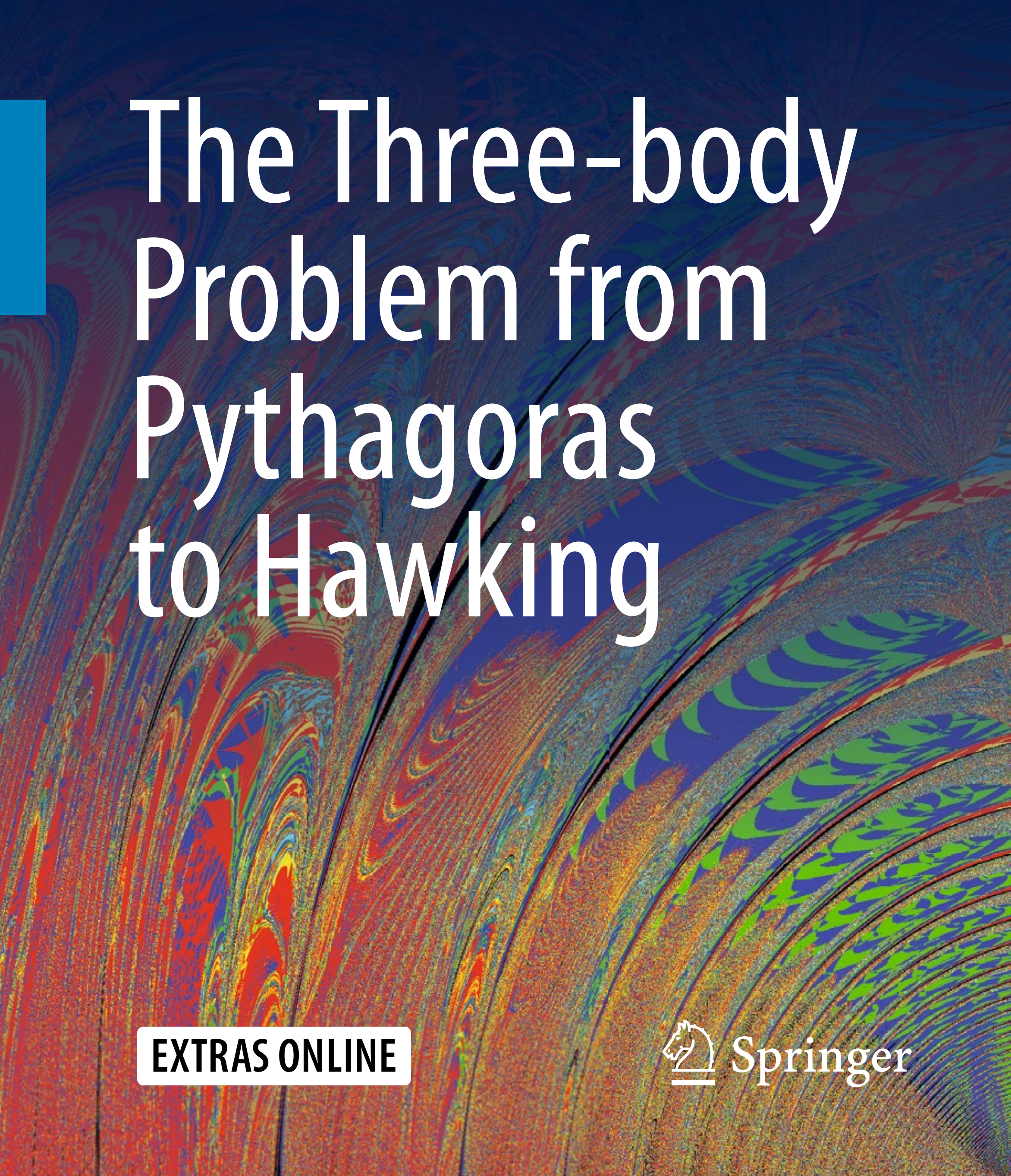
Mauri Valtonen · Joanna Anosova
Konstantin Kholshevnikov
Aleksandr Mylläri · Victor Orlov
Kiyotaka Tanikawa

The Three-body Problem from Pythagoras to Hawking



EXTRAS ONLINE

Springer

The Three-body Problem from Pythagoras to Hawking

Mauri Valtonen • Joanna Anosova •
Konstantin Kholshevnikov • Aleksandr Mylläri •
Victor Orlov • Kiyotaka Tanikawa

