Configuration Spaces
Geometry, Combinatorics and Topology

edited by

A. Bjorner, F. Cohen, C. De Concini,
C. Procesi and M. Salvetti

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Introduction

The present volume arises from the intensive research period "Configuration spaces: Geometry, Combinatorics and Topology" which took place at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, in May-June 2010.

The program was very intense and included two conferences, nine minicourses and two weekly seminars one of which organized by Filippo Callegaro and run by the younger participants.

The period covered a large number of different topics all centering around the notion of configuration spaces. These included among others:

1. Study of local systems on the complements of a hyperplane arrangements. Characteristic and resonance varieties. Cohomology and monodromy computations. Study of the fundamental group of complements of arrangements. Hodge theoretical aspects.
2. Qualitative and quantitative problems related to the study of partition functions. Relations with the theory of Box Splines and applications to index theory. Enumeration of lattice points in rational polytopes. Relations with toric geometry.
3. Invariants of braids and knots. Topological quantum field theory. Applications to low dimensional topology. Construction of representations of braid groups and related groups.
4. Applications of the theory of hyperplane arrangements and toric arrangements to the study of the combinatorics of matroids and generalizations.
5. Homotopy theory aspects of the study of configuration spaces and moduli spaces. Relations between cohomology of braid and mapping class groups.
6. Combinatorial aspects of the theory of Coxeter and Artin groups. Cohomological computations for both abelian and non abelian local systems.

7. Toric geometry. Moment angle complexes and applications. Geometry and topology of real and complex toric varieties.

Many of these topics were covered by the minicourses whose list we include here:

Fred Cohen: Moment-angle complexes, their stable structure and cohomology.

Eduard Looijenga: Aspects of the KZ system.

Stefan Papadima, Alex Suciu: Cohomology jumping loci and homological finiteness properties.

Claudio Procesi: Splines and partition functions

Dev Sinha: Hopf rings in topology and algebra.

Alexander Varchenko: The quantum integrable model of an arrangement of hyperplanes.

Michele Vergne: Remarks on Box splines.

Sergey Yuzvinsky: A short introduction to arrangements of hyperplanes

Sergey Yuzvinsky: Resonance varieties for arrangements and their relations to combinatorics and algebraic geometry

We feel that the papers appearing in this volume very well represent the spirit and the topics of the research period.

We very warmly thank both the authors of these papers and the other participants who very much contributed to the success of this activity.

Finally we wish to thank the staff of Centro di Ricerca Ennio De Giorgi and of the Scientific Section of Edizioni della Scuola Normale for a very efficient job both in running our period and in editing the present volume.

Pisa, July 2012

A. Bjorner, F. Cohen, C. De Concini,
C. Procesi and M. Salvetti

On the structure of spaces of commuting elements in compact Lie groups

Alejandro Adem* and José Manuel Gómez

Abstract. In this note we study topological invariants of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$, where π is a finitely generated abelian group and G is a compact Lie group arising as an arbitrary finite product of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$.

1 Introduction

Let $\mathcal{P}$ denote the class of compact Lie groups arising as arbitrary finite products of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. In this article we use methods from algebraic topology to study the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ where π denotes a finitely generated abelian group and $G \in \mathcal{P}$. Our main interest is the computation of invariants associated to these spaces such as their cohomology and stable homotopy type, as well as their equivariant K-theory with respect to the natural conjugation action. The natural quotient space under this action is the space of representations $\mathrm{Rep}(\pi, G)$, which can be identified with the moduli space of isomorphism classes of flat connections on principal G–bundles over M, where M is a compact connected manifold with $\pi_1(M) = \pi$. Thus our results provide insight into these geometric invariants in the important case when $\pi_1(M)$ is a finitely generated abelian group.

Our starting point is the observation (see [3]) that when $G \in \mathcal{P}$ and π is a finitely generated abelian group, the conjugation action of G on the space of homomorphisms $\mathrm{Hom}(\pi, G)$ satisfies the following property: for every element $x \in \mathrm{Hom}(\pi, G)$ the isotropy subgroup G_x is connected and of maximal rank. This property plays a central part in our analysis. Indeed, let $T \subset G$ be a maximal torus; in general if a compact Lie group

*Partially supported by NSERC.

G acts on a compact space X with connected maximal rank isotropy subgroups then there is an associated action of W on the fixed–point set X^T and many properties of the space X are determined by the action of W on X^T (see [3, 8]). For our examples this means that a detailed understanding of the W-action on the subspace $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$ can be used to describe key homotopy–theoretic invariants for the original space of homomorphisms.

