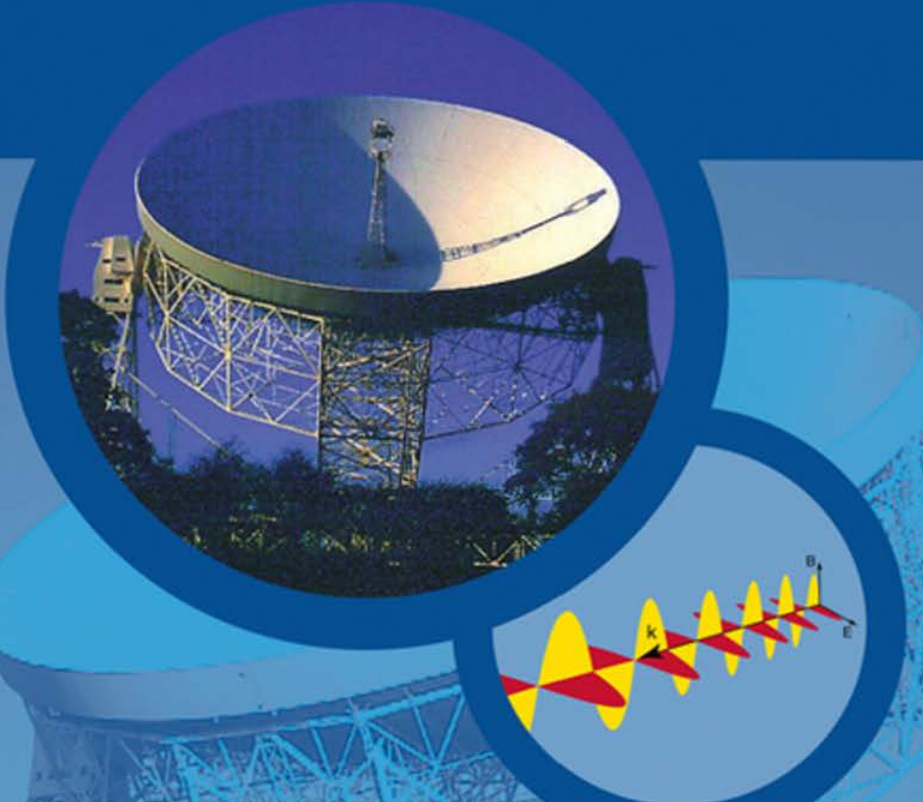
Electro-magnetism

SECOND EDITION

I. S. Grant & W. R. Phillips



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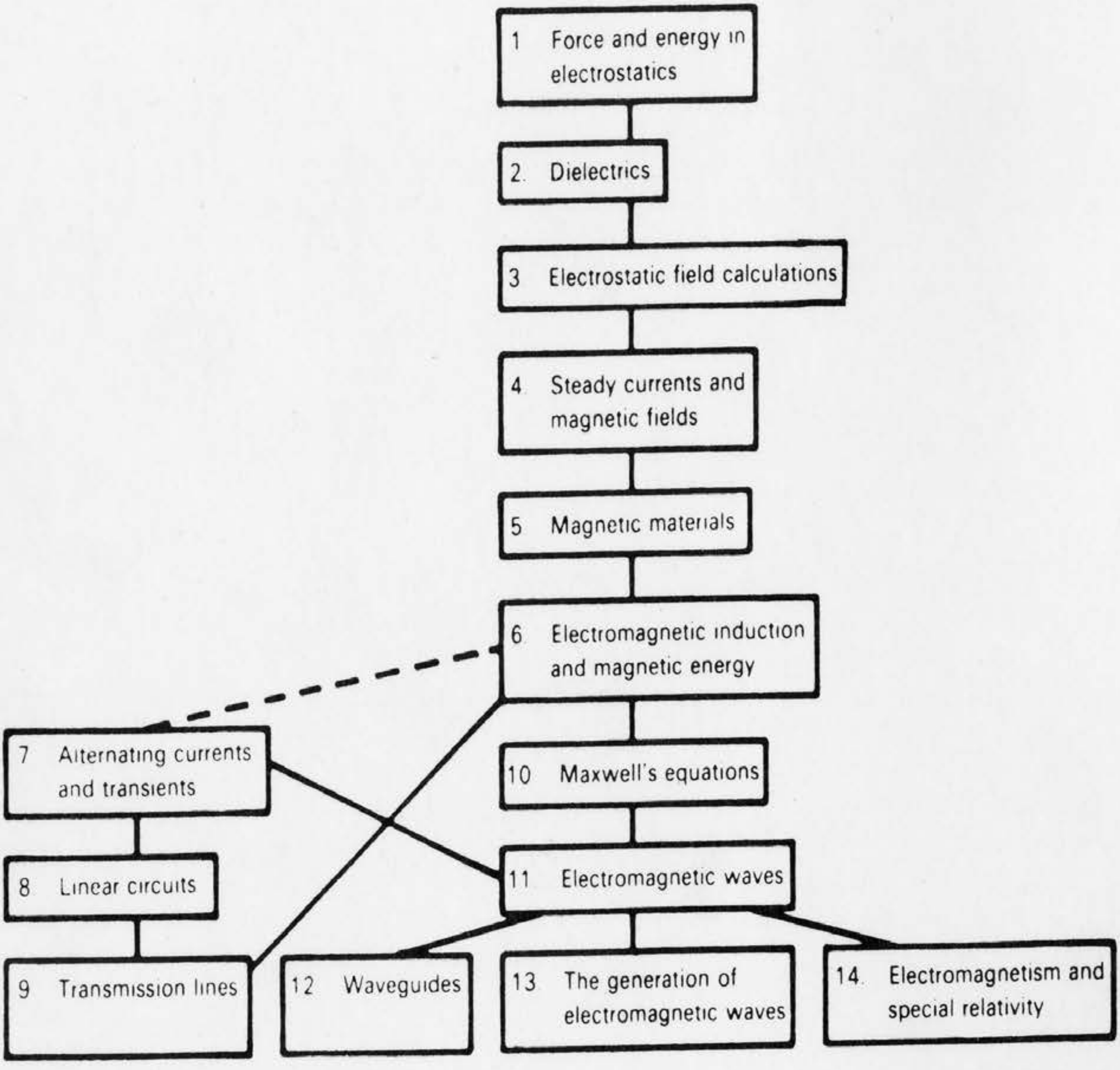

