Purabi Mukherji

Research Schools on Number Theory in India

During the 20th Century

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I dedicate this book to the memory of my father

Late Sushil Kumar Banerji

and my mother

Late Ranu Banerji

Foreword

It is with great pleasure that I write this foreword for the book, *Research Schools on Number Theory in India: During the 20th Century*, written by Dr. Purabi Mukherji. Since my expertise is neither number theory nor the history of science, I was quite apprehensive when Dr. Mukherji called to ask me if I would be willing to write a foreword for her book. After a little protest, however, I agreed.

I met Dr. Mukherji for the first time at the National Conference of Calcutta Mathematical Society in December 2015. At that first meeting, she told me that she was working on a project sponsored by the Indian National Science Academy (INSA), and we talked a little about the nature of this project. It was the start of a valuable friendship and we have talked to each other a lot since, trying to convince ourselves that the world around us perhaps will get a lot better one of these days and we may even live long enough to see it!

This is not the first book that Dr. Mukherji has authored. She co-authored a book, *History of the Calcutta School of Physical Sciences*, with Dr. Atri Mukhopadhyay. I was able to get hold of a copy of this book as soon as it was published and thoroughly enjoyed reading it. Given that I have no real sense of history of either science or anything else for that matter, I found this reading to be especially enlightening. Dr. Mukherji had called me some time ago telling me that for the INSA project, she was preparing a report on the research conducted on number theory in India during the twentieth century. I was very pleased to learn that this important project was in the hands of a very competent and knowledgeable person.

As I have said several times in many public forums, the research in number theory in India has been always of very high quality. I believe, it is for this reason, a very large number of bright young students have chosen to work in this area. Many of them today are counted among the leaders in this field. Unfortunately, there has been no serious account of the large body of work in the area of number theory during the twentieth century from around the country. For this reason, now that the INSA project has been turned into a book, I consider it to be a very timely and much needed addition to our literature on the subject.

Returning to the book itself—it was a fascinating and thoroughly engaging read. The organization of the book, from south to north followed by east to west, is very appealing. I was pleasantly surprised to see that there is a chapter on the applications of number theory to other areas. It has a complete list of all the research papers written during this period, which I am sure will be immensely useful to a student researching in this subject. Finally, as the author says and I agree, the book is written without using too much mathematical jargon to make it accessible to a general readership.

I intend to be coming back to this book ever so often. I am sure anyone who has some interest in the history of research in mathematics in India would enjoy reading this book as much as I did.

May 2020

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Preface

The great German mathematician Felix Klein had said "God is a geometer". Jacobi changed this and wrote "God is an arithmetician". Then came Kronecker and made the memorable comment: "God created the natural numbers and all the rest is the work of man". Number Theory, or the Theory of Numbers, has always held a special and unique position in the world of mathematics. Its fundamental place in the discipline has led to its sobriquet as "the queen of mathematics".

India had a long tradition of research in Number Theory since ancient times. Famous and eminent Indian mathematicians like Brahmagupta, Bhaskaracharya, Jayadeva, and others made notable contributions in this area. In the Western world, too, the all-time mathematical greats such as Euler, Gauss, Lagrange, Jacobi, and many others have made notable contributions in this area.

In India, research on Number Theory in modern times started with the advent of the iconic genius Srinivasa Ramanujan. His theories, conjectures, and questions on various topics of Number Theory inspired mathematicians both at home and abroad. From the early part of the twentieth century, inspired and influenced by Ramanujan, notable research was done in South India and Panjab. The establishment of Tata Institute of Fundamental Research (TIFR), in Bombay (now Mumbai), was a landmark event in the research activities related to Number Theory in the Indian context.

In this book, an attempt has been made to describe the gradual development of the major schools of research on Number Theory in South India, Panjab, TIFR, and the minor schools of Bengal and Bihar. A comprehensive bibliography of major number theorists of India during the twentieth century has been carefully compiled. Of course, the research contributions, which some of them have made after leaving India for good, have not been included.

A chapter has been devoted to the impact of the research made by Indian number theorists nationally and internationally. Finally, applications of some results developed by the indigenous number theorists in practical fields have been discussed. In the concluding chapter, a general discussion on the importance of Number Theory in the modern-day world of mathematics has been briefly discussed. A whole chapter has been devoted to the bibliographies of the number theorists whose works have been discussed in the text. It is valuable from the standpoint of historic documentation and will also serve the researchers in the field of Number Theory in their literature survey.