This approach can be used for example to obtain an explicit description of the number of path–connected components in $\mathrm{Hom}(\pi, G)$. Indeed we show that if $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group, then the number of path–connected components in $\mathrm{Hom}(\pi, G)$ equals the number of distinct orbits for the action of W on $\mathrm{Hom}(A, T)$

In [1] a stable splitting for the spaces of commuting n-tuples in G, $\mathrm{Hom}(\mathbb{Z}^n, G)$, was derived for any Lie group G that is a closed subgroup of $GL_n(\mathbb{C})$. Here we show that this splitting can be generalized to the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ when $G \in \mathcal{P}$ and π is any finitely generated abelian group. This is done by constructing a stable splitting on $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$ and proving that this splitting lifts to the space $\mathrm{Hom}(\pi, G)$. Suppose that $\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n)$, where $n \geq 0$ and $q_1, \ldots, q_n$ are integers. Here we allow some of the q_i's to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. Thus $\mathrm{Hom}(\pi, G)$ can be seen as the subspace of G^n consisting of those commuting n-tuples $(x_1, \ldots, x_n)$ such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. For $1 \leq r \leq n$ let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \leq m_1 < \cdots < m_r \leq n\}$. Given such a sequence $\mathfrak{m}$ let $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ be a quotient of π. Let $S_1(P_{\mathfrak{m}}(\pi), G)$ be the subspace of $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ consisting of those r-tuples $(x_{m_1}, \ldots, x_{m_r})$ in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ for which at least one of the x_{m_i}'s is equal to 1_G.

Theorem 1.1. *Suppose that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Then there is a G-equivariant homotopy equivalence*

$$\Theta : \Sigma\, \mathrm{Hom}(\pi, G) \to \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee_{\mathfrak{m} \in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G) \right).$$

In Section 4 we determine the homotopy type of the stable factors appearing in the previous theorem for certain particular cases. In particular we determine the stable homotopy type of $\mathrm{Hom}(\pi, SU(2))$ for any finitely generated abelian group.

Suppose now that G is any compact Lie group. The fundamental group of the spaces of homomorphisms of the form $\mathrm{Hom}(\mathbb{Z}^n, G)$ was computed in [7]. Let $\mathbb{1} \in \mathrm{Hom}(\mathbb{Z}^n, G)$ be the trivial representation. If $\mathbb{1}$ is chosen as the base point, then by [7, Theorem1.1] there is a natural isomorphism $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G)) \cong (\pi_1(G))^n$. Here we show that the methods applied in [7] can be used to compute $\pi_1(\mathrm{Hom}(\pi, G))$ for any choice of base point if we further require that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Write π in the form $\pi = \mathbb{Z}^n \oplus A$, with A a finite abelian group. Then the space of homomorphisms $\mathrm{Hom}(\pi, G)$ can naturally be identified as a subspace of the product $\mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G)$. Given $f \in \mathrm{Hom}(A, T)$ let

$$\mathbb{1}_f := \mathbb{1} \times f \in \mathrm{Hom}(\pi, G) \subset \mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G).$$

Every path–connected component in $\mathrm{Hom}(\pi, G)$ contains some $\mathbb{1}_f$ and thus it suffices to consider the elements of the form $\mathbb{1}_f$ as base points in $\mathrm{Hom}(\pi, G)$. With this in mind we have the following.

Theorem 1.2. *Let $\pi = \mathbb{Z}^n \oplus A$, with A a finite abelian group and let $G \in \mathcal{P}$. Suppose $f \in \mathrm{Hom}(A, T)$ and take $\mathbb{1}_f$ as the base point of $\mathrm{Hom}(\pi, G)$. Then there is a natural isomorphism $\pi_1(\mathrm{Hom}(\pi, G)) \cong (\pi_1(G_f))^n$ where $G_f = Z_G(f)$ is the subgroup of elements in G commuting with $f(x)$ for all $x \in A$.*

In Section 6 we study the equivariant K-theory of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ with respect to the conjugation action by G. When π is a finite group, then $\mathrm{Hom}(\pi, G)$ is the disjoint union of homogeneous spaces of the form G/H where H is a maximal rank subgroup. Using this it is easy to see that $K^*_G(\mathrm{Hom}(\pi, G))$ is a free module over the representation ring of rank $|\mathrm{Hom}(\pi, T)|$. This result can be generalized for finitely generated abelian groups of rank 1 in the following way.