The Three-body Problem from Pythagoras to Hawking

Springer

Mauri Valtonen
University of Turku
Turku, Finland

Joanna Anosova
University of Texas
Austin, Texas, USA

Konstantin Kholshevnikov
St.Petersburg State University
St.Petersburg, Russia

Aleksandr Mylläri
St.George's University
St.George's, Grenada, West Indies

Victor Orlov
St.Petersburg State University
St.Petersburg, Russia

Kiyotaka Tanikawa
National Astronomical Observatory
Tokyo, Japan

Additional material to this book can be downloaded from http://extras.springer.com.

ISBN 978-3-319-22725-2 ISBN 978-3-319-22726-9 (eBook)
DOI 10.1007/978-3-319-22726-9

Library of Congress Control Number: 2016935696

Cover illustration: Chaos in the three-body problem. This illustrates the sensitivity of the three-body problem to the exact initial configuration, represented by the position in the plane. Color corresponds to different geometric configurations after some three-body evolution. A small initial difference leads to very different orbital paths.

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

Writing a popular account of the three-body problem is a special challenge. The problem is as old as science in general, and contributions towards its solution have been made by an untold number of scientists. Yet we are not yet at the stage where we could declare that the problem has been solved. Another challenge is to try to convey some of the excitement of this problem to the general reader without the use of mathematics. For a problem which is studied in mathematics departments of many universities, it means, by necessity, major simplifications and often appealing to applications in the field of astronomy. Astronomical systems may be easier to visualize than purely mathematical constructions.

We have taken the historical approach. The three-body problem, the description of the motion of three celestial bodies under the action of their mutual gravitational pull, was first studied by Isaac Newton. In Chap. 1, we give a brief history of the problem prior to Newton and only to the extent that is relevant to Newton's work. There is much astronomical and mathematical science before Newton that we are not able to describe here. Some additions to the historical background come in Chap. 7 after we have learnt concepts that are important to the problem, such as the idea of chaos.

Newton's law of gravity is accurate enough for most astronomical calculations. However, the more accurate Einstein's law of gravity is necessary in many modern applications. In fact, the need to improve Newton's law became apparent only in the late nineteenth century, when it was realized that planet Mercury did not behave as expected by the solution of the three-body problem in Newton's theory. The last chapter describes more drastic changes to Newton's law, such as the laws governing black holes. They cannot be understood without Einstein's General Relativity, as his law of gravity is called.

Chapter 3 follows some steps in the evolution of the three-body problem. It includes, among others, the famous pre-Nobel competition for finding the answer and describes Poincaré and Sundman as leaders of two schools of thought on the nature of the solution. For Poincaré it was statistical at best, while Sundman claimed a fully deterministic solution. Both lines of enquiry have correspondence in the current work. Poincaré's solution leads to chaos theory, and further to enquiries about the nature of time, the subject of the fourth chapter.

In the next two chapters, we meet astronomical applications on two different scales, the Solar System and galaxies. Both of these systems have more than three bodies, more than three by far. However, it is possible to get first-order estimates of many processes using just three bodies in the calculations, as numerous scientists have demonstrated over the years. For example, two galaxies may be understood as two rigid bodies which provide the variable gravitational field for the study of the motion of a third body, such as a star. Repeating the process for many stars gives us an overview of how galaxies made of billions of stars may change their shape and other properties.

After the first round review, we take a more detailed look at the steps involved in the history of the three-body problem. It includes new frontiers and some of the recent results. Among frontiers are the systems involving black holes which are found in the final chapter. It takes us straight to the current efforts to prove black hole theorems. That is, we try to verify the concept of black holes that is derived from General Relativity.

It is not clear where one should start the history of the three-body problem. Pythagoras probably understood that Earth, Moon, and the Sun are three spherical celestial bodies whose exact alignments produce eclipses, lunar and solar. But only after the introduction of the force law between them did the three-body problem in the modern sense emerge. Newton's attempts to solve the three-body problem filled a good part of his famous work *Principia*. The three-body problem of today, with Einstein's law of gravity, may be used to test the so-called no-hair theorem of black holes. The no-hair theorem was first formulated by Israel, Carter, and Hawking, and a distant quasar composed of two black holes and a cloud of gas is the system currently under study. This represents an enormous range of scale. At the lower end we have the mass of the Sun and at the upper end more than ten billion suns. The objects of study can be near to us, like the Sun about 8 light minutes away, or 3.5 billion light years away in the case of the binary black hole system OJ287.

