Logic, Language, and the World

Volume I

The Internal Structure of Predicates and Names

Richard L. Epstein

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Advanced Reasoning Forum

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The Internal Structure of Predicates and Names

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Much of Chapter 43 on non-referring simple names appeared in Chapter XVI of my *Classical Mathematical Logic*.

Introduction

Modern formal logic surpasses in rigor, depth, and scope all that had previously been done in logic, it's said now.

But the limitations of modern formal logic are considerable and give the lie to that evaluation. The analysis of arguments, explanations, causal reasoning, and prescriptive reasoning—all part of the traditional scope of logic—depend on evaluating the strength of inferences, which is beyond modern formal methods that are concerned with only validity. Analyses of reasoning that depends on talk about mass or process lie outside the scope of modern formal logic, which is based on the assumption that the world is made up of things.

Even within the limits of the metaphysics and focus of modern formal logic, there is much that lies outside the formal methods that have been developed. This series of volumes is meant to extend the scope of what we can formalize, and in doing so see the real limitations of what can be done. Logic is presented as a tool to investigate the world through the medium and limitations of our language.

In the first section here I set out the standard of modern formal logic: classical predicate logic with equality. This is only a sketch, drawing on the full development in *An Introduction to Formal Logic*, which I refer to as Volume 0 of this series. Those familiar with that book need read only Chapter 3, the aside on the language of predicate logic on p. 19, and Chapter 8.

In the second section I show how we can extend classical predicate logic to formalize reasoning that involves adverbs and relative adjectives by viewing those as modifiers of simpler predicates. What we previously took to be atomic predicates, such as "barking loudly", can then have internal structure. Reasoning that involves conjunctions of terms, as in "Tom and Dick lifted the table", conjunctions of modifiers, conjunctions of predicates, and disjunctions of predicates can also be formalized by viewing them as part of the internal structure of atomic predicates.

The internal structure of names is the topic of the third and last section. Names for functions are used in classical predicate logic to form complex names, such as "sin (x^2)", which is what I present first. In our ordinary reasoning we use descriptions to form functions, such as "the wife of", and we use descriptions to form names, such as "the cat that scratched Zoe". To reason with those we need to take account of their internal structure, which we can do if we drop the assumption, basic to classical predicate logic, that every name must refer to a specific thing. Then we can devise formal logics as a guide to reasoning with simple, atomic names that do not refer.

The formal systems that are developed here are not just formalisms but are meant to help us understand how to reason well. Many worked examples show how we can use them. They also uncover limitations of the formal work. The analyses in the examples are tentative, presented with the hope of stimulating you to deeper and clearer analyses.

The work here proceeds by abstracting and creating formal models to formalize reasoning. By paying attention to the process of abstracting we gain insight into why we consider some reasoning to be good and some reasoning bad, and insight also into the deeper assumptions we make about the world on which our judgments rely. Questions about the metaphysics we assume for modern formal logic and the nature of formalizing have to be faced, most particularly the assumption that the world is made up of objects that we can name. Again, I can present only tentative answers, and often I can only pose a question about the relation of logic to language and how we use both to investigate the world. Some discussions are supplemented by my other work, and books and papers that I cite without attribution are mine. In the end I cannot tell you what is a thing. I cannot tell you what pointing and naming are. But by trying to clarify my ideas about things and naming I hope to lead you to clarify your ideas, to help us find a common basis on which to build and use modern formal logic.

The second volume of this series, *Time and Space*, is about how to extend modern formal logic to formalize reasoning that takes account of time and space; an overview of it is presented in my "Reflections on Temporal and Modal Logic". That volume shows in example after example that reasoning about process, mass, and change is outside the scope of modern formal logic not because of our lack of inventiveness in devising better formal systems but because the metaphysics of viewing the world as made up of things cannot encompass a conception of the world as made up of mass or process. The third volume, *The World as Process*, shows how we can talk and reason about the world as process, not just informally but with a rigorous formal system; an overview of that is presented in my "The World as Process".

Others use logic as a bulwark against the mysteries. They build a wall within which reason reigns and live within the cities built of logic. I use logic as a way into the mysteries, using reason where I can to lead me to the boundary beyond which reason has no sway if we are to enter. Logic is the path, not the end. There is no end but only a continual beginning.

Background

1 Propositions, Inferences, and Formal Logic

We would like rules to guide us in finding truths and, when that is not straightforward, to determine what follows from given assumptions: if this were true, then that would follow. To do this, we must agree on what it is that is true or false.

Proposition A *proposition* is a written or uttered part of speech used in such a way that it is true or false, but not both.

By "uttered" I include silent uttering to oneself, what we might call thinking of the sentence.[1]

Some say that propositions are abstract objects or thoughts that can be shared by all people and that what I have defined here are physical linguistic representatives of propositions. But those who hold such views reason using linguistic propositions, which can serve as a common basis on which to begin our work, as I discuss in Appendix 1.

Typically, we identify equiform words for our reasoning, where what we deem to be equiform depends on the uses we are making of the words. Similarly, we identify equiform propositions, though for those it is more difficult to be clear about what is or is not important for reasoning.

Words and propositions are types Throughout any particular discussion equiform words can be treated as the same for our reasoning. We identify them and treat them as if they were the same word. Equiform propositions, too, will be identified and treated as the same for our reasoning. Briefly, *a word or a proposition is a type*.