Theorem 1.3. *Suppose that $G \in \mathcal{P}$ is simply connected and of rank r. Let $\pi = \mathbb{Z} \oplus A$ where A is a finite abelian group. Then $K^*_G(\mathrm{Hom}(\pi, G))$ is a free $R(G)$-module of rank $2^r \cdot |\mathrm{Hom}(A, T)|$.*

It turns out that $K^*_G(\mathrm{Hom}(\pi,G))$ is not always free as a module over $R(G)$. In fact, as was pointed out in [3], the $R(SU(2))$-module $K^*_{SU(2)}(\mathrm{Hom}(\mathbb{Z}^2, SU(2)))$ is not free. However, $K^*_{SU(2)}(\mathrm{Hom}(\mathbb{Z}^2, SU(2))) \otimes \mathbb{Q}$ turns out to be free as a module over $R(SU(2)) \otimes \mathbb{Q}$. The next theorem shows that a similar result holds for all the spaces of homomorphisms that we consider here.

Theorem 1.4. *Suppose that $G \in \mathcal{P}$ is of rank r and that π is a finitely generated abelian group written in the form $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group. Then $K^*_G(\mathrm{Hom}(\pi, G)) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank $2^{nr} \cdot |\mathrm{Hom}(A, T)|$.*

The layout of this article is as follows. In Section 2 some general properties of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ are determined. In Section 3 we study the cohomology groups with rational coefficients of these spaces. In Section 4 Theorem 1.1 is proved and some explicit examples are computed. In Section 5 the fundamental group of the spaces $\mathrm{Hom}(\pi, G)$ are computed for any choice of base point. Finally, in Section 6 we study the problem of computing $K_G^*(\mathrm{Hom}(\pi, G))$, where G acts by conjugation on $\mathrm{Hom}(\pi, G)$.

Both authors would like to thank the Centro di Ricerca Matematica Ennio De Giorgi at the Scuola Normale Superiore in Pisa for inviting them to participate in the program on Configuration Spaces: Geometry, Combinatorics and Topology during the spring of 2010.

2 Preliminaries on spaces of commuting elements

Let π be a finitely generated discrete group and G a Lie group. Consider the set of homomorphisms from π to G, $\mathrm{Hom}(\pi, G)$. This set can be given a topology as a subspace of a finite product of copies of G in the following way. Fix a set of generators $e_1, \ldots, e_n$ of π and let F_n be the free group on n-letters. By mapping the generators of F_n onto the different e_i's we obtain a surjective homomorphism $F_n \to \pi$. This surjection induces an inclusion of sets $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(F_n, G) \cong G^n$. This way $\mathrm{Hom}(\pi, G)$ can be given the subspace topology. It is easy to see that this topology is independent of the generators chosen for π. In case π happens to be abelian, then any map $F_n \to \pi$ factors through $F_n \to \mathbb{Z}^n \to \pi$ yielding an inclusion of spaces $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G) \hookrightarrow G^n$. Thus the space of homomorphisms $\mathrm{Hom}(\pi, G)$ can be seen as a subspace of the space of commuting n-tuples in G, $\mathrm{Hom}(\mathbb{Z}^n, G)$.

In this note we collect some facts about these spaces of homomorphisms in the particular case that π is a finitely generated abelian group and G belongs to a suitable family of Lie groups. We are mainly interested in the following family of Lie groups.

Definition 2.1. Let $\mathcal{P}$ denote the collection of all compact Lie groups arising as finite cartesian products of the groups $SU(r)$, $U(q)$ and $Sp(k)$.

Whenever G belongs to the family $\mathcal{P}$ the space of homomorphisms $\mathrm{Hom}(\pi, G)$ satisfies the following crucial condition as we prove below in Proposition 2.3.

Definition 2.2. Let X be a G-space. The action of G on X is said to have connected maximal rank isotropy subgroups if for every $x \in X$, the isotropy group G_x is a connected subgroup of maximal rank; that is, for every $x \in X$ we can find a maximal torus T_x in G such that $T_x \subset G_x$.