Vladimir Titov from St. Petersburg State University (Russia) has prepared animations illustrating choreographies in the three-body systems. These animations and other add-on materials can be found on the book web page http://extras.springer.com.

We would like to thank Sverre Aarseth, Nick Ourusoff, and Renate and Henrik Appelqvist for reading the manuscript and for helpful comments. We thank also Laura Garbolino, Heike Hartmann, Göran Östlin, Mark Hurn, Kathryn Shaw, Alan Harris, Annika Augustsson, Jaana Tegelberg, Sylvio Ferraz Mello, Joerg Waldvogel, Arthur Chernin, Gene Byrd, Harry Lehto and Martin Gutknecht for help with illustrations.

transform, or build upon this figure or a part thereof, you must distribute your work under the same license as the original. Figure captions give bibliographic data of the source with name(s) of the originator(s); it may also be an http address.

We hope that this small review will stimulate interest in the reader, and for those with mathematical knowledge, further enquiries to the mysteries of the three-body problem.

Turku, Finland	Mauri Valtonen
Austin, TX	Joanna Anosova
St. Petersburg, Russia	Konstantin Kholshevnikov
Grenada, West Indies	Aleksandr Mylläri
St. Petersburg, Russia	Victor Orlov
Tokyo, Japan	Kiyotaka Tanikawa
September 2015	

Contents

Chapter 1
Classical Problems

Impossible Problems to Solve

In the history of mathematics there are a number of problems which have agitated the imagination of the greatest minds for centuries. Three problems proved elusive: the problem of squaring the circle,[1] the doubling the cube,[2] and the trisecting the angle.[3] The problems are to be solved purely by using a compass and an unmarked ruler, a straightedge. Ferdinand von Lindemann proved in 1882 that the first problem has no solution while Pierre Wantzel showed in 1837 that the solutions of the latter two problems are also impossible.

Another problem of the same category is the three-body problem. It is as old as the other three. It deals with the motions of three celestial bodies such as the Earth, the Sun and the Moon. The solution is required, for example, to predict solar eclipses. At the time of the eclipse, the Moon moves in front of the Sun, and blocks the sunlight, causing darkness lasting about 6 min. The three celestial bodies are then lined up in the order of Earth-Moon-Sun. The solar eclipse is visible only on a narrow strip of the Earth's surface, and thus it is quite possible that an individual never sees a solar eclipse in his or her life. But when it happens, it is an awe-inspiring experience, and we can only imagine what a terrific effect it has had on ancient people. A lunar eclipse is much more common. There the Moon drifts into the shadow of the Earth, and so cannot receive the sunlight. It may be observed from the whole hemisphere where the Moon is visible at that time (see Fig. 1.1).

[1] Squaring of the circle refers to finding the area of a circle of a given radius. In modern terms, it is the question of finding the exact value of π.

[2] The problem of doubling the cube, also called the Delian problem, is to find the length of the side of a cube which makes its volume twice as big as the original cubic volume. In modern terms, it amounts to the determination of $\sqrt[3]{2}$.

[3] Dividing a given angle in three equal parts using only a compass and a ruler.