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[1] In my previous work I defined a proposition to be a written or uttered declarative sentence that we agree to view as true or false, explaining that agreements need not be explicit (see *Propositional Logics* and *Predicate Logic*). I now think it is better to talk about how we use parts of speech. Also, to define "proposition" in terms of the notion of a declarative sentence is bad for two reasons. First, it is circular, for we typically define a declarative sentence to be one that is a proposition or is true or false. For example, Edward Sapir says:

> The sentence has, like the word, a psychological as well as a merely logical or abstracted existence. Its definition is not difficult. It is the linguistic expression of a proposition.
>
> *Language*, Chapter 2, paragraph 12

Second, we will consider parts of speech in artificial languages that we will want to treat as true or false, and, though they may formalize declarative sentences, they are not what we could normally call declarative sentences. Compare what Jean Buridan says in his *Summulae de Dialectica*, 9.6, Third Sophism:

> Even a barrel hoop hanging in front of a tavern is a proposition, "for it is equivalent in its signification to the conventionally signficative utterance that someone might yell at the entrance of the tavern: 'Wine is sold here!' "

Some say that types are abstract objects, in accord with their belief that propositions are abstract. In that case the assumption that words and propositions are types concerns which inscriptions and utterances (which is what we actually use in our reasoning) represent or express or point to the same abstract thing.

A proposition is true or false. But what does that mean? That is a big question which will occupy us throughout this book. For now, I will assume only that we have enough idea of what it means for a proposition to be true for us to begin our studies.

Propositions are true or false. An inference is what we use to say that one proposition follows from one or more others.

Inference An *inference* is a collection of two or more propositions—one of which is designated the *conclusion* and the others the *premises*—that is intended by the person who sets it out as either showing that the conclusion follows from the premises or investigating whether that is the case.

Some say that inferences, too, are abstract things. But all who reason use linguistic inferences, and it is those we can study whether or not we consider them to be representatives of abstract entities.[2]

When does an inference show that the conclusion follows from the premises? That depends in part on what kind of reasoning we are analyzing. Different conditions apply depending on whether we are concerned with arguments, explanations, mathematical reasoning, reasoning about cause and effect, or conditionals.[3] In our work here, we will consider an inference good only if it is valid.

Valid inferences An inference is *valid* means that there is no way the world could be in which the premises are true and conclusion false at the same time.

For example, the following is valid:

(1) Ralph is a dog.
All dogs bark.
Therefore, Ralph barks.

I can't prove that to you. At best I rephrase it in other words. If you understand English, it's clear that it's valid.

Similar inferences are also valid:

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2 Intent is crucial in determining whether what has been uttered is an inference, as can be seen in hundreds of examples in *Critical Thinking*. The examples of inferences in this book should be understood as prefaced by "imagine that someone has put forward the following inference".

3 See *Reasoning in Science and Mathematics*, *The Fundamentals of Argument Analysis*, and *Cause and Effect*, *Conditionals*, *Explanations*.

Dick is a student.
All students study hard.
Therefore, Dick studies hard.

Suzy is a cheerleader.
All cheerleaders have a liver.
Therefore, Suzy has a liver.

It would facilitate our reasoning if we could clarify in what way these inferences are similar, for we have the intuition that we don't need to know anything about cheerleaders, or students, or dogs to see that they are valid. Somehow, it is the forms of the propositions in these inferences that matter.

Formal logic *Formal logic* is the analysis of inferences for validity in terms of the structure of the propositions appearing in the inference and the analysis of propositions for truth in terms of their structure.

Now consider:

Fido is a dog
Therefore, Fido barks.

To show that this is not valid, we show that there is a way the world could be in which the premise is true and conclusion false: Fido could be a basenji, a kind of dog that can't bark.

To invoke a way the world could be, a *possibility*, in the evaluation of an inference, we use a description when we wish to reason together. A description of the world is a collection of claims: we suppose that this, and that, and this are true. We do not require that we give a complete description of the world, for no one is capable of presenting such a description nor would anyone be capable of understanding one if presented. By using collections of claims to describe or to stand in for possibilities, we need not commit ourselves to a possibility being something real, such as a world in which I am not bald.

But what qualifies some collections of claims as describing a possibility and others not? Regardless of how we conceive of possibilities, we always seem to agree that a description of a way the world could be must be consistent. That is, it cannot have or entail a contradiction. It must be *logically possible*. So there is no way the world could be in which there is a square circle. But there seems to be no contradiction inherent in postulating that a dog could give birth to a donkey: it is logically possible.[4]

[4] Some logicians have formulated how to reason when the information we have is or might be inconsistent. A few have argued that contradictions, such as there being square circles, are possible. But such an assumption is not needed for reasoning around contradictions, as I show in "Paraconsistent Logics with Simple Semantics", and it would leave us with no semantic basis from which to start our analysis of possibilities, as I discuss in "Truth and Reasoning".

So a collection of claims describes a way the world could be if it neither contains nor entails a contradiction. Yet that requires knowing what it means for a collection of claims to entail another claim, which is what we are trying to understand. We find ourselves in a circle.

One way to extricate ourselves from this circle is to investigate parts of our reasoning, picking out just this or that kind of reasoning relative to restricted semantic and syntactic assumptions that allow for clarity of analysis, developing a formal logic. Then we can have a clearer notion of possibility and of valid inference for that kind of reasoning. As we extend our investigations to allow for more kinds of reasoning, we will have fuller analyses of logical possibilities and valid inferences.