Proposition 2.3. *Suppose that π is a finitely generated abelian group and $G \in \mathcal{P}$. Then the conjugation action of G on* $\mathrm{Hom}(\pi, G)$ *has connected maximal rank isotropy subgroups.*

Proof. Choose generators $e_1, \ldots, e_n$ of π. As pointed out above we can use these generators to obtain an inclusion of G-spaces $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$. Given this inclusion it suffices to show that the conjugation action of G on $\mathrm{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy groups. In [3, Example 2.4] it was proven that the action of G on $\mathrm{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy subgroups if and only if $\mathrm{Hom}(\mathbb{Z}^{n+1}, G)$ is path–connected. The proposition follows by noting that $\mathrm{Hom}(\mathbb{Z}^k, G)$ is path–connected for all $k \geq 0$ whenever $G \in \mathcal{P}$. □

Suppose that a compact Lie group G acts on a space X with connected maximal rank isotropy subgroups. Choose a maximal torus T in G and let W be the Weyl group. By passing to the level of T-fixed points, the action of G on X induces an action of the Weyl group W on X^T. Many properties of the action of G on X are determined by the action of W on X^T as explained in [8] and in some situations the former is completely determined by the latter up to isomorphism. For example, we can use this approach to produce G-CW complex structures on the spaces of homomorphisms as is proved next.

Corollary 2.4. *Suppose that π is a finitely generated abelian group and $G \in \mathcal{P}$. Then* $\mathrm{Hom}(\pi, G)$ *with the conjugation action has the structure of a G-CW complex.*

Proof. Since π is a finitely generated abelian group it can be written in the form $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group. Let $X := \mathrm{Hom}(\pi, G)$ with the conjugation action of G. Note that $X^T = \mathrm{Hom}(\pi, G)^T = T^n \times \mathrm{Hom}(A, T)$. Since $\mathrm{Hom}(A, T)$ is a discrete set, it follows that X^T has the structure of a smooth manifold on which W acts smoothly. In particular, by [9, Theorem 1] it follows that X^T has the structure of a W-CW complex. Since the conjugation action of G on X has connected maximal rank isotropy subgroups then by [3, Theorem 2.2] it follows that this W-CW complex structure on X^T induces a G-CW complex on X. □

This approach can also be used to determine explicitly the structure of these spaces of homomorphisms whenever π is a finite abelian group.

Proposition 2.5. *Suppose that π is a finite abelian group and $G \in \mathcal{P}$. Then there is a G-equivariant homeomorphism*

$$\Phi : \mathrm{Hom}(\pi, G) \to \bigsqcup_{[f] \in \mathrm{Hom}(\pi, T)/W} G/G_f.$$

Here $[f]$ runs through a system of representatives of the W-orbits in $\mathrm{Hom}(\pi, T)$ *and each G_f is a maximal rank subgroup with $W(G_f) = W_f$.*

Proof. Consider the G-space $X := \mathrm{Hom}(\pi, G)$. Note that $X^T = \mathrm{Hom}(\pi, T)$ is a discrete set endowed with an action of W. By decomposing X^T into the different W-orbits we obtain a W-equivariant homeomorphism

$$X^T \cong \bigsqcup_{[f] \in \mathrm{Hom}(\pi,T)/W} W/W_f.$$

Here $[f]$ runs through a set of representatives for the action of W on $\mathrm{Hom}(\pi, T)$. For each $f \in \mathrm{Hom}(\pi, T)$ let G_f denote the subgroup of elements in G commuting with $f(x)$ for all $x \in \pi$. This group is a maximal rank subgroup in G as $T \subset G_f$. Moreover, by [8, Theorem 1.1] it follows that $W(G_f) = W_f$. Also note that if we let G act on the left on the homogeneous space G/G_f then $(G/G_f)^T = W/W_f$. Let

$$Y = \bigsqcup_{[f] \in \mathrm{Hom}(\pi,T)/W} G/G_f.$$

The left action of G on Y has maximal rank isotropy and there is a W-equivariant homeomorphism $\phi : X^T \to Y^T$. By [8, Theorem 2.1] there is a unique G-equivariant extension $\Phi : X \to Y$ of ϕ and this map is in fact a homeomorphism. □

3 Rational cohomology and path–connected components

In this section we explore the set of path connected components and the rational cohomology groups of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$.

