

Configuration Spaces Geometry, Combinatorics and Topology

edited by

A. Björner, F. Cohen, C. De Concini,
C. Procesi and M. Salvetti



EDIZIONI
DELLA
NORMALE

14

CRM
SERIES

 | Centro
di Ricerca
Matematica
Ennio De Giorgi

Configuration Spaces

Geometry, Combinatorics and Topology

edited by

A. Bjorner, F. Cohen, C. De Concini,
C. Procesi and M. Salvetti



EDIZIONI
DELLA
NORMALE

© 2012 Scuola Normale Superiore Pisa

ISBN 978-88-7642-430-4

ISBN 978-88-7642-431-1 (eBook)

Contents

Authors' affiliations	xv
Introduction	xix
Alejandro Adem and José Manuel Gómez	
On the structure of spaces of commuting elements in compact Lie groups	
1	Introduction 1
2	Preliminaries on spaces of commuting elements 4
3	Rational cohomology and path-connected components 6
4	Stable splittings 8
5	Fundamental group 16
6	Equivariant K -theory 22
6.1	Finite abelian groups 22
6.2	Abelian groups of rank one 23
6.3	Finitely generated abelian groups 25
	References 26
Meirav Amram, David Garber and Mina Teicher	
On the fundamental group of the complement of two real tangent conics and an arbitrary number of real tangent lines	
1	Introduction 27
2	Braid monodromy factorizations 30
3	The computation of the fundamental groups 34
	References 46
Dimitry Arinkin and Alexander Varchenko	
Intersection cohomology of a rank one local system on the complement of a hyperplane-like divisor	
	49
	References 53

Anthony P. Bahri, Martin Bendersky, Frederick R. Cohen
and Samuel Gitler

A survey of some recent results concerning polyhedral products

1	Introduction	55
2	Definitions	57
3	Four examples	62
4	Decompositions of suspensions	64
5	Applications to cohomology rings	68
6	Proof of a classical decomposition	72
7	Problems	75
	References	76

Enrique Artal Bartolo, Jose Ignacio Cogolludo-Agustín
and Anatoly Libgober

Characters of fundamental groups of curve complements and orbifold pencils

1	Introduction	81
2	Preliminaries	84
	2.1 Characteristic varieties	84
	2.2 Essential coordinate components	87
	2.3 Alexander invariant	87
	2.4 Orbicurves	90
	2.5 Orbifold pencils on quasi-projective manifolds	91
	2.6 Structure of characteristic varieties (revisited)	92
	2.7 Zariski pairs	93
3	Examples of characters of depth 3: Fermat Curves	94
	3.1 Fundamental group	94
	3.2 Essential coordinate characteristic varieties	96
	3.3 Marked orbifold pencils	99
4	Order two characters: augmented Ceva	101
	4.1 Ceva pencils and augmented Ceva pencils	102
	4.2 Characteristic varieties of \mathcal{C}_i , $i = 6, 7, 8, 9$	102
	4.3 Comments on independence of pencils	105
5	Curve arrangements	106
	References	107

Nicole Berline and Michèle Vergne

Analytic continuation of a parametric polytope and wall-crossing

		111
1	Introduction	112
2	Definition of the analytic continuation	126
	2.1 Some cones related to a partition polytope	126

2.2	Vertices and faces of a partition polytope	127
2.3	The Brianchon-Gram function	129
3	Signed sums of quadrants	131
3.1	Continuity properties of the Brianchon-Gram function	131
3.2	Polarized sums	135
4	Wall-crossing	139
4.1	Combinatorial wall-crossing	139
4.2	Semi-closed partition polytopes	143
4.3	Decomposition in quadrants	146
4.4	Geometric wall-crossing	147
4.5	An example	147
5	Integrals and discrete sums over a partition polytope . . .	150
5.1	Generating functions of polyhedra and Brion's theorem	150
5.2	Polynomiality and wall-crossing for integrals and sums	152
5.3	Paradan's convolution wall-crossing formulas . . .	157
6	A refinement of Brion's theorem	159
6.1	Brianchon-Gram continuation of a face of a partition polytope	162
7	Cohomology of line bundles over a toric variety	167
	References	171