M. Valtonen et al., *The Three-body Problem from Pythagoras to Hawking*,
DOI 10.1007/978-3-319-22726-9_1

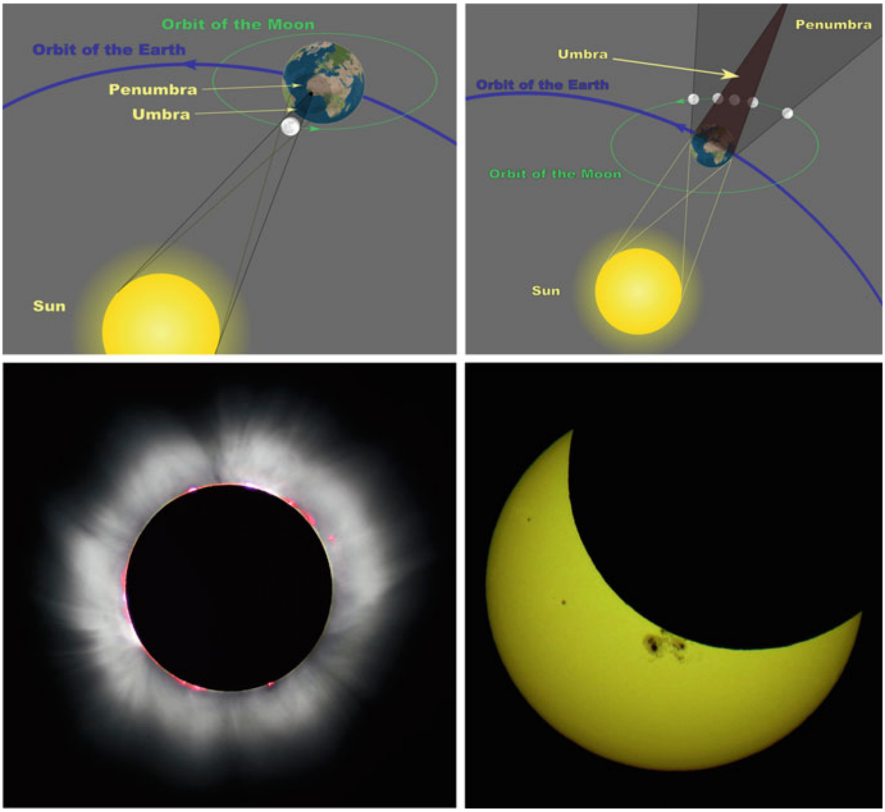


Fig. 1.1 The positions of the Sun, the Moon and the Earth during a solar eclipse (*upper left*) and a lunar eclipse (*upper right*). The bodies are not drawn in scale. During the solar eclipse, the observer on Earth is in the full shadow of the Moon called umbra. The partial shadow is called penumbra. It is wider than the full shadow, and therefore a partial solar eclipse (*lower right*) is seen over a wider geographic region than the total solar eclipse. In the total solar eclipse, for a few minutes the Sun is completely covered by the Moon, and only the fainter outer layers of the Sun are seen outside the Moon's limb (*lower left*) (Credit: Wikipedia Creative commons and (*lower left*) Luc Viatour/www.Lucnix.be)

The failure to predict eclipses has cost lives. According to a legend, the Imperial Chinese astronomers He and Xi failed to predict a solar eclipse (perhaps on October 22, 2134 BC), and were beheaded for it. The health and success of emperors were thought to depend on anticipating the eclipses, and the astronomers had put their ruler in danger.[4]

[4] According to the legend, the emperor Chung Kang relied on his astronomers to track and interpret heavenly motions. It was a serious job. Eclipses were believed to be caused by a dragon eating the Sun, and were bad omens for the emperor. The monster had to be frightened away with drums, gongs and arrows fired into the sky. When two state astronomers, He and Xi got drunk and failed to

But a solar eclipse may also lead to a happy ending. According to Greek historian Herodotus, a solar eclipse (probably in the late afternoon of May 28, 585 BC) happened during the war between the Medes and the Lydians. The Median king Cyaxares ruled the present day Iran and Eastern Turkey while the Lydian king ruled the Western part of the present Turkish territory, neighboring the coastal Ionian towns settled by the Greek. After 5 years of undecided war yet another battle ensued near the Halys river. When the darkness suddenly came, both sides laid down their weapons and stopped the fight, as the gods had spoken and warned the kings by the removal of daylight. The new border between the two kingdoms was agreed at Halys river, and to seal the peace, the Princess of Lydia married the Prince of the Media.

Herodotus and other ancient sources (among them the generally reliable Eudemus of Rhodes, the author of History of Astronomy) say that this solar eclipse was predicted by the Ionian astronomer Thales of Miletus (c. 621 BC–c. 546 BC) and he made a public announcement of it in advance to the Ionians.[5]

The successful prediction of eclipses requires the solution of the three-body problem. Had He and Xi been negligent in their calculation and Thales been more careful? No, there was no general method for solving the problem at the time. We do not have solid evidence that the problem had even been stated correctly in those days.