This flow diagram shows the main logical connections between chapters. Any specific chapter can be understood if all the chapters above it which are connected by a line have been covered. Chapters 7 and 8 on A. C. Theory may be tackled before the earlier chapters provided that Kirchhoff's rules are assumed. Strictly speaking Kirchhoff's rules depend on earlier material, this is indicated by the dotted line joining Chapters 6 and 7.

Electromagnetism

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Second Edition

I. S. Grant
W. R. Phillips

Department of Physics
University of Manchester

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Editors' preface to the Manchester Physics Series

The Manchester Physics Series is a series of textbooks at first degree level. It grew out of our experience at the Department of Physics and Astronomy at Manchester University, widely shared elsewhere, that many textbooks contain much more material than can be accommodated in a typical undergraduate course; and that this material is only rarely so arranged as to allow the definition of a shorter self-contained course. In planning these books we have had two objectives. One was to produce short books: so that lecturers should find them attractive for undergraduate courses; so that students should not be frightened off by their encyclopaedic size or their price. To achieve this, we have been very selective in the choice of topics, with the emphasis on the basic physics together with some instructive, stimulating and useful applications. Our second objective was to produce books which allow courses of different lengths and difficulty to be selected, with emphasis on different applications. To achieve such flexibility we have encouraged authors to use flow diagrams showing the logical connections between different chapters and to put some topics in starred sections. These cover more advanced and alternative material which is not required for the understanding of latter parts of each volume.

Although these books were conceived as a series, each of them is self-contained and can be used independently of the others. Several of them are suitable for wider use in other sciences. Each Author's Preface gives details about the level, prerequisites, etc., of his volume.

The Manchester Physics Series has been very successful with total sales of more than a quarter of a million copies. We are extremely grateful to the many students and colleagues, at Manchester and elsewhere, for helpful criticisms and stimulating comments. Our particular thanks go to the authors for all the work they have done, for the many new ideas they have contributed, and for discussing patiently, and often accepting, the suggestions of the editors.

Finally, we would like to thank our publishers, John Wiley & Sons Ltd, for their enthusiastic and continued commitment to the Manchester Physics Series.

D. J. Sandiford
F. Mandl
A. C. Phillips
February 1997

Preface to the Second Edition

The basic content of undergraduate courses in electromagnetism does not change rapidly, and the range of topics covered in the second edition of this book is almost the same as in the first edition. We have made a few additions, for example by giving a fuller treatment of circuit analysis and by discussing the dispersion of electromagnetic waves. Some material which now seems outdated has been removed, and illustrative examples have been modernized.