2 Classical Propositional Logic

The simplest formal logics are *propositional logics* in which the only structure of propositions we consider is how they are formed by combination from other propositions. Traditionally, we confine our interest to four ways of combining or making propositions from others using the *connectives* "and", "or", "not", and "if . . . then . . .". We adopt symbols for our abstractions of the ordinary language connectives:

$\neg$ *negation* for "not"

$\vee$ *disjunction* for "or"

$\wedge$ *conjunction* for "and"

$\rightarrow$ *the conditional* for "if . . . then . . ."

These are *formal connectives*. So we might write "Juney is a dog $\wedge$ Juney barks" in place of "Juney is a dog and Juney barks". We might write "Tom sang $\rightarrow$ Dick played the piano" in place of "If Tom sang, then Dick played the piano". These phrases with formal connectives are not propositions until we say how we will understand these symbols.

To start, we need to set out how we can form new propositions from given ones using formal connectives. By defining a *formal language* we can give the structure of all the propositions we can form from any propositions we start with. First we take *propositional symbols* or *variables*, p_0 , p_1 , . . . that can stand for any propositions, though the intention is that they'll stand for ones that don't contain a formal connective or word that we would formalize as a formal connective. We then need *metavariables* to stand for any of the propositional symbols or complex expressions we'll form from those; we'll use $A, B, C, A_0, A_1, A_2, \ldots$. The analogue of a sentence in English is a *well-formed formula* (*wff*), which we define inductively.

Wffs and the formal language $L(p_0, p_1, \ldots, \neg, \rightarrow, \wedge, \vee)$

Vocabulary

propositional variables $p_0 \;\; p_1 \;\; \ldots$

connectives $\neg \;\; \rightarrow \;\; \wedge \;\; \vee$

parentheses) (

Well-formed formulas (*wffs*)

Each of (p_0), (p_1), (p_2), (p_3), . . . is a wff of *length* 1.

If A is a wff of length n, then $(\neg A)$ is a wff of length $n + 1$.

If A and B are wffs and the maximum of their lengths is n, then $(A \rightarrow B)$, $(A \wedge B)$, and $(A \vee B)$ are wffs of length $n + 1$.

A concatenation of symbols of the vocabulary is a *wff* just in case it is a wff of length n for some $n \geq 1$.

Wffs of length 1 are *atomic*; all others are *compound*.

No formal wff such as "$p_0 \wedge \neg p_1$" is a proposition. Only when we agree on how we understand the formal connectives and then assign propositions to the variables, such as "p_0" stands for "Ralph is a dog" and "p_1" stands for "Four cats are sitting in a tree," do we have a formula "Ralph is a dog $\wedge$ $\neg$ (four cats are sitting in a tree)" that can be true or false.

Realizations and semi-formal languages A *realization* is an assignment of propositions to some or all of the propositional symbols. The *realization of a formal wff* is the formula we get when we replace the propositional symbols appearing in the formal wff with the propositions assigned to them; it is a *semi-formal wff*. The *semi-formal language* given by a realization is the collection of realizations of the formal wffs.

I'll use the same metavariables A, B, C, A_0, A_1, A_2, . . . to stand for semi-formal wffs, too, and p, q to stand for *atomic propositions* that realize the propositional symbols. We abbreviate "A if and only if B" as "A iff B", which means "if A, then B; and if B then A", and we write "$A \leftrightarrow B$" for "$(A \rightarrow B) \wedge (B \rightarrow A)$".

I've been using quotation marks around parts of speech or formal symbols to show that I'm talking about the linguistic item or symbol and not using it in the ordinary way. For example, I write "Dick" to indicate that I'm talking about the word and not using the word to refer to someone as when I write: Dick is a student. In what follows I'll often write formal symbols without quotation marks when it's clear I'm talking about the formal symbol, as when I say that $\wedge$ is a connective.

We've described the linguistic forms we'll study. This is the *syntax* of the theory of reasoning we're developing. We did so without any talk of what the formal symbols mean, other than knowing that they will be abstractions of certain English words or phrases, and also without any talk of the meanings or truth or falsity of the formulas of a semi-formal language, that is, the *semantics* of semi-formal languages. We have separated syntax from semantics, and this is what we must do if we want a simple inductive definition of the formal language that we can use in proofs about the language and semi-formal languages. Explicitly, we make the following assumption.

Form and meaningfulness What is grammatical and meaningful is determined solely by form and what primitive parts of speech are taken as meaningful. For a given semi-formal language every wff is a proposition.

There is an additional way we want to make the semantics independent of the syntax. Given a semi-formal wff, once the semantic values of the whole are determined, then any other proposition that has the same semantic values can be substituted for it in any logical analysis. That is, semantically it does not matter that "$\wedge$" appears in a formula such as "Ralph is a dog $\wedge$ dogs bark" except for how that determines the semantic values of the whole. Explicitly, we make the following assumption.

Division of form and content If two propositions have the same semantic values, they are indistinguishable in any semantic analysis regardless of their form.

What are the semantic values? We've agreed to view each proposition as having a *truth-value*, that is, as being true or false. Often we want to take into account in our reasoning other semantic values such as subject matter, or the ways in which a proposition could be known to be true, or the likelihood of a proposition being true or false, or what things a proposition refers to, or what time it is meant to be about. But here, we'll make an assumption that will lead to the simplest formal system of propositional logic.

The classical abstraction for propositional logic The only semantic aspect of a proposition that matters to our reasoning is its truth-value.