Whether his prediction of the solar eclipse is fact or fiction, Thales may still have been the first scientist to understand the basic causes of eclipses and that they involve three celestial bodies. The next step in understanding the problem is probably due to Pythagoras of Samos (ca. 572–497 BC), a student of Thales, and

---

predict an eclipse, the emperor had no time to prepare a response. Although the Sun apparently survived the dragon's attack, the pair were beheaded.

[5] Thales may have travelled to Babylonia in his youth and gained access to the extensive records of astronomical observations which dated from the time of the ruler Nabonassar (747 BC). By that time the Babylonians, just as Chinese in their own quarters, and many others, had been recording celestial events for several 1000 years. These records formed the basis for predicting lunar eclipses, and to some extent, solar eclipses. The methods may have been already known before 585 BC, even though written evidence for this knowledge has survived only from later centuries.

After centuries of continuous monitoring of celestial events, a period of 18 years and 10–11 days (called the Saros cycle) was discovered in lunar eclipses, after which similar eclipses start to repeat themselves. Another shorter cycle is 47 months long. Thales may have witnessed, or at least heard of, a nearly total solar eclipse in Babylonia on May 18th, 603 BC. If he suspected that also solar eclipses follow the Saros cycle, he could have predicted a solar eclipse on May 28, 585 BC. Alternatively, he may have known that 23.5 months after a lunar eclipse a solar eclipse has a high probability. This period is exactly one half of the 47 month lunar eclipse cycle, and he must have understood that the opposite alignment of the Earth-Moon-Sun happens half-way through this cycle. He most likely observed the July 4, 587 BC lunar eclipse which would have lead to the same predicted date. Perhaps he knew of both methods which gave him confidence. Anyway, he and the warriors were lucky in that the Halys river battle happened to be on the narrow strip, about 270 km wide, where the eclipse was total. A more common occurrence of a partial eclipse where the Sun is only partly covered by the Moon, is seen over a wider region, but it is not such an eerie and chilling experience as the total eclipse.

his followers called Pythagoreans. Pythagoreans believed that the Earth and the Sun and the Moon were spherical bodies, all in motion in space. When they occasionally line up, eclipses appear. This was clearly a step forward in the formulation of the three-body problem; it remained to demonstrate how the Babylonian records of eclipses are explained in this system, but that was far beyond the powers of the Pythagoreans.

The first recorded explanation of the eclipses was given by Anaxagoras of Smyrna. He has been given the honor of having brought the new scientific ideas of Ionian towns to Athens. He claimed that the Sun is a hot rock and that Moon, also a rock, is illuminated by reflected light from the Sun. He maintained, quite correctly, that in a solar eclipse the Moon goes between the Earth and the Sun, and that in a lunar eclipse the Moon is in the shadow of the Earth. All this was too much for the religious Athenians for whom the Sun was a God. Anaxagoras nearly ended up with a long prison sentence, but was saved by his influential friend Pericles, who spoke for him in the trial. While waiting for his trial Anaxagoras started working on the problem of squaring the circle, the first Greek scientist known to have attacked this problem. Finally he was freed but was forced to return to Ionia.

Therefore, who made the first statement of the three-body problem is not exactly known, but if we put it to Pythagoras, we are not far off. He definitely gave the insight that mathematics is needed to solve the problem, as we will learn below.

In the current form the three-body problem was first formulated by the father of modern science, the Englishman Isaac Newton (1642–1727) in Cambridge. The problem is to determine the relative motion of three material points interacting under the Newton's law of universal gravitation.[6] A material point in mechanics refers to a body whose size and rotation may be neglected, as not influencing the mutual attractions. Newton gave the problem in his famous treatise *Mathematical Principles of Natural Philosophy* (*Philosophie Naturalis Principia Mathematica* in Latin) in 1687. Despite the simplicity of the formulation, the best mathematical minds could not find an acceptable solution in the general case.

Nevertheless, centuries of efforts by many outstanding mathematicians were not in vain. In the latter part of eighteenth century two partial solutions to the problem were discovered. The Swiss Leonhard Euler (1707–1783) while working in Berlin in 1763 found that three bodies can always be on a rotating straight line and published this result in Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae in St. Petersburg. Italian/French Joseph Louis Lagrange (1736–1813) while working in Paris in 1772 found that three bodies can always be at the vertices of a rotating equilateral triangle. They were the two most outstanding mathematicians of the time. Figure 1.2 shows the trajectories of motion in these two cases when the masses of all bodies are equal, but solutions of the Euler and Lagrange cases exist for any mass values.