We have made many small changes in presentation which we hope will make the argument clearer to readers. We gratefully acknowledge the help of all those who have suggested ways of improving the text. We are particularly indebted to Dr. R. Mackintosh and his colleagues at the Open University for a host of detailed suggestions. The adoption of this book as the text for the new 'third-level' Open University course on electricity and magnetism led to a careful scrutiny of the first edition by the course team. This has resulted, we believe, in changes which make the book more useful for students. Any errors or obscurities which remain are our responsibility.

Manchester
January, 1990

I. S. GRANT
W. R. PHILLIPS

Preface to the First Edition

This book is based on lectures on classical electromagnetism given at Manchester University. The level of difficulty is suitable for honours physics students at a British University or physics majors at an American University. A-level or high school physics and calculus are assumed, and the reader is expected to have some elementary knowledge of vectors. Electromagnetism is often one of the first branches of physics in which students find that they really need to make use of vector calculus. Until one is used to them, vectors are difficult, and we have accordingly treated them rather cautiously to begin with. Brief descriptions of the properties of the differential vector operators are given at their first appearance. These descriptions are not intended to be a substitute for a proper mathematical text, but to remind the reader what div, grad and curl are all about, and to set them in the context of electromagnetism. The distinction between macroscopic and microscopic electric and magnetic fields is fully discussed at an early stage in the book. It is our experience that students do get confused about the fields **E** and **D**, or **B** and **H**. We think that the best way to help them overcome their difficulties is to give a proper explanation of the origin of these fields in terms of microscopic charge distributions or circulating currents.

The logical arrangement of the chapters is summarized in a flow diagram on the inside of the front cover. Provided that one is prepared to accept Kirchhoff's rules and the expressions for the e.m.f.s across components before discussing the laws on which they are based, the A.C. theory in Chapters 7 and 8 does not require any prior knowledge of the earlier chapters. Chapters 7 and 8 can therefore be used at the beginning of a course on electromagnetism. Sections of the book which are starred may be omitted at a first reading, since they do not contain material needed in order to understand later chapters.

We should like to thank the many colleagues and students who have helped with suggestions and criticisms during the preparation of this book; any errors which remain are our own responsibility. It is also a pleasure to thank Mrs Margaret King and Miss Elizabeth Rich for their rapid and accurate typing of the manuscript.

May, 1974
Manchester, England.

I. S. Grant
W. R. Phillips

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CHAPTER 1

Force and energy in electrostatics

The only laws of force which are known with great precision are the two laws describing the gravitational forces between different masses and the electrical forces between different charges. When two masses or two charges are stationary, then in either case the force between them is inversely proportional to the square of their separation. These inverse square laws were discovered long ago: Newton's law of gravitation was proposed in 1665, and Coulomb's law of electrostatics in 1785. This chapter is concerned with the application of Coulomb's law to systems containing any number of stationary charges. Before studying this topic in detail, it is worth pausing for a moment to consider the consequences of the law in the whole of physics.

In order to make full use of our knowledge of a law of force, we must have a theory of mechanics, that is to say, a theory which describes the behaviour of an object under the action of a known force. Large objects which are moving at speeds small compared to the speed of light obey very closely the laws of classical Newtonian mechanics. For example, these laws and the gravitational force law together lead to accurate predictions of planetary motion. But classical mechanics does not apply at all to observations made on particles of atomic scale or on very fast-moving objects. Their behaviour can only be understood in terms of the ideas of quantum theory and of the special theory of relativity. These two theories have changed the framework of discussion in physics, and have made possible the spectacular advances of the twentieth century.

It is remarkable that while mechanics has undergone drastic amendment, Coulomb's law has stood unchanged. Although the behaviour of atoms does not fit the framework of the old mechanics, when the Coulomb force is used with the theories of relativity and quantum mechanics, atomic interactions are explained with great precision in every instance when an accurate comparison has been made between experiment and theory. In principle, atomic physics and solid state physics, and for that matter the whole of chemistry, can be derived from Coulomb's law. It is not feasible to derive everything in this way, but it should be borne in mind that atoms make up the world around us, and that its rich variety and complexity are governed by electrical forces.

1.1 ELECTRIC CHARGE

Most of this book applies electromagnetism to large-scale objects, where the atomic origin of the electrical forces is not immediately apparent. However, to emphasize this origin, we shall begin by consideration of atomic systems. The simplest atom of all is the hydrogen atom, which consists of a single proton with a single electron moving around it. The hydrogen atom is stable because the proton and the electron attract one another. In contrast, two electrons repel one another, and tend to fly apart, and similarly the force between two protons is repulsive*. These phenomena are described by saying that there are two different kinds of *electric charge*, and that like charges repel one another, whereas unlike charges are attracted together. The charge carried by the proton is called *positive*, and the charge carried by the electron *negative*.