Since the book is written from the viewpoint of the “history of science”, mathematical expressions and technical jargons have been avoided as much as possible. The language has been made lucid so that general readers with an attraction for scientific developments in India can easily understand the exposition.

Kolkata, India
January 2020

Purabi Mukherji

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Contents

1 Historical Prelude and Introduction 1

2 Indian Schools of Research on Number Theory 5
2.1 South Indian School of Research on Number Theory (1910–1950) 6
2.1.1 Srinivasa Ramanujan (1887–1920) 6
2.1.2 K. Ananda Rau (1893–1966) 28
2.1.3 Subbayya Sivasankaranarayana Pillai (1901–1950) 32
2.1.4 T. Vijayaraghavan (1902–1955) 41
2.2 The Panjab School of Research on Number Theory (1920–1999) 45
2.2.1 Sarvadaman Chowla (1907–1995) 45
2.2.2 Hansraj Gupta (1902–1988) 53
2.2.3 Ram Prakash Bambah 62
2.3 Trends of Research on Number Theory in Bengal and Bihar 72
2.3.1 Number Theorists of Bengal 72
2.3.2 Number Theorists in Bihar 78
2.4 TIFR School of Number Theory 79
2.4.1 Komaravolu Chandrasekharan (1920–2017) and the Initial Years of the TIFR 79
2.4.2 Kollagunta Gopalaiyer Ramanathan (K. G. Ramanathan) (1920–1992) 83
2.4.3 Srinivasacharya Raghavan (1934–2014) 87
2.4.4 Kanakanahalli Ramachandra (1933–2011) 90
2.4.5 Chakravarti Padmanabhan Ramanujam (1938–1974) 96
2.4.6 S. Srinivasan (1943–2005) 98
2.4.7 Tarlok Nath Shorey 99
2.4.8 Ramachandran Balasubramanian 103
2.4.9 Ayyadurai Sankaranarayanan 105

2.4.10 Sukumar Das Adhikari . . . 106
2.4.11 Dipendra Prasad . . . 107
2.4.12 K. Soundararajan . . . 108
References . . . 109

3 Impact of the Research of the Indian Number Theorists on Modern Mathematics . . . 111
3.1 Rogers–Ramanujan Identities . . . 114
3.2 Hypergeometric Series . . . 115
3.3 Waring's Problem . . . 116

4 Epilogue . . . 125
4.1 Practical Applications . . . 125
4.1.1 Crystallography . . . 125
4.1.2 Statistical Mechanics . . . 126
4.1.3 Computation of the Value of π (Pai) . . . 126
4.1.4 Ramanujan Graphs . . . 127
4.1.5 Lattice . . . 127
4.1.6 Theoretical Physics . . . 128
4.2 Concluding Remarks . . . 128
Bibliography . . . 129

References . . . 177

Chapter 1
Historical Prelude and Introduction

Since the beginning of the recorded history, numbers have fascinated the human mind. In every known civilization, mathematics related to numbers was practiced. In ancient Egypt, the famous *Rhind* (or *Ahmes*) *Papyrus* was written in around 1700–1600 BC. They are safely stored in the British Museum in London. Information about the kind of mathematics practiced in ancient Egypt can be obtained from them. They deal with numbers as expressed by ancient Egyptian mathematicians. Cuneiform tablets recovered from Mesopotamia display evidences about the mathematics practiced there at the end of the third millennium BC. It confirms that arithmetic was quite developed at that time. In China, by the eleventh century BC, mathematics had been developed independently. They had developed number sets, comprising very large numbers and negative numbers, decimals and binary systems, and many more important theories and techniques. They were also quite proficient in calculating the value of pi (or π), division, and root extraction. Pascal's triangle, as known in the Western world, was known to the Chinese for at least three centuries even before Pascal was born (1303 CE). It is mentioned in the work of Chu Shih-Cheih.

Another major civilization was the ancient Indian civilization that prospered around the Indus Valley in India. Here, mathematics has a very rich history going back to at least or about 3000 BC. This tradition continued for centuries. The concept of zero as a number in its own right, as is now generally accepted internationally, was first developed in India. This is a seminal contribution. It should be specially noted that the transformation of zero from a simple placeholder to a number in its own right gives credence to the belief that the subcontinent had a prevalent highly developed mathematical culture. Apart from that the Indian contribution in the fields of arithmetic, negative numbers, geometry, trigonometry, and astronomy are also remarkable.