Carl-Friedrich Bödigheimer and Ulrike Tillmann

**Embeddings of braid groups into mapping class groups
and their homology**

1	Introduction	173
2	Non-geometric embeddings	174
2.1	Mirror construction	175
2.2	Szepietowski's construction	177
2.3	Geometric embedding and orientation cover . . .	178
2.4	Geometric embedding and mirror construction .	179
2.5	Operadic embedding	179
2.6	More constructions	180
3	Proving non-geometricity	180
4	Action of the braid group operad	182
5	The maps in stable homology	184
6	Calculations in unstable homology	186
	References	190

Filippo Callegaro, Frederick R. Cohen and Mario Salvetti
**The cohomology of the braid group B_3 and of $SL_2(\mathbb{Z})$
 with coefficients in a geometric representation**

1	Introduction	193
2	General results and main tools	195
3	Main results	198
4	Methods	204
5	Principal congruence subgroups	207
6	Topological comparisons and speculations	208
	References	209

Daniel C. Cohen, Michael Falk and Richard Randell
Pure braid groups are not residually free

1	Introduction	213
2	Automorphisms of the pure braid group	214
3	Epimorphisms to free groups	216
4	Epimorphisms of the lower central series Lie algebra	223
5	Pure braid groups are not residually free	224
	References	229

Alexandru Dimca and Gus Lehrer

**Hodge-Deligne equivariant polynomials and monodromy
 of hyperplane arrangements**

1	Introduction	231
2	Equivariant Hodge-Deligne polynomials	235
3	Localization at the singular locus	237
4	Monodromy of Milnor fibers of line arrangements	240
	4.1 Proof of Theorem 1.1	240
	4.2 Proof of Theorem 1.3	241
	4.3 Proof of Theorem 1.5	242
5	Computation of Hodge-Deligne polynomials for line arrangements	242
A	Appendix	246
	A.1 The setting	246
	A.2 Background in p -adic Hodge theory	246
	A.3 Cohomology and eigenvalues of Frobenius	247
	A.4 Filtrations and comparison theorems	247
	A.5 Katz's theorem	248
	A.6 Equivariant theory	250
	A.7 Further remarks	251
	References	252

Michael J. Falk and Alexander N. Varchenko
**The contravariant form on singular vectors
of a projective arrangement**

1	Introduction	255
2	Flag complex and contravariant form of a central arrangement	259
3	Projective OS algebra and flag space	262
4	Singular subspace and contravariant form for projective arrangements	263
5	Dehomogenization	264
6	Transition functions	269
	References	271

Chad Giusti and Dev Sinha
**Fox-Neuwirth cell structures
and the cohomology of symmetric groups**

1	Introduction	273
2	The Fox-Neuwirth cochain complexes	274
	2.1 Fox-Neuwirth cells	275
	2.2 The cochain complexes	279
3	The cohomology of BS_4	281
	3.1 Block symmetry and skyline diagrams	283
	3.2 Geometry and characteristic classes	284
4	Hopf ring structure and presentation	285
	4.1 Cup product structure	286
	4.2 Hopf ring structure	288
	4.3 Hopf ring presentation	289
5	The skyline basis	290
6	The cohomology of BS_∞ as an algebra over the Steenrod algebra	292
	6.1 Nakaoka's theorem revisited	292
	6.2 Steenrod structure	293
	6.3 Madsen's theorem revisited	295
	References	296

Eddy Godelle and Luis Paris
Basic questions on Artin-Tits groups

1	Introduction	299
2	Conjectures	301
3	From free of infinity Artin-Tits groups to Artin-Tits groups	303
	References	310

Anthony Henderson

Rational cohomology of the real Coxeter toric variety of type A

1	Introduction	313
2	The De Concini–Procesi model	315
3	Poset homology and the proof of Theorem 1.1	319
4	Comparison with the moduli space of stable genus-zero curves	322
	References	325

Hidehiko Kamiya, Akimichi Takemura and Hiroaki Terao

Arrangements stable under the Coxeter groups

1	Introduction	327
2	Main results	328
3	Examples	333
3.1	Catalan arrangement	333
3.2	Coxeter group of type A_{m-1} and restricted all-subset arrangement	339
3.3	Coxeter group of type A_{m-1} and unrestricted all-subset arrangement	343
3.4	Mid-hyperplane arrangement	347
3.5	Signed all-subset arrangement	349
	References	352