The only question then is in what way, if any, the truth-value of the whole depends on its form and the semantic values of its parts. We might abstract very little from ordinary English and say that "and" has so many different kinds of uses that there is no regular relation between the truth-values of "Ralph is a dog" and "Ralph barks" that determines the truth-value of "Ralph is a dog $\wedge$ Ralph barks". If we do that, we'll have a poor guide for how to reason. Rather, we abstract considerably by making the following assumption.

Compositionality The semantic values of the whole are determined by its form and the semantic values of its parts.

So with the classical abstraction, the truth-value of the whole is a function of the truth-values of its parts and of nothing else. Of the various ways we could interpret the formal symbols in accord with these assumptions, we adopt the following:

$\neg$ A is true iff A is false.

A $\wedge$ B is true iff both A is true and B is true.

A $\vee$ B is true iff A is true or B is true or both are true.

A $\rightarrow$ B is true iff A is false or B is true.

Note that I've used "and", "or", "not", and "if . . . then" to explain the evaluations of the formal connectives. This is not circular. We are not defining or giving meaning to "and", "or", "not", "if . . . then . . ." but to $\wedge$, $\vee$, $\neg$, $\rightarrow$. I must assume you understand English.

Suppose now that we have a realization. We agree that the sentences assigned to the propositional variables, for example, "Ralph is a dog" for p_0, are propositions, that is, each has a truth-value. It is a further agreement to say which truth-value each has. Whether an atomic proposition is true or false is not for us as logicians to decide. We assign truth-values to the atomic propositions of the formal language by any method. Then we extend those to the compound propositions of the semi-formal language by the definitions above, where I use T to stand for "true" and F for "false"

Models A *model* is a semi-formal language, a *valuation* v that assigns truth-values to the atomic propositions, and the extension of that assignment to all formulas of the semi-formal language via the classical truth-tables. If $\mathsf{v}(A) = \mathsf{T}$, A is *true in the model*, and we write $\mathsf{v} \vDash A$, read as "v validates A". If $\mathsf{v}(A) = \mathsf{F}$, A is *false* in the model, and we write $\mathsf{v} \nvDash A$, read as "v does not validate A".

Models, then, are the possibilities that classical propositional logic recognizes. For those possibilities to characterize validity, there have to be enough of them.

Sufficiency of the collection of models For any realization, any assignment of truth-values to the atomic propositions defines a model.

Valid inferences For a collection of formal wffs Γ and a formal wff A, the inference Γ *therefore* A is *valid* means that there is no model in which all the wffs in Γ are true and A is false. In that case we say that A is a *semantic consequence* of Γ, or that the pair Γ, A is a *semantic consequence*.

A semi-formal inference is valid if it is the realization of a formal inference that is valid. An inference in ordinary English is valid if there is a formalization of it on which we feel certain we'd all agree is valid.

We write $\Gamma \vDash A$ for "A is a semantic consequence of Γ", which we also read as "Γ validates A". We write $\Gamma \nvDash A$ when it's not the case that $\Gamma \vDash A$.

We can formalize "Ralph is a dog or Ralph isn't a dog" as "Ralph is a dog $\vee$ $\neg$ (Ralph is a dog)". This is true regardless of whether "Ralph is a dog" is true or "Ralph is a dog" is false. Indeed, any semi-formal proposition of the form $A \vee (\neg A)$ is true, as you can check. The form of such propositions, relative to the assumptions of classical propositional logic, guarantees their truth.

Tautologies A formal wff is a *tautology* or *valid* iff in every model its realization is evaluated as true. In that case we write $\vDash A$.

A semi-formal proposition is a tautology iff it is the realization of a wff that is a classical tautology.

A proposition in ordinary English is a tautology if there is a good formalization of it that is a tautology.

Classical propositional logic The formal language, the definitions of realization, models, tautology, and semantic consequence together comprise *classical propositional logic*.

Another way to characterize classical propositional logic is syntactically with an axiom system. Briefly, letting Γ, Σ, Δ, and subscripted versions of those stand for collections of wffs, semi-formal propositions, or ordinary language propositions, according to context, we have the following definition.

Proofs Given a collection of wffs, called the *axioms*, that are taken to be self-evidently true due to their form, a *proof* or *derivation of* B is a sequence $A_1, \ldots, A_n$ such that A_n is B and each A_i is either an axiom or is a result of applying a rule of the system to one or more of the preceding A_j's. If there is a proof of B, we say that B is a *theorem*, and we notate that as $\vdash B$.

A *proof of a proposition* B *from some wffs* Γ is a sequence $A_1, \ldots, A_n$ such that A_n is B and each A_i is an axiom, or is a wff from Γ, or is a result of applying a rule of the system to one or more of the preceding A_j's. In there is a proof of B from Γ, then we say that B is a *syntactic consequence of* Γ, which we notate as $\Gamma \vdash B$. If B is not a syntactic consequenceof A, we write $\Gamma \nvdash B$.

We can write $\dfrac{A_1, \ldots, A_n}{B}$ to mean that $A_1, \ldots, A_n \vdash B$.

Consistency, completeness, and theories

Γ is *consistent* iff for every A either $\Gamma \nvdash A$ or $\Gamma \nvdash \neg A$.

Γ is *complete* iff for every A, either $\Gamma \vdash A$ or $\Gamma \vdash \neg A$.

Γ is *a theory* iff for every A, if $\Gamma \vdash A$, then A is in Γ.

Soundness and completeness of an axiomatization

Given a syntax, a semantics, and an axiom system:

The axiomatization is *sound* means for every A, if $\vdash A$ then $\vDash A$.

The axiomatization is *complete* means for every A, $\vdash A$ iff $\vDash A$.