In India, during the Vedic period (approximately in 1200 BC), there is enough documentary evidence to prove that the mathematicians of those times were strong in dealing with arithmetical problems of different types. They were competent to handle arithmetic of fractions and surds and found good rational approximations

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to irrational numbers like $\sqrt{2}$. In Vedic texts, numbers were usually expressed as combinations of powers of ten.

The *Shulba Sutras* written in 800 BC are the oldest geometrical treatises available in the subcontinent. The arithmetic content of these *Shulba Sutras* consists of rules for finding Pythagorean triples such as (3, 4, 5), (5, 12, 13), (8, 15, 17), and (12, 35, 37). Actually, Baudhayana gave a statement in terms of sides and diagonals of a rectangle which, later, the Greek mathematicians presented as the *Pythagorean theorem.* The triples mentioned above arose out of a religious ritual, where every Hindu home was required to have three fires burning at three different alters of square, circular, and semicircular shapes, having the same area. These conditions led to certain types of "Diophantine" problems.

Around the seconnd century BC, mathematicians of India were aware of the "binary system", as one may see in *Pingala Chanda Sutra.* The concept of modern-day Pascal's triangle is also found in *Pingala Chanda Sutra* 1500 years before the Chinese invention. Later, over the centuries, many Hindu and Jaina mathematicians made notable contributions in the fields of arithmetic.

Aryabhata 1 (476–550 AD) computed the value of π correct to four decimal places. He also clearly stated that it was but an approximate value only. He also dealt with continued fractions and discovered algorithms for finding approximate square roots and cube roots. The so-called "generalized Euclidean algorithm" related to g.c.d. is actually due to him.

The famous Hindu mathematician Brahmagupta (sevenh century AD) in *Brahmasphutasiddhanta* initiated rules for working with negative numbers. The rules for the four basic arithmetical operations namely, addition, subtraction, multiplication, and division, were initiated by him. He was the first mathematician to state formally that any number added to its own negative gives rise to zero. The solution method of the so-called "Pell's equation" is now internationally attributed to Brahmagupta and his famous principle of *Bhavana,* which was later rediscovered in Europe by Euler.

Mahavira Acharya (800–870 AD) and Virasena (eighh century AD) were Jaina mathematicians who dealt with 2-adic orders. Sridhara (870–930 AD) who lived in West Bengal authored a treatise named *Pati-ganita.* There he dealt with fractions and gave eight rules for operations involving zero. He also initiated methods of summation of different arithmetic and geometric series.

Aryabhata II (920–1000 AD), in his astronomical treatise named *Maha-Siddhanta*, discussed in detail the methods of numerical mathematics (*ankganit*) and solution of indeterminate equations (*kuttaka*).

Bhaskara II (1114–1185 AD) made substantial contributions to arithmetical and geometrical progressions. He called a fraction whose denominator is zero *khahara* and gave an allegorical description of this entity as comparable with the infinite immutable God, from whom everything is created without reducing Him, while everything merged within Him after destruction but without increasing Him. In modern mathematical terminology, this corresponds to "division by zero is infinity". He was also familiar with a positive number which had two square roots. He worked on surds and developed methods for solving various kinds of equations. His technique of solving the general form of "Pell's equation" and the general indeterminate quadratic equation using the *chakravala* method (originally developed by Jayadeva of about 200 years earlier) is highly acclaimed. He also did commendable work on indeterminate cubic, indeterminate quartic, and indeterminate higher-order polynomial equations. He was successful in computing the value of π correct to five decimal places.

So, it is evident that starting from very ancient times till the twelfth century, Indian mathematicians had an unbroken tradition of working with numbers and related problems. This tradition was successfully followed and much furthered by the Kerala School of Madhava of Sangamagrama from the fourteenth century to the seventeenth century.

Coming to more recent times of all branches of mathematics, the greatest mathematical genius of the nineteenth century, C. F. Gauss (1777–1855) hailed "number theory" as the queen of mathematics. The natural and pertinent question would be why this particular branch of mathematics is so highly adulated. Probably, the reason for this glorification of the said topic is because of the fact that in this particular branch of mathematics, it is possible to ask questions in very simple terms: easily understandable by anyone but not so easily answerable by all. In contrast, in any other branch of mathematics, asking a really meaningful, non-trivial question, a long list of definitions is a mandatory prerequisite.