---

[6] $F = Gm_1m_2/r^2$ where F is the gravitational force of attraction between two bodies of masses m_1 and m_2, separated by distance r from each other. G is the universal gravitational constant.

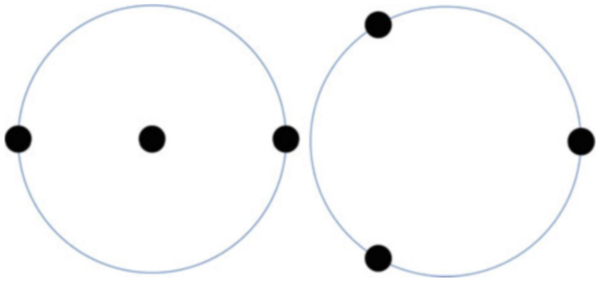


Fig. 1.2 Three equal bodies on a straight line (*left*) and in the corners of an equilateral triangle (*right*). The lines trace the orbital paths of these stable triple systems which could be for example three stars in space. The first one was discovered mathematically by Leonhard Euler, the second one by Joseph Louis Lagrange

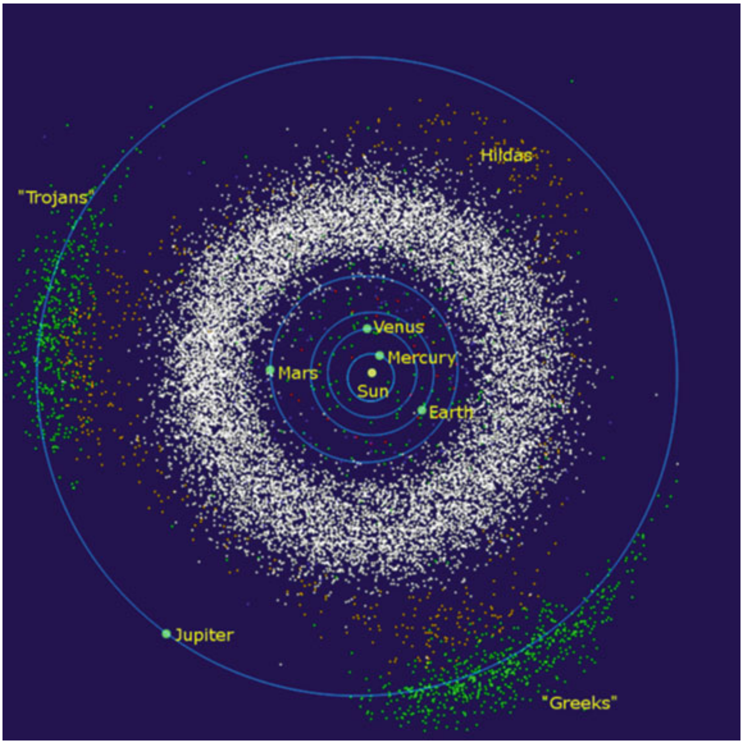


Fig. 1.3 Trojans and Greeks are two large groups of objects that share the orbit of the planet Jupiter around the Sun. Relative to Jupiter, each asteroid belonging to these groups is close to one of Jupiter's two stable Lagrangian points lying 60° ahead of the planet in its orbit, and 60° behind (Credit: Wikimedia Commons)

In the Solar System we observe two large groups of asteroids that are placed close to the points of the Lagrange solutions in the system Sun-Jupiter-asteroid. They are called Trojans and Greeks (all big asteroids in each group are named after heroes of the Trojan War) (Fig. 1.3).

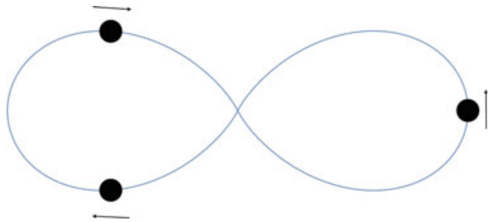


Fig. 1.4 Three equal bodies chase each other in a stable orbit shaped like figure eight. It was discovered mathematically by Cristopher Moore. Such an orbit is possible for three equal stars, but a system like this has never been seen nature