The magnitude and direction of the force between two stationary particles, each carrying electric charge, is given by Coulomb's law. The law summarizes four facts:

(i) Like charges repel, unlike charges attract.
(ii) The force acts along the line joining the two particles.
(iii) The force is proportional to the magnitude of each charge.
(iv) The force is inversely proportional to the square of the distance between the particles.

The mathematical statement of Coulomb's law is:

$$\mathbf{F}_{21} \propto \frac{q_2 q_1}{r_{21}^3} \mathbf{r}_{21}. \tag{1.1}$$

The vector $\mathbf{F}_{21}$ in Figure 1.1 represents the force on particle 1 (carrying a charge q_1) exerted by the particle 2 (carrying charge q_2). The line from q_2 to q_1 is represented by the vector $\mathbf{r}_{21}$, of length r_{21}: since the unit vector along the direction $\mathbf{r}_{21}$ can be written $\mathbf{r}_{21}/r_{21}$, Equation (1.1) is an *inverse square law* of force,

*If the protons are separated by a distance less than 10^{-14} m, they are affected by the very short range nuclear forces. Unlike gravitational and electrical forces, nuclear forces are not known precisely. However, in the study of atoms one does not need to know anything about nuclear forces beyond the fact that they are strong enough to bind together the constituent parts of the atomic nucleus.

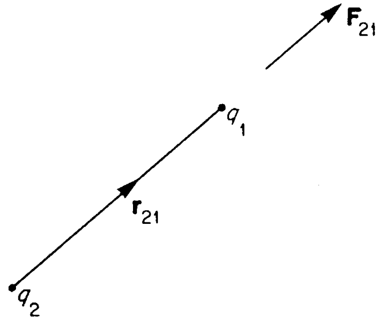

Figure 1.1. The force between two charges.

although r_{21}^3 appears in the denominator. Notice that the equation automatically accounts for the attractive or repulsive character of the force if q_1 and q_2 include the sign of the charge. When the charges q_1 and q_2 are both positive or both negative, the force on q_1 is along $\mathbf{r}_{21}$, i.e. it is repulsive. On the other hand, when one charge is positive and the other negative, the force is in the direction opposite to $\mathbf{r}_{21}$, i.e. it is attractive.

To complete the statement of the force law, we must decide what units to use, and hence determine the constant of proportionality in Equation (1.1). We shall use SI (Système International) units, which are favoured by most physicists and engineers applying electromagnetism to problems involving large-scale objects. A different system of units, called the Gaussian system, is frequently used in atomic physics and solid state physics, and it is an unfortunate necessity for students to become reasonably familiar with both systems. (The two systems of units are discussed in Appendix A.) In SI units, Coulomb's law is written as

$$\boxed{\mathbf{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{r_{21}^3}\mathbf{r}_{21}} \tag{1.2}$$

where

q_1 and q_2 are measured in coulombs,

$\mathbf{r}_{21}$ is measured in metres

and

$\mathbf{F}_{21}$ is measured in newtons.

The magnitude of the unit of charge, which is called the *coulomb*, is actually defined in terms of magnetic forces, and we shall leave discussion of the definition until Chapter 4. The factor 4π in the constant of proportionality in Coulomb's law is introduced in order to simplify some important equations which we shall

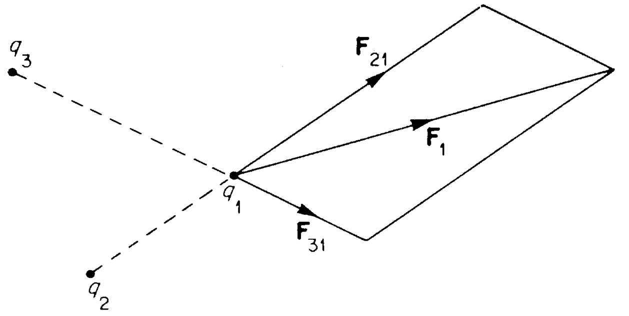

Figure 1.2. How electrostatic forces are added when there are more than two charges.

meet later. The constant ε_0, which is called the *permittivity of free space*, has the value

$$\varepsilon_0 = 8.8541878 \times 10^{-12} \text{ coulomb}^2 \text{ newton}^{-1} \text{ m}^{-2}.$$

The value of ε_0 is not determined experimentally, but has been defined in a way which makes the SI system of units self-consistent: the relation between electrical units and other units is discussed in Appendix A.

