Lecture Notes in Applied and Numerical Harmonic Analysis

$$\widehat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Kathy D. Merrill

Generalized Multiresolution Analyses

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LN-ANHA Series Preface

The *Lecture Notes in Applied and Numerical Harmonic Analysis (LN-ANHA)* book series is a subseries of the widely known *Applied and Numerical Harmonic Analysis (ANHA)* series. The Lecture Notes series publishes paperback volumes, ranging from 80 to 200 pages in harmonic analysis as well as in engineering and scientific subjects having a significant harmonic analysis component. *LN-ANHA* provides a means of distributing brief-yet-rigorous works on similar subjects as the *ANHA* series in a timely fashion, reflecting the most current research in this rapidly evolving field.

The *ANHA* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modem harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, bio-medical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in applicable topics such as the following, where harmonic analysis plays a substantial role:

Bio-mathematics, bio-engineering, and bio-medical signal processing;
Communications and RADAR;
Compressive sensing (sampling) and sparse representations;
Data science, data mining and dimension reduction;
Fast algorithms;
Frame theory and noise reduction;
Image processing and super-resolution;
Machine learning;
Phaseless reconstruction;
Quantum informatics;
Remote sensing;
Sampling theory;
Spectral estimation;
Time-frequency and time-scale analysis – Gabor theory and wavelet theory.

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, for example, the concept of “function.” Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor’s set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener’s Tauberian theorem in

Fourier analysis not only characterizes the behavior of the prime numbers but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

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Chapter 1
Introduction

The history of wavelets is a story that demonstrates the power of collaboration between different specialties within mathematics, physics, engineering, and computer science. Many parallel beginnings of wavelets emerged during the twentieth century, as researchers in all of these fields applied and modified Fourier's 1807 decomposition of functions into canonical pieces. (See [21] or [42] for details.) Applications in signal and image processing drove innovation, while theory from harmonic analysis and theoretical physics broadened and deepened the ideas. One early example of the fruitfulness of this interaction between application and theory coincided with the naming of wavelets in the early 1980s. Geophysical engineer J. Morlet and theoretical physicist A. Grossman collaborated to adapt windowed Fourier transform functions into a tool they called a wavelet [27]. Their paper already incorporated abstract harmonic analysis, a theme of this book, in the form of representations of the $ax + b$ group. A second important example of collaboration between application and theory happened in the mid-1980s. Stéphane Mallat brought his knowledge of pyramidal algorithms in image processing to a collaboration with Yves Meyer, who brought insights from harmonic analysis dating back to the work of Littlewood-Paley and Calderon. The result was a formulation of the concept of multiresolution analyses, the ancestor of this book's topic, and the basis of a construction method for wavelets [34, 41].

Since these beginnings, tools from classical harmonic analysis involving the standard Fourier transform have been an essential part of the theoretical side of the interactions that built and refined wavelets. Abstract harmonic analysis, while not quite so constant a presence in wavelet theory, has reappeared regularly, after being introduced into the beginnings of the subject by Grossman and Morlet. For example, in 1988 Feichtinger and Gröchenig [24] used group representations to unify Gabor and wavelet transforms into a single theory [28], and later Hartmut Führ extended their work to more general settings [25].

In 1995, Baggett, Carey, Moran, and Ohring [3] introduced a representation theoretic approach to the study of multiresolution structures. This volume will

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describe the resulting strand of research, which is centered around the concept of a generalized multiresolution analysis. The use of abstract harmonic analysis to further develop this concept has been carried out by several groups during the late 1990s and into the first two decades of 2000. This work has led to more powerful and more general methods to build new wavelets, and also to an ability to use wavelets in broader contexts.

1.1 Wavelets and Multiresolution Structures

Historically, wavelets have used the operations of dilation and translation to decompose functions into canonical pieces, so that each piece has a prescribed location and level of zoom. Ideally, the pieces decay rapidly outside of a narrow focus, and thus overcome the Fourier system's weakness of being non-localized. Multiresolutions have been used to break the space up according to level of zoom, with a subspace V_j containing elements zoomed in up to level j.

The development of multiresolution structures has roots in subband filtering in signal processing and pyramid algorithms in image reconstruction [15], and such structures occur naturally in these and other applied fields. For example, pattern recognition requires the ability to examine an image at different resolutions, since patterns of varying sizes are often being sought [35]. The use of multiresolutions in computer vision applications is encouraged by evidence showing that the human visual system itself uses multiresolution signal processing [1]. Further, the decomposition of a signal into a coarser component together with a detail component at each level enables a recursive method perfectly adapted to computer reconstruction. Conversely, the recursive structure behind dynamical systems such as iterated function systems can be naturally adapted to build multiresolution structures in that context. (See, for example, Chapter 6, and for a different approach, [36].)