For a long time this was all we had. Then in year 1993 the American mathematician Cristopher Moore discovered the figure eight ("8") orbit when the three bodies are equal, by using computer calculation, and subsequently it was proven rigorously correct by two mathematicians, French Alain Chenciner and American Robert Montgomery. In this case the three bodies move along a closed curve shaped like figure eight (Fig. 1.4). This solution is periodic, i.e., after a certain amount of time called the period, the three bodies take up the same positions and have the same velocities as at the initial moment of time. It is clear that this is true also after two periods, three periods and so on. The two previously known three-body solutions also have the same periodic property.[7]

The triple system with periodic movements reminds us of an ideal pendulum that oscillates forever. Of course, in reality it is impossible to create an eternal pendulum because of frictional forces, but it can be described mathematically. The motion of the mathematical pendulum has an exact solution; it performs so called harmonic oscillations.

In 1912 the Finnish mathematician Karl Sundman (1873–1949) in Helsinki showed how to construct the solution of the general three-body problem using a series of numbers called terms, to be added to each other, with the total number of terms running into infinity. In practice, one may add only a finite number of terms and hope for the best. However, in 1930 the French astronomer David Belorizky in Paris showed that the finite number has to be as big as the power 80,000 of ten in order to solve the Lagrange's triangular three body problem with the usual accuracy of astronomical observations, and just for one sixth of the period in time. This number of terms is insanely large; the number 1 is followed by 80,000 zeros. In comparison, the total number of atoms in the observable universe is "only" 1 followed by 80 zeros. Even if we use this enormous number of terms, we still do not have an exact solution. Thus we could say that the work of Sundman and Belorizky has proven the impossibility of solving the three-body problem, except for the special situations illustrated above.

[7] Two of the authors (JA and VO) came close to the discovery of the figure "8" stable orbit a decade before Moore. They would have had to pursue the orbit longer to prove the case which was not yet possible.

The proof that a problem does not have an exact mathematical solution does not signify that it is meaningless or that there does not exist ways to get the answers by less exact means. This is true of the three "classical" mathematical problems. Already in Antiquity there were methods of squaring of a circle (i.e., calculating the area of a circle), doubling a cube (i.e., finding the side of a cube which has twice the volume of the original cube) or dividing an angle in three equal parts. But they were not mathematically exact, in the sense defined by the ancient Greek geometers.

The story of the origin of the problem of doubling the cube, called the Delian problem, according to Eratosthenes goes as follows.

> God proclaimed to the Delians through the oracle that, in order to get rid of a plague, they should construct an altar double that of the existing one. Their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.

Whether this is true or not is an open question, but definitely there was a plague in Athens around 430 BC that killed about a quarter of the population. An early attempt of the solution of the problem around this time was made by Hippocrates; thus it is at least possible that the doubling of the cube problem arouse in this way.

In the same manner we may ask if the three-body problem has put the modern scientists in shame. Perhaps not. Let us move our sights to modern observers, aiming their telescopes to a star-like point in the sky called OJ287. It is too faint to be seen by naked eye, and records of it start only in 1891 when the photography of the sky had recently begun. What these records show is peculiar flaring of light, often more than doubling the brightness from one night to another. The explanation for this peculiarity came from the solution of a three-body problem, a system consisting of two black holes and a cloud of gas which orbit each other. The system is far away, so far that it takes light signals 3.45 billion years to come to our telescopes (light travels from the Moon to us in just over 1 s). Therefore we cannot watch the motions directly, the three bodies are too close to each other in the sky to be seen separately, but from the solution of the three-body problem we may calculate that from time to time the smaller black hole collides with the cloud of gas and makes it radiate with huge brightness.

Just like the three-body problem may be used to predict the exact time of a solar eclipse (we will come to details in later chapters), the three-body solution predicts the times of the big flares in OJ287. They are not so common; only two big flares occur in 12 years. The latest predicted flare was to take place on September 13, 2007, and it was to happen just before sunrise. After sunrise, and even an hour before it, the Sun overpowers any faint light from the stars, thus the measurement of OJ287 had to be made quickly, right after it had risen above the horizon but when the Sun was still below the horizon. It was a difficult task, and required a coordinated effort of astronomers in Japan, China, Turkey, Greece, Bulgaria, Poland, Finland, Germany, Great Britain and Spain. Why astronomers in all these countries were needed was because each astronomer has his/her own sunrise, first in