Electrostatic forces are two-body forces, which means that the force between any pair of charges is unaltered by the presence of other charges in their neighbourhood*. In a system containing many charges, the electrostatic force between each pair is given by Coulomb's law. To find the total force on any one particle, one simply makes a vector sum of the forces it experiences due to all the others separately. This rule is illustrated in Figure 1.2 for a system of three charges. The forces on q_1 due to the presence of the charges q_2 and q_3 are $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ and the total force on q_1 is

$$\begin{aligned}\mathbf{F}_1 &= \mathbf{F}_{21} + \mathbf{F}_{31} \\ &= \frac{q_2 q_1}{4\pi\varepsilon_0 r_{21}^3}\mathbf{r}_{21} + \frac{q_3 q_1}{4\pi\varepsilon_0 r_{31}^3}\mathbf{r}_{31}.\end{aligned}$$

In general, the force $\mathbf{F}_j$ on a charge q_j due to a number of other charges q_i is

$$\mathbf{F}_j = \sum_{i \neq j} \mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}^3} \mathbf{r}_{ij}.$$

The symbol $i \neq j$ under the summation signs indicates that the summation for charge i is over all the other charges j, but of course not including charge i itself.

*The forces between complete atoms are many-body forces, since the force between two atoms does depend on where other atoms are situated. The many-body nature of the force arises because the distribution of the constituent charged particles within each atom is changed by the presence of other atoms. But if the distribution of all the charge within each atom were specified, then the total force could be found by adding the forces between each pair of charges.

This equation can be written in another way in terms of the position vectors of the charges with respect to a fixed origin O. If the position vectors of the charges $q_1, q_2 \ldots q_i \ldots$ are $\mathbf{r}_1$, $\mathbf{r}_2 \ldots \mathbf{r}_i \ldots$ then the vector joining charges i and j is $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$. The total force on q_j is thus

$$\mathbf{F}_j = \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_j - \mathbf{r}_i|^3} (\mathbf{r}_j - \mathbf{r}_i). \tag{1.3}$$

A trivial example of the application of Equation (1.3) is in working out the electrostatic forces exerted by atomic nuclei containing many protons on the electrons surrounding them. Nuclei are much smaller than atoms, and for this purpose can be regarded as point charges. Equation (1.3) then tells us that the attractive force between an electron and a nucleus containing Z protons is Z times as great as that between an electron and a single proton.

It turns out that apart from the sign, the charge carried by electrons and protons is the same, and has the magnitude $e = 1.602 \times 10^{-19}$ coulombs*: the charge on the proton is $+e$, that on the electron is $-e$. The strength of atomic interactions is governed by the size of the electronic charge e. Although e is a very small number when expressed in coulombs, this does not imply that electrostatic forces are feeble. On the contrary, they are immensely strong. For example, electrostatic forces are responsible for the great strength of solids under compression. When neighbouring atoms are close together, their electron clouds begin to overlap, and the mutual repulsion of these clouds opposes any compressing force.

Another example of the strength of the electrostatic force acting on the atomic scale is given by the experiment which led Rutherford to propose the nuclear model of the atom. He found that when swiftly moving α-particles are allowed to collide with gold atoms, they are sometimes deflected through 180°, implying that a strong force is at work. The force is just the electrostatic repulsion experienced by an α-particle when it chances to approach close to the nucleus of a gold atom. Let us calculate the magnitude of the force. An α-particle is a helium nucleus, containing two protons and carrying a charge $+2e$, and the gold nucleus carries a charge $+79e$. In Rutherford's experiment, the α-particles were energetic enough to approach within 2×10^{-14} m of the nucleus (still well outside the range of the nuclear forces). Substituting in Equation (1.3), the repulsive force at this distance is

$$F = \frac{2 \times 79 \times e^2}{4\pi\varepsilon_0 (2 \times 10^{-14})^2} \simeq 90 \text{ newtons.}$$

This force, acting within a single atom, is nearly as much as the weight of a mass of 10 kilogrammes!

*The value of e (unlike ε_0) does depend on the results of measurements, and there is an uncertainty associated with it. The best value of e, adopted in 1986, is $e = (1.602\,177\,33 \pm 0.000\,000\,49) \times 10^{-19}$ coulombs. For most practical purposes this error is so small that it is of no consequence.