The "theory of numbers" (or number theory) is a branch of mathematics that deals with properties and relationships of numbers, especially the positive integers. Number theory can be broadly classified as "analytic" and "algebraic". "Analytic number theory" is a branch of number theory that uses methods and theories from mathematical analysis to solve problems about the integers. On the other hand, "algebraic number theory" studies algebraic structures related to algebraic integers.

From the world perspective, the twentieth century has been witness to an explosion to number theory-related research work. Along with classical and analytic number theory, scholars started exploring specialized subfields, such as algebraic number theory, geometric number theory, and combinatorial number theory.

Some of the greatest mathematical minds of modern times, such as Leonhard Euler (1707–1783), Carl F. Gauss (1777–1855), J. L. Lagrange (1736–1813), P. De Fermat (1601–1665), C. G. J. Jacobi (1804–1851), E. Waring (1734–1778), S. Ramanujan (1887–1920), P. Erdös (1913–1996), and many others have worked on various problems related to number theory.

Since this book is devoted to the development of the "schools of research" on number theory in India during the twentieth century, the focus will be the discussion on Indian number theorists of the time and their contributions in developing such centers of research in the field.

Chapter 2
Indian Schools of Research on Number Theory

The development that took place in the research activities on number theory in India during the twentieth century is quite vast and a chronological study of the same has been undertaken during the course of the present exercise. The different schools of research developed during the period have been identified, and detailed discussions on the notable contributions of famous number theorists of India have been done. The research schools that developed during the first half of the twentieth century have been dealt with first. The ones that developed roughly during the second half of that century have been discussed later.

The two major schools that developed in the first half of the twentieth century are, respectively, South Indian School and the Panjab School of research on number theory. A small group of researchers from Bengal and Bihar also did carry out some research in the said discipline during that time. It would of course be pertinent to admit at the outset that compared to the South Indian and Panjab Schools of research on number theory, the contribution of the Bengal–Bihar schools was pretty meager. But for the sake of historical records, it is of some relevance.

In the second half of the twentieth century, the Panjab School continued to flourish. But the most outstanding center of research on number theory was established in the newly founded Tata Institute of Fundamental Research (TIFR) in Mumbai. Detailed discussions on all these centers have been done and they form the main text of the book. The analysis is made on a chronological basis to unravel the historical significance of the whole exercise.

P. Mukherji, *Research Schools on Number Theory in India*,
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2.1 South Indian School of Research on Number Theory (1910–1950)

Here, the discourse is about the development of number theory-related research work that was initiated for the first time by Indian mathematicians born in the southern part of India. The first half of the twentieth century was a witness to phenomenal growth in the field, and brief discussions about the main contributors in the field are presented below.

2.1.1 Srinivasa Ramanujan (1887–1920)

Any discussion related to number theory has to start with Srinivasa Ramanujan. He was one of the world's finest mathematicians and his contributions to number theory unambiguously make him the greatest number theorist of the world. He has been the fountainhead of inspiration to generations of mathematicians in India as well as the world. He has been a key figure who has drawn number theorists both nationally and internationally to prove his conjectures and theorems and solve problems posed by him directly or indirectly connected to number theory. In a general way, too, S. Ramanujan was a major influence in the field of scientific research in India during the early years of the twentieth century. A remark by the Nobel Laureate S. Chandrasekhar is quite relevant in this context. He wrote:

> Let me turn to the role of Ramanujan in the development of science in India during the early years of this century (twentieth century). Perhaps the best way I can give you a feeling for what Ramanujan meant to the young men going to schools and colleges during the period 1915–1930 is to recall the way in which I first learnt of Ramanujan's name….

Then he related how he came to know about Ramanujan's premature death and his great achievements during his stay at Cambridge. What impressed Chandrasekhar most was that even though Ramanujan was brought up in abject poverty and in a scientifically sterile environment, he had succeeded in conquering all the adversities and established himself as one of the most original mathematicians of the century. Chandrasekhar felt that those facts were more than enough to encourage aspiring young Indian students to break the bonds of their intellectual confinement and establish themselves at a spectacular height as Ramanujan had done. Chandrasekhar further commented:

> The twenties and thirties were a period when young Indians were inspired for achievement and accomplishment by the men they saw among them.