The axiomatization is *strongly complete* means for every Γ and A, $\Gamma \vDash A$ iff $\Gamma \vdash A$.

Here is an axiom system for classical propositional logic, where A, B, C stand for any wffs of the formal language.

Classical Propositional Logic

1. $\neg A \to (A \to B)$
2. $B \to (A \to B)$
3. $(A \to B) \to ((\neg A \to B) \to B)$
4. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
5. $A \to (B \to (A \wedge B))$
6. $(A \wedge B) \to A$
7. $(A \wedge B) \to B$
8. $A \to (A \vee B)$
9. $B \to (A \vee B)$
10. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

rule $\dfrac{A,\ A \to B}{B}$ *modus ponens*

In Appendix 7, I prove the following.

Strong completeness of the axiomatization of classical predicate logic
For any wffs Γ and any wff A, $\Gamma \vdash A$ iff $\Gamma \vDash A$.

3 Formal Theories of Reasoning Well and Limitations of Propositional Logic

Classical propositional logic is meant to serve as a guide to us in our reasoning. We begin by considering the use of certain ordinary language sentence connectives and various examples of inferences. We try to describe, by formalizing, what we see as correct ways to evaluate inferences. Then we say that these are indeed the correct ways to evaluate inferences *relative to the assumptions we have made*. If the theory disagrees with an intuition of ours about what is true or what follows from what, we either give up that intuition in the belief that the assumptions on which we based our theory are correct and that the theory formalizes those well, or we show in what way the formal theory is inadequate to deal with that kind of reasoning due to other aspects of form or meaning of propositions that are crucial in that reasoning.

Creating and evaluating models or theories of reasoning or indeed any human activity or any science can be done well only by restricting our attention to some aspects of our experience and ignoring others.[5] It is beyond our ability to take account at one time of all of experience—if that phrase even makes sense. We cannot pay attention to all we encounter in the world at any one time, nor would we wish to. We cannot pay attention even to all aspects of what we and others say. It is nonsense when Donald Davidson proclaims his more than Leibnizian dream of a perfect calculus of meaning:

> I dream of a theory that makes the transition from ordinary idiom to canonical notation purely mechanical, and a canonical notation rich enough to capture, in its dull and explicit way, every difference and connection legitimately considered the business of a theory of meaning.[6]

These issues become clearer when we consider the limitations of propositional logic. I'll discuss only classical propositional logic, though similar remarks apply to other propositional logics. Consider the inference:

(1) Ralph is a dog.
All dogs bark.
Therefore, Ralph barks.

This is valid. There are no propositional connectives in it. So each proposition in the inference is atomic for propositional logic. Hence, the form of it in propositional logic is: p_1, p_2, therefore p_3. In a model we can assign truth or falsity to each atomic proposition independently of all others, and hence we can have a model in which

[5] See "On Models and Theories" and "Prescriptive Theories?". See particularly the section "The method of reflective equilibrium" in the latter article for a case study of what goes wrong when formal theories are evaluated solely by their consequences.
[6] "The Logical Form of Action Sentences", p. 115.

"Ralph is a dog" is true, "All dogs bark" is true, and "Ralph barks" is false. The inference is not valid in classical propositional logic.

Yet we all know that such an assignment of truth-values cannot be. It is not possible for the premises of (1) to be true and its conclusion false. So we cannot formalize the inference in classical propositional logic, for an informally valid inference should be formalized as a valid formal inference according to our criteria of formalization. There must be some aspect of these propositions that is not accounted for in classical propositional logic.

It might be just the meaning of the words in this inference. But then we think of the other inferences we looked at in Chapter 1:

Dick is a student.
All students study hard.
Therefore, Dick studies hard.

Suzy is a cheerleader.
All cheerleaders have a liver.
Therefore, Suzy has a liver.

We see a pattern. We note that the repetition of certain words in certain places in the inference matters to the evaluation of whether it is valid. In the second example it matters that the name "Suzy" appears in the first premise and the conclusion. It matters that the words "cheerleader" and "liver" appear in certain places. And the word "all" is crucial to the reasoning. The internal structure of the propositions in these inferences is significant for our reasoning, and hence the semantic aspects of parts of propositions matter, too.

4 The Language of Predicate Logic

We want to parse the internal structure of what we took to be atomic propositions in order to give structural analyses of examples of reasoning like those we saw in the last chapter. We have to relate parts of propositions—words and phrases—to our experience in order to attribute semantic values to them. But what parts and what experience?

The examples we saw in the last chapter are notably about things: dogs, Ralph, students, Dick, cheerleaders, Suzy. We have lots of words for things: chairs, tables, rocks, people, trees, . . . and lots of names for particular things. We organize our experience through our language in terms of things. Not all of our experience is in terms of things, for we also talk about water and mud, about the burning of a flame in a fireplace and the push of the wind, which don't seem to be things. But enough of our talk and our reasoning can be understood as about things for us to make the one big assumption on which modern formal logic is based.

Things, the world, and propositions The world is made up at least in part of things. The only propositions we will be interested in are those that are about things.

What do we mean by "thing"? We seem to be able to agree that rocks, people, dogs, tables, chairs, and trees are things. What we consider most basic about them is not what they're made of nor whether we happen to be looking at them, but only that they are individuals: this rock, that person, this dog, that tree. A thing—whatever it is—is individual and distinct from all else in the world. Yes, a thing may be composed of other things or masses, but what makes it a thing is that it is a whole, a distinct individual. How odd that sounds, for we seem to be saying over and over what we have no way to say except by saying "a thing" or "an individual".