Normally we do not notice that electrostatic forces are so powerful, because matter is usually electrically neutral, carrying equal amounts of positive and negative charge. Not only are large lumps of matter electrically neutral, but the positive charge on the nucleus of a single isolated atom is precisely cancelled out by the negative charge of the surrounding electrons. So far as we know the cancellation is exact, and it has been shown experimentally that the magnitude of the net residual charge on a neutral atom is less than $10^{-20}e$. This is very remarkable, since apart from their electrical behaviour, protons and electrons are totally dissimilar particles. Many charged particles besides electrons and protons have been discovered by nuclear physicists, and all* share the property of carrying charges $\pm e$. It follows that the total charge carried by any piece of matter must be an integral multiple of the electronic charge e. A situation like this one, in which a physical quantity is not allowed to have a continuous range of values, but is restricted to a set of definite discrete values, is referred to as a quantum phenomenon. No one knows why electric charge should obey this quantum rule; it is an experimental fact. Nevertheless, because the rule is universal in its application, we can be sure that the electronic charge is a physical quantity of fundamental importance.

1.2 THE ELECTRIC FIELD

We have seen that the total force on a charged particle is the vector sum of the forces exerted on it by all other charges. Usually there is an enormous number of charged particles present in real matter. When considering the forces acting on any one of them, it is helpful to distract attention from the multitude of sources contributing to the net force by introducing the concept of the *electric field*. If a charge q experiences a force $\mathbf{F}$, then the ratio $\mathbf{F}/q$ is called the electric field at the point where q is located. The dimensions of electric field are [force] [charge]$^{-1}$, and in SI units the electric field is measured in newton coulomb^{-1}. (An equivalent unit, which will be explained when we come to deal with electrostatic energy, is the volt m^{-1}; 1 volt m^{-1} $\equiv$ 1 newton coulomb^{-1}.)

The electric field acting on q can be expressed in terms of the magnitudes of the other charges in the neighbourhood of q and their relative positions with respect to q. Let us assume that q is a test charge which can be put anywhere, and that its magnitude is very small, so that it exerts negligible forces on the other charges and can be moved about without altering their positions. Now we can evaluate the electric field at a point P, caused by an assembly of charges q_i, by placing the test charge at P. In Figure 1.3 P has a position vector $\mathbf{r}$, and the charges q_i have position vectors $\mathbf{r}_i$ with respect to an origin at O. The force on the test charge is

*There are particles called *quarks* carrying charges which are fractions of e. However, quarks are never observed separately, but always combine to form particles which do not have fractional charge.

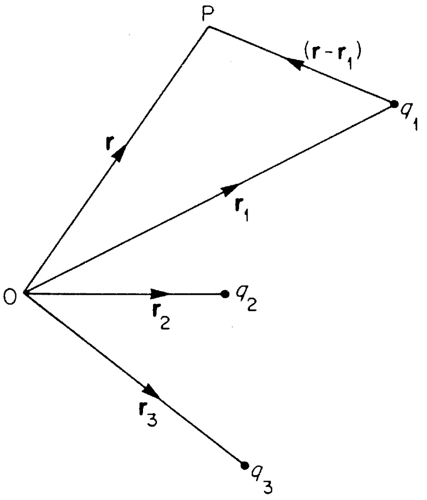

Figure 1.3. Vectors used in the definition of the electric field.

given by Equation (1.3) as

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{qq_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i),$$

where $(\mathbf{r} - \mathbf{r}_i)$ represents the vector joining q_i to the point P. The factor q is common to all terms in the sum, and it follows that

$$\frac{\mathbf{F}}{q} = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i).$$

The magnitude q of the test charge does not appear on the right-hand side of this equation. We can therefore allow q to become vanishingly small—then we are quite sure that the presence of the test charge does not modify the position of the other charges. Let us call the electric field at the point P with position vector $\mathbf{r}$ $\mathbf{E}(\mathbf{r})$.

Then

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}}{q},$$

or

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i).} \tag{1.4}$$

Equation (1.4) is the definition of the electric field $\mathbf{E}(\mathbf{r})$, and it contains no reference to a test charge q. The test charge was introduced because it illustrates that the electric field is a force per unit charge, and because it helps one to visualize the electric field if one imagines a test charge which can be moved around to sample the strength of the field at any position. The electric field $\mathbf{E}(\mathbf{r})$ is a function of position; just as the value of a function $f(x)$ is determined by the argument x, so the value of $\mathbf{E}(\mathbf{r})$ is given by Equation (1.4) in terms of its argument, which is the position vector $\mathbf{r}$. The function $\mathbf{E}(\mathbf{r})$ is itself a vector, specifying the direction as well as the magnitude of the force per unit charge on a point charge at $\mathbf{r}$. The electric field is only the first of a number of functions of position which are useful in electromagnetism. Those functions of position which are themselves vectors are called *vector fields.* When discussing vector fields we shall frequently omit any reference to the argument, writing the electric field, for example, simply as '$\mathbf{E}$', leaving it to be understood that the field is a function of position. Whenever there is some uncertainty about the position at which the function is to be evaluated, we shall always write out the vector field in full, including the argument.