The life story of Srinivasa Ramanujan is fairly well known to the scientific community in India. Still, for the completeness of historical facts, a brief outline of the life history of this iconic Indian mathematician is given below.

Ramanujan (Fig. 2.1) was born on December 22, 1887, at Erode in Tamil Nadu

Fig. 2.1 Srinivasa Ramanujan (1887–1920)

in a Brahmin family. His parents belonged to the financially weaker section of the society and had to work hard to make both ends meet. At the age of 7, he was sent to a primary school situated in the town of Kumbakonam. He stood first in the Tanjore district primary examination held in 1897. This helped him to get a half-free studentship in the Town High School at Kumbakonam. He studied there from 1898 to 1903. In 1904, he passed the Matriculation examination, conducted by the University of Madras.

Right from his school days, Ramanujan showed extraordinary intuition and astounding proficiency in various branches of mathematics such as arithmetic, algebra, geometry, number theory, and trigonometry. The senior mathematics teacher of the school was so confident about Ramanujan's mathematical prowess that every year he entrusted young Ramanujan with the task of preparing a conflict-free timetable for the whole school. While still in his fourth form, Ramanujan mastered Loney's *Trigonometry* (Part II). During the school years, Ramanujan won many prizes for his outstanding performances in mathematics.

To augment the meager family income, every year Ramanujan's mother took in a few students as boarders. These undergraduate students noticed the young boy's unusual fascination for mathematics and out of kindness for him, they gave him an elementary introduction to all branches of mathematics. In 1903, with the help of friends from the local government college, Ramanujan was able to procure G. S. Carr's *A Synopsis of Elementary Results*, a book on Pure mathematics. The book, published in 1896, contained propositions, formulae, and methods of mathematical analysis with brief hints. Ramanujan started working tirelessly to solve the problems given in the book. The more he worked, the more he became involved with mathematics. A biographer of Ramanujan, Prof. P. V. Seshu Aiyar, in his edited book has stated:

> It was this book which awakened his genius. He set himself to establish the formulae given therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned.[1]

The fact is that while proving formulae from Carr's book, Ramanujan discovered many new ones. The self-taught genius thus laid for himself the foundations for pursuing higher mathematics. During 1904–1907, he also started noting down the new formulae in his now famous *Notebook*. After completing school education, Ramanujan won a junior scholarship for his proficiency in mathematics and English composition. This enabled him to join the First Examination in Arts (FA) course at the Government Arts College at Kumbakonam. Due to his obsessive preoccupation with mathematics, he neglected the other subjects of the course and failed. As a consequence, he lost his scholarship and was not promoted to the senior FA class either. The next year, in 1906, Ramanujan for the first time left his hometown and moved over to Madras and got admitted into a college there to continue the FA. Due to ill health, he had to discontinue his studies. In 1907, he appeared for the FA examination as a private candidate. Though he secured full marks in mathematics, he failed in the other subjects. That was the end of his formal education.

In spite of endless struggle with abject poverty, Ramanujan still maintained an unfailing commitment to mathematics. Those were years of great adversity for him. Despite the stresses and strains of day-to-day existence, Ramanujan continued noting down the mathematical results conjectured or discovered by him in *Notebooks*. Years later after Ramanujan was no more, his mentor and friend Prof. G. H. Hardy (1877–1947) had lamented and wrote:

> The years between 18 and 25 are the critical years in a mathematician's career.... During his five unfortunate years (1907–1912), his genius was misdirected, sidetracked and to a certain extent distorted.[2]

Finally, from 1910 onwards a few lovers of mathematics in Madras (present-day Chennai) took notice of this exceptionally talented mathematician. The earliest contributions of Ramanujan were communicated by Prof. Seshu Aiyar to the Journal of the Indian Mathematical Society in the form of "questions". They were published in 1911. Ramanujan's first research paper on number theory also appeared in the same issue of the journal and was titled "*Some Properties of Bernoulli Numbers*" [Sect. I, (SR. 1)]. In the paper, he stated eight theorems concerning the arithmetical properties of Bernoulli numbers of which he proved three, two were stated as corollaries, and three were just conjectures. In 1912, on the advice of Prof. Seshu Aiyer,

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[1] G. H. Hardy, P. V. Seshu Aiyer and B. M. Wilson (Eds.).*Collected Papers by Srinivasa Ramanujan.* New York: Chelsea. 1962.

[2] G. H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Works.* New York: Chelsea. 1940.