the ones called "Garden of Eden" are those that are unreachable from any initial configuration or, in other words, those with no predecessor configuration. When every configuration of a CA has a unique predecessor, the CA is called *reversible*. Some problems in physics, such as the motion of particles in an ideal gas (see Section 6.1.2.1 on the lattice gas model) or the Ising model of alignment of magnetic charges (see Section 6.1.3), can be simulated by reversible CA. Given the transition function of a CA, whereas the question of reversibility is decidable in the one-dimensional case [Amoroso & Patt, 1972], it is undecidable for two-dimensional CA [Kari, 1990].

1.3 Second-Order Cellular Automata

As explained in Section 1.2.3, the transition function f of a CA takes as arguments the current state of a cell and the current states of its neighboring cells in the lattice. This can be expressed mathematically in the following way:

$$c(t+1) = f\left(c_{\mathcal{N}(c)}(t)\right),$$

where $\mathcal{N}(c)$, the neighborhood of $c \in C$, represents the set of cells formed by c and its neighboring cells. This definition corresponds to the so-called *first-order CA*, which constitute the ordinary CA in the literature.

Second-order CA extend first-order CA by allowing the transition function f to be also dependent on the states of cells at time $t-1$. This is formalized as follows:

$$c(t+1) = f\left(c_{\mathcal{N}(c)}(t), c(t-1)\right),$$

which means that the next state of a cell $c \in C$ depends on the current states of the cells in its neighborhood $\mathcal{N}(c)$ and on the previous state of c. An interesting example of second-order CA can be found in Section 6.1.1.3 in the context of wave propagation modeling through the wave equation.