Suppose that G is a compact connected Lie group and let T be a maximal torus in G. Assume that G acts on a space X of the homotopy type of a G-CW complex with maximal rank isotropy subgroups. Consider the continuous map

$$\begin{aligned} \phi : G \times X^T &\to X \\ (g, x) &\mapsto gx. \end{aligned}$$

Since G acts on X with maximal rank isotropy subgroups for every $x \in X$ we can find a maximal torus T_x in G such that $T_x \subset G_x$. As every pair of maximal tori in G are conjugate it follows that for every $x \in X$ we can find some $g \in G$ such that $gx \in X^T$. This shows that ϕ is a surjective map. The normalizer of T in G, $N_G(T)$ acts on the right on $G \times X^T$ by

$(g, x) \cdot n = (gn, n^{-1}x)$ and the map ϕ is invariant under this action. Thus ϕ descends to a surjective map

$$\begin{aligned}\varphi : G \times_{N_G(T)} X^T = G/T \times_W X^T &\to X \\ [g, x] &\mapsto gx\end{aligned}$$

The map φ is not injective in general. Indeed, as was proven in [4], given $x \in X$ there is a homeomorphism $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$, where G_x^0 denotes the path–connected component of G_x containing the identity element. Let $\mathbb{F}$ be a field with characteristic relatively prime to $|W|$. Then as observed in [4] the space $G_x^0/N_{G_x^0}(T)$ has $\mathbb{F}$–acyclic cohomology. The Vietoris-Begle theorem shows that φ induces an isomorphism in cohomology with $\mathbb{F}$-coefficients. As a consequence we obtain the following proposition (first proved in [4]).

Proposition 3.1. *Suppose that G is a compact connected Lie group acting on a spaces X with maximal rank isotropy subgroups. If $\mathbb{F}$ is a field with characteristic relatively prime to $|W|$ then*

$$H^*(X; \mathbb{F}) \cong H^*(G/T \times_W X^T; \mathbb{F}) \cong H^*(G/T \times X^T; \mathbb{F})^W.$$

Remark 3.2. Suppose that G acts on X with *connected* maximal rank isotropy groups. As pointed out above the map φ is not injective in general since $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$ for $x \in X$. Under the given hypothesis we have $G_x^0 = G_x$. By [8, Theorem 1.1] the assignment $(H) \mapsto (WH)$ defines a one to one correspondence between the set of conjugacy classes of isotropy subgroups of the action of G on X and the set of conjugacy classes of isotropy subgroups of the action of W on X^T. Thus the different isotropy subgroups of the action of W on X^T determine how far the map φ is from being injective. In particular, if W acts freely on X^T then φ is a continuous bijection and thus a homeomorphism if for example X^T is compact.

Suppose now that $G \in \mathcal{P}$ and let π be a finitely generated abelian group. By Proposition 2.3 the conjugation action of G on $\mathrm{Hom}(\pi, G)$ has connected maximal rank isotropy subgroups. In this case $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$. As a consequence of the previous result the following is obtained.

Corollary 3.3. *Suppose that $G \in \mathcal{P}$ and let π be a finitely generated abelian group. Then there is an isomorphism $H^*(\mathrm{Hom}(\pi, G); \mathbb{Q}) \cong H^*(G/T \times \mathrm{Hom}(\pi, T); \mathbb{Q})^W$.*

As an application of Corollary 3.3 the following can be derived.

Corollary 3.4. *Suppose that* $G \in \mathcal{P}$ *and let* π *be a finitely generated abelian group written in the form* $\pi = \mathbb{Z}^n \oplus A$*. Then the number of path–connected components in* $\mathrm{Hom}(\pi, G)$ *equals the number of different orbits of the action of* W *on* $\mathrm{Hom}(A, T)$

4 Stable splittings

In this section we show that the fat wedge filtration on a finite product of copies of G induces a natural filtration on the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$. It turns out that this filtration splits stably after one suspension whenever π is a finitely generated abelian group and $G \in \mathcal{P}$.

Suppose that π is a finitely generated abelian group. Using the fundamental theorem of finitely generated abelian groups π can be written in the form

$$\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n),$$

where $n \geq 0$ and $q_1, \ldots, q_n$ are integers. Here we allow some of the q_i's to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. This way we can see $\mathrm{Hom}(\pi, G)$ as the subspace of G^n consisting of those commuting n-tuples $(x_1, \ldots, x_n)$ such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. The fat wedge filtration on G^n induces a natural filtration on the space of homomorphisms $\mathrm{Hom}(\pi, G)$. To be more precise, for each $1 \leq j \leq n$ let

$$\begin{aligned} S_j(\pi,G) = &\{(x_1, \ldots, x_n) \in \mathrm{Hom}(\pi, G) \subset G^n \mid x_i = 1_G \\ &\text{for at least } j \text{ of the } x_i\text{'s}\}. \end{aligned}$$