Since each thing is distinct from all else, each is in some way distinguishable from all else. What we mean by saying that we can distinguish each thing we are talking about from all others will be determined in part by the kind of things we are talking about. Equally, it will determine what we consider to be a thing. There is no fixed answer to our question of what we mean by "distinguishable" that we can agree on for all things. Nonetheless, this is where we will start, refining and comparing our notions of things and distinguishability as we proceed in our work.

Consider now the proposition "Spot is a dog". This is about Spot. What we're saying about him is just the rest of the proposition: "is a dog". Generally, if a proposition has one or more names in it, we can take the names out and label that as the "about", what we'll call a "predicate". For example, we can parse the proposition "Spot loves Dick" as composed of two names "Spot" and "Dick" and a predicate "— loves —", or we can take just one of the names out and get a predicate "Spot loves —" or "— loves Dick".

Names and predicates A *name* is a word (or phrase) that we intend to use to pick out a single thing. A *predicate* is any incomplete phrase with specified gaps such that when the gaps are filled with names the phrase becomes a proposition.

Some people think that predicates, like propositions, are abstract. But in their reasoning they use what we have defined as predicates, though they consider those to be only representatives or expressions of abstract predicates.

What further structure of propositions will we recognize? Since we continue to take propositions as fundamental in our reasoning, we can continue to use the ways of forming new propositions from old ones with the formal connectives of propositional logic. We'll formalize "Ralph is a dog and Ralph barks" as "Ralph is a dog $\wedge$ Ralph barks", and so we have a predicate "— is a dog $\wedge$ — barks".

We talk not only of specific things but of things in general. Doing so, we say how many: seventeen, at least one, no more than forty-seven, many, almost all, each and every. To begin, as is traditional, let's restrict ourselves to considering just two ways: those that can be assimilated to talk about some thing or things, and those that can be assimilated to talk about all things.

These, then, are the parts we will use to parse propositions: names, predicates, propositional connectives, and ways to say "some" or "all". This is enough to begin. To make clear how we will parse propositions, we need to set out a formal language that will specify the structures we'll consider.

We start with symbols for names: $c_0, c_1, \ldots$. But we don't have names for everything we want to talk about, nor is it worth our time to name each thing prior to reasoning about it, even if that were possible. Rather, as in ordinary English, we can use temporary names. We say "that" and point, and if in the context of our conversation it's clear we mean to pick out the lamp on my table, then "that" functions as a temporary name to pick out the lamp. Formally, we can use the symbols $x_0, x_1, \ldots$ as *variables* to play the role of temporary names.

For our formalizations of "all" and "every", we'll use "$\forall$", and for our formalizations of "some" or "there is" or "there exists", we'll use "$\exists$". These are the *universal* and *existential quantifiers*. Then to formalize "Something barks" we can use variables to write "$\exists x_0$ (x_0 barks)", where the predicate is "— barks" and the first use of x_0 is for the "thing" in "something" and the second is for the pronoun when we rewrite the informal proposition as "There is something such that it barks". Similarly, we can write "Everything breathes" as "$\forall x_{17}$ (x_{17} breathes)".

Variables also serve to allow for cross-referencing. For example, if we wish to formalize "Everyone loves itself", we'd use the predicate "— loves —" intending for both blanks to be filled with a name of the same thing, as in "$\forall x_1$ (x_1 loves x_1)".

Predicates can differ depending on how many blanks they have. If there is one blank, as in "—barks", we say the predicate is *unary*; if there are two blanks, as in "—loves —", we say the predicate is *binary*; if there are three blanks, as in

"— and — are the parents of —", we say the predicate is *ternary*; if there are n blanks, we say the predicate is *n-ary*, or its *arity* is n. It might seem we don't normally use predicates that are even 4-ary, but we do, as in "Spot chased Puff towards Dick and away from Suzy". It's hard, though, to think of an example where we would use a 47-ary predicate. But mathematicians do. It's a harmless generalization to allow for predicates of any arity in our formal work, a generalization that allows us not to worry in the middle of our reasoning whether we've got all the tools we need. So we'll take as formal symbols for predicates $P_0^1, P_0^2, P_0^3, \ldots, P_1^1, P_1^2, P_1^3,$ $\ldots$, where the superscript tells us the arity of the predicate symbol, and the subscript tells us which predicate symbol it is in the list. Since it's usually clear what the arity of a predicate symbol is when we use it, I'll normally not write the superscript.

We need to be able to talk about parts of the formal language:

i, j, k, n, and subscripted versions of those stand for counting numbers;

x as well as y, z, w, and subscripted versions of the latter stand for variables;

u, v, and subscripted versions of those stand for *terms* (names or variables);

A, B, C, and subscripted versions of those stand for variables;

P, Q, and subscripted versions of Q stand for atomic predicates.

Wffs and the formal language[7] $L(\neg, \rightarrow, \wedge, \vee, \forall, \exists, P_0, P_1, \ldots, c_0, c_1, \ldots)$

Vocabulary *predicate symbols* P_i^n for $n \geq 0$ and $i \geq 1$, where i is the arity

name symbols $c_0, c_1, \ldots$ } *terms*
variables $x_0, x_1, \ldots$ }

propositional connectives $\neg, \rightarrow, \wedge, \vee$

quantifiers $\forall, \exists$

Punctuation *parentheses* ( ) *comma* , *blank* —

Well-formed formulas (*wffs*)

i. If P is a k-ary predicate symbol and $u_1, \ldots, u_k$ are terms, then

$$(P(—, \ldots, —)(u_1, \ldots, u_k))$$

is a wff of length 1. The term u_i *fills the* i^{th} *blank in* P (reading from the left). If u_i is a variable, it is *free* in the wff; if it is a name symbol, it is not free.

ii. If A is a wff of length n, then $(\neg A)$ is a wff of length $n + 1$.
An occurrence of a variable in $(\neg A)$ is free iff it is free in A.[8]

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[7] I explain the unusual parts of this particular definition in an aside on p. 19.