Wavelets and multiresolution structures were originally developed in $L^2(\mathbb{R})$ with translations by integers and dilation by 2, using the unitary operators $\tau_n f(x) = f(x-n)$ and $\delta f(x) = \sqrt{2} f(2x)$. Later, they were extended to $L^2(\mathbb{R}^N)$ with translations by the integer lattice and dilation $\delta_A f(x) = \sqrt{|\det A|} f(Ax)$ by an *expansive* (all eigenvalues have absolute value greater than 1) integer matrix A. The work of Baggett, Carey, Moran, and Ohring [3] generalized these operations to interrelated operators on an abstract Hilbert space. In particular, they replaced standard translations by a countable, discrete, not necessarily abelian group Γ of unitary operators on a separable Hilbert space H, and replaced dilation by another unitary operator δ on H such that $\delta^{-1}\Gamma\delta \subset \Gamma$. They called such a collection of operators an *Affine Structure*. In this context, they defined a wavelet:

Definition 1.1 A (orthonormal) wavelet in a Hilbert space H is a finite set $\{\psi_1, \psi_2, \cdots, \psi_L\} \subset H$ such that $\{\delta^j(\gamma(\psi_i))\}$ forms an orthonormal basis of H, with $-\infty < j < \infty$, $\gamma \in \Gamma$, and $1 \le i \le L$.

The authors in [3] then generalized the Mallat/Meyer definition of a multiresolution analysis not only to allow operators on an abstract Hilbert space, but also to replace the requirement of an orthonormal basis of translates in the core subspace by that of the core subspace being invariant under the action of Γ. They called this structure simply a *multiresolution*; it was later given the name *generalized multiresolution analysis* in [4], which is the name we will use in this book. We will follow the convention of reserving the name *multiresolution analysis* for such a structure that also has a scaling vector as defined below.

Definition 1.2 A generalized multiresolution analysis (GMRA) of a Hilbert space H, relative to Γ and δ, is a collection $\{V_j\}_{j=-\infty}^{\infty}$ of closed subspaces of H that satisfy:

1. $V_j \subseteq V_{j+1}$ for all j.
2. $\delta(V_j) = V_{j+1}$ for all j.
3. $\cup V_j$ is dense in H and $\cap V_j = \{0\}$.
4. V_0 (called the *core subspace*) is invariant under the action of Γ.

A multiresolution analysis (MRA) is a collection $\{V_j\}_{j=-\infty}^{\infty}$ of closed subspaces of H that satisfy conditions 1–3 above, together with

4′. There exists a scaling vector $\phi \in V_0$ such that $\{\gamma\phi : \gamma \in \Gamma\}$ gives an orthonormal basis for V_0.

Condition (4) allows the application of the theory of group representations to prove theorems about the inter-relationship between wavelets and multiresolution structures. In [3], the authors showed that all (orthonormal) wavelets have associated GMRAs. This makes GMRAs a natural tool to pair with wavelets. While they found an example of a GMRA that can have no associated orthonormal wavelet, they also used an analysis of the representation of the group Γ on V_0 to show that all abstract MRAs for groups Γ where $\delta^{-1}\Gamma\delta$ is of finite index d in Γ do have associated wavelets, and established that the number L of wavelet functions must equal $d - 1$.

A parallel line of research was begun in 1994 by de Boor, DeVore, and Ron [22], with the study of shift invariant spaces, spaces invariant under integer translations. They exploited the concept of a range function, introduced by Helson [29], to determine conditions under which a finitely generated shift invariant space has a generating set that is orthogonal or stable. Ron and Shen [44] later introduced techniques involving Gramian fibers, functions on $\mathbb{R}^N$ whose values are nonnegative Hermitian matrices with coefficients that are inner products that sum over translates of L^2 functions. They also used what they called quasi-affine systems to overcome the fact that sets of positive dilates of wavelets are not shift invariant [45]. We will point out connections to this parallel approach throughout the book.

In the late 1990s and early 2000s, other researchers proposed different generalizations of Mallat and Meyer's multiresolution analysis that were based on the concept of a frame.

Definition 1.3 A frame for a Hilbert space H is a collection of vectors $\{v_1, v_2, \cdots\}$ for which there exist positive frame bounds A and B with $A\|v\|^2 \leq \sum_i |\langle v, v_i\rangle|^2 \leq B\|v\|^2$ for all $v \in H$. A Parseval frame (or normalized tight frame) is a frame with bounds $A = B = 1$.