In Equation (1.4) each charge q_i appears just once on the right-hand side. If the charge q_i were the *only* charge present, the field would be

$$\mathbf{E}_i(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{r}_i)q_i}{|\mathbf{r}-\mathbf{r}_i|^3}, \tag{1.5}$$

and we can rewrite Equation (1.4) as

$$\mathbf{E}(\mathbf{r}) = \sum_i \mathbf{E}_i(\mathbf{r}). \tag{1.6}$$

In other words, the total electric field is the sum of the electric fields due to each charge separately. This is an example of the *Principle of Superposition.* As applied to electrostatic fields, the Principle of Superposition states that if we have two electrostatic fields due to different groups of charged particles, the fields must be added together (superimposed) to find the field due to all the particles in both groups. This follows immediately from Equation (1.6). We shall find that the Principle of Superposition applies to all the different kinds of field which occur in electromagnetism.

Now let us investigate the properties of the electric field in the neighbourhood of isolated point charges. From Equation (1.5), the magnitude of the field at a distance r from a positive point charge $+q$ is $q/4\pi\varepsilon_0 r^2$, and the field points away from the charge. The field around a negative point charge $-q$ has the same magnitude, but it points towards the charge. In Figure 1.4 the direction of the field around positive and negative charges is indicated by the arrowed lines. These continuous lines, everywhere following the direction of the field, are called *lines of force* or *field lines.* Lines of force begin on positive charges and end on negative charges, but they may also go to infinity without termina-

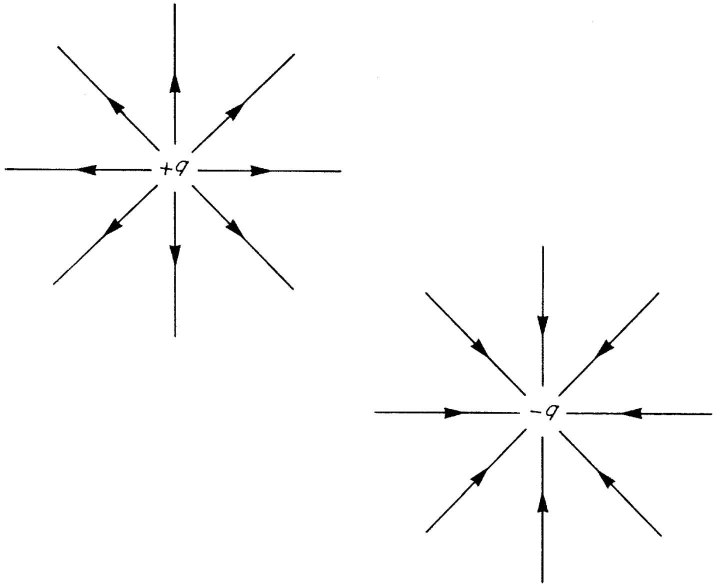

Figure 1.4. Field lines around point charges.

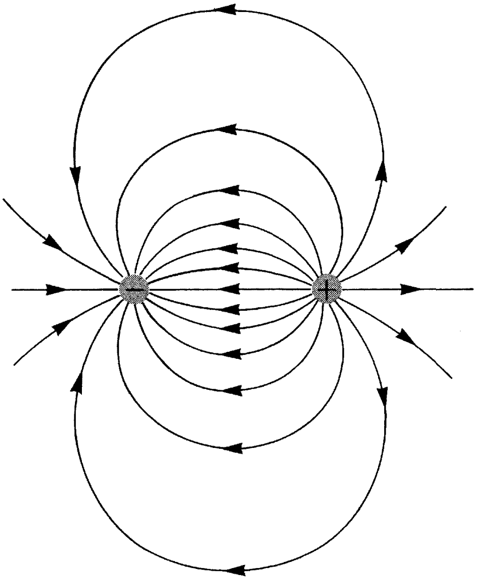

Figure 1.5. Field lines around an electric dipole.

ting, as in Figure 1.4. Notice that the lines of force are close together near the point charges where the field is strong, and far apart at large distances where the field is weak.