[8] This is an abbreviated statement which in full should read:

The i^{th} occurrence of a variable in $(\neg A)$ reading from the left is free in $(\neg A)$
iff the i^{th} occurrence of a variable in A reading from the left is free.

The succeeding steps of the definition can be made more precise in the same way.

iii. If A and B are wffs and the maximum of the lengths of A and B is n, then each of $(A \to B)$ and $(A \wedge B)$ and $(A \vee B)$ is a wff of length $n + 1$.

An occurrence of a variable in $(A \to B)$ is free iff the corresponding occurrence of the variable in A or in B is free, and similarly for $(A \wedge B)$ and $(A \vee B)$.

iv. If A is a wff of length n and some occurrence of x is free in A, then each of $(\forall x\, A)$ and $(\exists x\, A)$ is a wff of length $n + 1$.

An occurrence of a variable in either $(\forall x\, A)$ or $(\exists x\, A)$ is free iff the variable is not x and the corresponding occurrence in A is free.

A concatenation of symbols of the vocabulary is a *wff* iff it is a wff of length n for some $n \geq 1$.

A wff of length 1 is *atomic*. All other wffs are *compound*.

In $(\forall x\, A)$ the initial $\forall x$ has *scope* A and *binds* each free occurrence of x in A, and similarly for $(\exists x\, A)$.

A wff is *closed* if there is no occurrence of a variable free in it; otherwise it is *open*.

We adopt a convention on informally deleting parentheses:

- The parentheses around atomic wffs and the outer parentheses around the entire wff can be deleted.
- Parentheses between successive quantifiers at the beginning of a wff may be deleted.
- $\neg$ binds more strongly than $\wedge$ and $\vee$, which bind more strongly than $\to$. And $\forall x$, $\exists x$ bind more strongly than any of those.
- A conjunction or disjunction without parentheses is understood as associating the conjuncts or disjuncts to the left.
- Square brackets may be used in place of parentheses.

A term is free for a variable

$A(x)$ means x occurs free in A (other variables may also be free in A).

$A(u/x)$ is the formula that results by replacing every free occurrence of x in A by the term u (unless we say that it replaces only some). We say that $A(u/x)$ is the result of *substituting u for x*.

The variable y is *free for an occurrence of x in* A if that occurrence of x is free and does not lie within the scope of an occurrence of $\forall y$ or $\exists y$. It is *free for x in* A if y is free for every free occurrence of x in A.

A formula of the formal language is not a proposition; it is the form of a proposition. Only when we assign predicates to the predicate symbols in it and names to the name symbols in it can we have a proposition. For example, consider:

$$P_0^1(—)\,(c_2) \wedge \forall x_3\,(P_0^1(—)\,(x_3) \rightarrow P_2^1(—)\,(x_3))$$

We can assign "— is a dog" to $P_0^1(—)$, "— barks" to $P_2^1(—)$, "Ralph" to c_2, and get:

$$(\text{— is a dog})\,(\text{Ralph}) \wedge \forall x_3\,((\text{—is a dog})\,(x_3) \rightarrow (\text{— barks})\,(x_3))$$

This is a proposition when we fix on a particular interpretation of the formal connectives, as we did in Chapter 2, and fix on a way to understand the quantifiers and variables, which we'll do in the next chapter.

Realizations and semi-formal languages An ordinary language name or predicate is *simple* iff it contains no part we could formalize as a name, predicate, propositional connective, variable, or quantifier, or combination of those.

A *realization* of the formal language is an assignment of simple names to none, some, or all of the name symbols and simple predicates to at least one of the predicate symbols. The *realization of a formal wff* is what we get when we replace the formal symbols in it with the parts of ordinary language that are assigned to them; it is a *semi-formal wff*. The *semi-formal language* for a realization is the realizations of all formal wffs.

The realization of a predicate symbol is the simplest predicate we can have in our semi-formal language. It has no structure relative to our other vocabulary except for the placement of blanks.

I'll use the same metavariables for parts of a semi-formal language; for example, A, B, C, A_0, A_1, . . . can stand for semi-formal wffs. The terminology of the formal language will be understood to apply to semi-formal formulas, so we can say that the formula "((— barks) (x_1))" is open, and a simple predicate realizing a predicate symbol is *atomic*.

In a semi-formal language we divide the vocabulary into three parts.

Categorematic vocabulary, logical vocabulary, and punctuation

The *categorematic* vocabulary of a semi-formal language consists of the predicates that realize the predicate symbols and the names that realize the name symbols.

The *syncategorematic* or *logical* vocabulary of a semi-formal language consists of the formal symbols $\forall, \exists, \neg, \rightarrow, \wedge, \vee, x_0, x_1, \ldots$.

Punctuation is that part of the vocabulary that is not meant to formalize anything but is used only to facilitate reading wffs: blanks, commas, and parentheses.

Categorematic parts of the formal language joined by logical vocabulary and punctuation are categorematic.