This way we obtain a filtration of $\mathrm{Hom}(\pi, G)$

$$\begin{aligned} \{(1_G, ..., 1_G)\} = S_n(\pi, G) &\subset S_{n-1}(\pi, G) \subset \cdots \subset S_0(\pi, G) \\ &= \mathrm{Hom}(\pi, G). \end{aligned} \tag{4.1}$$

Note that each $S_j(\pi, G)$ is invariant under the conjugation action of G. In particular each $S_j(\pi, G)$ can be seen as a G-space that has connected maximal rank isotropy subgroups. On the level of the T-fixed points the filtration (4.1) induces a filtration of $\mathrm{Hom}(\pi, G)^T$

$$\begin{aligned} \{(1_G, ..., 1_G)\} = S_n(\pi, G)^T &\subset S_{n-1}(\pi, G)^T \subset \cdots \subset S_0(\pi, G)^T \\ &= \mathrm{Hom}(\pi, G)^T. \end{aligned} \tag{4.2}$$

For each $1 \leq i \leq n$ consider $\mathrm{Hom}(\mathbb{Z}/q_i, T) = \{t \in T \mid t^{q_i} = 1\}$. Note that each $\mathrm{Hom}(\mathbb{Z}/q_i, T)$ is a space endowed with the action of W. Whenever $q_i = 0$ we have $\mathrm{Hom}(\mathbb{Z}/q_i, T) = T$ and if $q_i \neq 0$ then $\mathrm{Hom}(\mathbb{Z}/q_i, T)$ is a discrete set. Since T is abelian it follows that

$$\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T) = \mathrm{Hom}(\mathbb{Z}/q_1, T) \times \cdots \times \mathrm{Hom}(\mathbb{Z}/q_n, T).$$

Moreover, the filtration (4.2) is precisely the fat wedge filtration of $\mathrm{Hom}(\pi, G)^T$ where we identify $\mathrm{Hom}(\pi, G)^T$ with the above product. It is well known that the fat wedge filtration on a product of spaces splits stably after one suspension. More precisely, for each $0 \leq j \leq n-1$ we can find a continuous map

$$r_j : \Sigma S_j(\pi, G)^T \to \Sigma S_{j+1}(\pi, G)^T$$

in such a way that there is a homotopy h_j between $r_j \circ \Sigma(i_j)$ and $1_{\Sigma(S_{j+1}(\pi,G)^T)}$. Here

$$i_j : S_{j+1}(\pi, G)^T \to S_j(\pi, G)^T$$

denotes the inclusion map. Moreover, both the map r_j and the homotopy h_j can be arranged in such a way that they are W-equivariant. The W-action that we have in sight is the diagonal action of W on the product $\mathrm{Hom}(\mathbb{Z}/q_1, T) \times \cdots \times \mathrm{Hom}(\mathbb{Z}/q_n, T)$. Consider the action of G on $\Sigma \mathrm{Hom}(\pi, G)$ with G acting trivially on the suspension component. This action has connected maximal rank isotropy subgroups and $(\Sigma \mathrm{Hom}(\pi, G))^T = \Sigma \mathrm{Hom}(\pi, T)$. By [8, Theorem 2.1] we can find a unique G-equivariant extension

$$R_j : \Sigma S_j(\pi, G) \to \Sigma S_{j+1}(\pi, G)$$

of r_j and a unique G-equivariant homotopy H_j between $R_j \circ \Sigma(I_j)$ and $1_{\Sigma(S_{j+1}(\pi,G))}$ extending h_j. Here $I_j : S_{j+1}(\pi, G) \to S_j(\pi, G)$ as before denotes the inclusion map.

Let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \leq m_1 < \cdots < m_r \leq n\}$. Note that $J_{n,r}$ contains precisely $\binom{n}{r}$ elements. Given such a sequence $\mathfrak{m}$, there is an associated abelian group $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ obtained as a quotient of π and also a G-equivariant projection map

$$P_{\mathfrak{m}} : \mathrm{Hom}(\pi, G) \to \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$$
$$(x_1, ..., x_n) \mapsto (x_{m_1}, ..., x_{m_r}).$$

The above can be used to prove the following theorem.

Theorem 4.1. *Suppose that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Then there is a G-equivariant homotopy equivalence*

$$\Theta : \Sigma \mathrm{Hom}(\pi, G) \to \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee_{\mathfrak{m} \in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G) \right).$$