We can also draw diagrams of lines of force to illustrate the electric field when there are many charges present. Lines of force are continuous, except where they terminate on positive or negative charges, and they never cross one another, since the direction of the field is unique at every point. One can often get a rough idea of the field around a distribution of charges simply by sketching lines of force, and without doing any mathematics. For example, Figure 1.5 shows the lines of force near a pair of point charges of equal magnitude, one positive and one negative. Such a pair of equal and opposite charges is called an electric dipole. Very close to each charge, the field is almost the same as for isolated point charges, but the field lines starting off at the positive charge curve round to finish at the negative charge. The diagram gives an indication of the strength of the field as well as its direction, because lines of force are always densely packed in regions where the field is strong. The total number of lines on the diagram is not significant; if twice as many lines were drawn terminating on each charge, regions of relatively high or low density of lines of force would still correspond to regions of high or low field strength. The example of the electric dipole is an important one and later on we shall obtain a mathematical expression for the field at some distance from a dipole.

1.3 ELECTRIC FIELDS IN MATTER

1.3.1 The atomic charge density

So far electric fields have been dealt with in terms of idealized particles which are stationary and which carry point charges. This is not satisfactory if we want to discuss the electric field within an actual atom, since electrons are certainly not stationary point charges. Indeed, according to quantum mechanics the position of an electron cannot even be sharply defined. Instead of imagining the electron as a point, it is more realistic to regard its charge as being smeared out in a cloud around the nucleus. The electric field in and around a hydrogen atom, for example, behaves as though only part of the electronic charge is contained by any section of the electron cloud. Let us write $\rho_{el}(\mathbf{r})\delta\tau$ for the charge contained within a small volume $\delta\tau$ situated at the point with position vector $\mathbf{r}$. The quantity $\rho_{el}(\mathbf{r})$ is the *charge density* of the electron cloud, i.e. its charge per unit volume at $\mathbf{r}$. The charge density is another function of position, but unlike the electric field it is a scalar quantity, fully specified by its magnitude at each point. Scalar functions of position such as the charge density are called *scalar fields*. As with the vector fields, we shall often omit the argument of scalar fields, writing the electron charge density just as ρ_{el}, for example. The total charge $(-e)$ carried by the electron in a hydrogen atom is equal to the sum of the charges contained in all the small volumes $\delta\tau$ making up the electron cloud. In the limit as the volumes $\delta\tau$ become infinitesimal, the sum becomes an integral; integrating over a volume V large enough to contain the whole atom, we write

$$\int_V \rho_{\mathrm{el}}(\mathbf{r})\,\mathrm{d}\tau = -e.$$

Here we have introduced a shorthand notation for the volume integral, which is really a triple integral over the three components needed to specify position vectors $\mathbf{r}$ within the volume V. In Cartesian coordinates $\mathrm{d}\tau$ is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, the volume of the rectangular box enclosed by sides of length $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$. The volume of integration V must include the whole atom, but since $\rho_{\mathrm{el}}(\mathbf{r})$ is zero outside the atom, it can be made as large as we please without affecting the result. If we extend V to cover the whole of space, in Cartesian coordinates the integral becomes

$$\int_{\text{all space}} \rho_{\mathrm{el}}(\mathbf{r})\,\mathrm{d}\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_{\mathrm{el}}(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -e.$$

In atoms more complex than hydrogen, the electron clouds of different electrons overlap, and a volume element may contain parts of the charge of more than one electron. Even the charge carried by the nucleus should be represented by a charge density since although the nucleus is small compared with the whole atom, it does have a definite size. Now we define the atomic charge density $\rho_{\mathrm{atomic}}(\mathbf{r})$ in a piece of matter as the net charge density at $\mathbf{r}$, including positive contributions from nuclei. Where the charge densities associated with different particles overlap, they are added together to form the *atomic charge density*. The atomic charge density has large positive values inside the nucleus of each atom and is negative in the electron cloud. The volume integral $\int_V \rho_{\mathrm{atomic}}(\mathbf{r})\mathrm{d}\tau$ represents the net charge within V: thus for a volume V containing an electrically neutral piece of matter, the value of the integral must be zero.

1.3.2 The atomic electric field

The electric field due to an assembly of point charges q_i was found by making a vector sum of the fields due to each charge separately:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{(\mathbf{r}-\mathbf{r}_i)\,q_i}{|\mathbf{r}-\mathbf{r}_i|^3}.$$

Applying the same procedure to a continuous charge density ρ_{atomic}, the sum becomes an integral, and the *atomic electric field* at a point P with position vector $\mathbf{r}$ is

$$\mathbf{E}_{\mathrm{atomic}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{(\mathbf{r}-\mathbf{r}')\,\rho_{\mathrm{atomic}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}\tau'. \tag{1.7}$$