We would like to formalize much of our reasoning in our system. That includes reasoning we do about the system, which involves predicates such as "— is a wff", "— is a proposition", "— is true", and names for parts of the language.

However, serious problems arise if we allow for these to be in a semi-formal language. If we realize the predicate symbol $P_0^1(—)$ as "— is true" and the name symbol "c_0" as a name for the semi-formal wff "$\neg$ (— is true) (c_0)", we'll have a version of the *liar paradox*: "This sentence is not true". Resolving whether or if that is true or false is a tortuous issue.[9] We can avoid that problem by imposing a sharp distinction between reasoning in our system, that is, our logic, and reasoning about our system, that is, our *metalogic*.

Metalogic vs. logic No name symbol can be realized as a name of any wff or part of a wff of the semi-formal language. No predicate symbol can be realized as a predicate that can apply to wffs or parts of wffs of the semi-formal language.

With the quantifier "$\exists$" we can formalize "there is" or "there exists" but only by treating those phrases as stipulating how many: at least one. However, in English we also use "exists" as a predicate, as in "Ralph exists". Should we allow that "— exists" can realize a predicate? If we do, then we can have a formalization of "There is something that doesn't exist" as "$\exists x\, (\neg$ (— exists) $(x))$". Resolving how to evaluate such sentences will require choices that do not seem essential to our basic work. We'll defer such issues to the last section of this book, adopting the following.

Existence and $\exists$ Assertions about existence can be formalized only by using the quantifier $\exists$. The phrase "— exists" and other predicates informally equivalent to it are excluded from our realizations.

Unlike propositional logic, in predicate logic not every formula of a semi-formal language is a proposition. Consider, for example:

(1) (—barks) $(x_1) \rightarrow$ (—is a cat) (x_2)

This is not a proposition even if we've settled on how to understand the connectives, variables, and quantifiers. It's a proposition only when we say what "x_1" and "x_2" stand for, which cannot be done within the formal language. So it is only when we add quantifiers to (1), making it a closed formula, that we have a proposition. For example, we could have $\forall x_1\, \exists x_2\, ((\text{—barks})(x_1) \rightarrow (\text{—is a cat})(x_2))$. Once we've fixed on how we'll understand the formal symbols, this is a proposition. *Only closed formulas of the semi-formal language are propositions.*

Some mathematicians and logicians view open formulas, such as $x < y \rightarrow \neg\, (y < x)$, as propositions, understanding the free occurrences of variables to be universally quantified. This is confusing because we're never sure whether someone is talking about an open formula or a proposition. Here we'll be explicit, writing, for example, $\forall x\, \forall y\, (x < y \rightarrow \neg\, (y < x))$. We can transform any open formula into a proposition in this way.

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[9] See Chapters IV and XXII of *Classical Mathematical Logic*.

The universal closure of a wff Let $x_{i_1}, \ldots, x_{i_n}$ be a list of all the variables that occur free in A in *alphabetical order*, that is, $i_1 < \cdots < i_n$. The *universal closure* of A is: $\forall \ldots \text{A} \equiv_{\text{Def}} \forall x_{i_1} \cdots \forall x_{i_n} \text{A}$.

Aside: *Unusual features of this formal language*

This definition of the formal language differs from most in two respects. The first concerns the use of variables. Variables typically have three roles in a formal language of predicate logic: (i) they indicate what is being quantified, (ii) they serve as temporary names, and (iii) they index blanks in a predicate. The first two roles work together in our semantic analysis of quantification, as we'll see in the next chapter. But the third role creates two problems.

First, if we say that "x_1 is a dog" is or stands for a predicate, then so does "x_2 is a dog", and something must be said about why those are or stand for the same predicate. We can't say that it's because the variable is just a placeholder, because that isn't clear, and why can two different placeholders give the same predicate? After all, in "x_1 is a dog $\wedge$ x_2 is a dog" we don't say that these are the same predicate. Nor can we say that "it doesn't matter what variable we use" until we've explained variables, and that depends on already knowing what predicates are. A second problem, which will become clear later in the text, is that the use of variables without blanks makes it difficult to distinguish a predicate modifier from a propositional operator. So in the formal language here, the blanks are retained in the predicates. Though in English, blanks can appear in various places in a predicate, for the formal language I write the blanks following the predicate symbol, separated by commas, as in $\text{P}_2^3(-,-,-)$. Blanks are retained in semi-formal wffs, too, as in "(— is a dog) (Ralph)".

The second difference is that the usual definition of a formal language allows for superfluous quantifications such as $\forall x_3\, (\text{P}_0^1(\text{c}_0))$. The rationale for including such formulas is that to do so simplifies the definition of the formal language, allowing a definition of bound and free variables to be made after the definition of the language. But the disadvantage is that semi-formal languages then contain formulas such as "$\forall x_3$ (Ralph is a dog)" that would correspond to the nonsensical "For everything, Ralph is a dog". The semantics for such uses of superfluous quantifiers simply ignore the quantifier, treating that wff as equivalent to "Ralph is a dog". That is not consonant with our normally treating nonsense as false, as I discuss in "Truth and Reasoning". The advantages of not allowing superfluous quantification, beyond ridding our semi-formal languages of nonsense, are significant:

We need no axiom schemes for superfluous quantification.

Many proofs about the language are simplified by no longer having to treat cases of superfluous quantification separately.

All variable-binding operators can be treated uniformly.

So in this formal language and in others that follow, no superfluous quantifiers are allowed. This requires that the definition of what it means for a variable to be bound or free has to be incorporated into the definition of the formal language.