Toshitake Kohno

Quantum and homological representations of braid groups

1	Introduction	355
2	Lawrence-Krammer-Bigelow representations	356
3	Discriminantal arrangements	358
4	KZ connection	363
5	Solutions of KZ equation by hypergeometric integrals	365
6	Relation between LKB representation and KZ connection	367
	References	371

Andrey Levin and Alexander Varchenko

Cohomology of the complement to an elliptic arrangement

1	Introduction	373
2	Cohomology of an elliptic discriminantal arrangement	374
3	Differential forms of a discriminantal arrangement	375
3.1	Combinatorial space	375
3.2	Rational representation	375
3.3	Theta representation.	376

4	Transversal elliptic hyperplanes	377
4.1	Elliptic hyperplanes in E^k	377
4.2	Intersection of $\ell \leq k$ transversal elliptic hyperplanes	378
4.3	Differential forms of k transversal hyperplanes in \mathcal{E}	379
5	Arbitrary elliptic arrangement	383
5.1	An elliptic arrangement	383
5.2	Differential k -forms of an elliptic arrangement	384
5.3	Cohomology of the complement	385
	References	387

Ivan Marin

**Residual nilpotence for generalizations
of pure braid groups**

1	Introduction	389
2	Artin groups and Paris representation	391
2.1	Preliminaries on Artin groups	391
2.2	Paris representation	392
2.3	Reduction modulo h	394
3	Braid groups of complex reflection groups	395
	References	399

Davide Moroni, Mario Salvetti and Andrea Villa

**Some topological problems on the configuration spaces
of Artin and Coxeter groups**

1	Introduction	403
2	General pictures	404
2.1	Case \mathbf{W} finite	404
2.2	Case \mathbf{W} infinite	406
2.3	Algebraic complexes for Artin groups	410
2.4	CW -complexes for Coxeter groups	413
2.5	Algebraic complexes for Coxeter groups	414
3	Applications	415
3.1	Cohomology of Artin groups over non-abelian representations	415
3.2	The genus problem for infinite type Artin groups	418
	References	429

John Shareshian and Michelle L. Wachs

**Chromatic quasisymmetric functions
and Hessenberg varieties**

1	Introduction	433
---	------------------------	-----

1.1	Preliminaries	436
2	q -Eulerian numbers and toric varieties	439
2.1	Action of the symmetric group	439
2.2	Expansion in the basis of fundamental quasisymmetric functions	440
2.3	Unimodality	443
2.4	Smirnov words	444
3	Rawlings major index and generalized Eulerian numbers	445
4	Chromatic quasisymmetric functions	448
4.1	Stanley's chromatic symmetric functions	448
4.2	A quasisymmetric refinement	450
4.3	Schur and power sum decompositions	453
4.4	Fundamental quasisymmetric function basis decomposition	454
5	Hessenberg varieties	455
	References	458

Alexander I. Suci

**Geometric and homological finiteness
in free abelian covers**

1	Introduction	461
1.1	Finiteness properties	461
1.2	Characteristic varieties and Ω -sets	462
1.3	Comparing the Ω -sets and the Σ -sets	463
1.4	Formality, straightness, and resonance	464
1.5	Applications	464
2	Characteristic and resonance varieties	466
2.1	Jump loci for twisted homology	466
2.2	Tangent cones and exponential tangent cones	467
2.3	Resonance varieties	468
2.4	Tangent cone and resonance	470
2.5	Formality	470
2.6	Straightness	471
3	The Dwyer–Fried invariants	471
3.1	Betti numbers of free abelian covers	472
3.2	Dwyer–Fried invariants and characteristic varieties	473
3.3	Straightness and the Ω -sets	474
4	The Bieri–Neumann–Strebel–Renz invariants	475
4.1	A finite type property	475
4.2	The Σ -invariants of a chain complex	476
4.3	The Σ -invariants of a CW-complex	477
4.4	Generalizations and discussion	477

4.5	Novikov homology	479
4.6	Σ -invariants and characteristic varieties	479
5	Relating the Ω -invariants and the Σ -invariants	480
5.1	Finiteness properties of abelian covers	481
5.2	Upper bounds for the Σ - and Ω -invariants	481
5.3	The straight and formal settings	483
5.4	Discussion and examples	483
6	Toric complexes	484
6.1	Toric complexes and right-angled Artin groups	485
6.2	Jump loci and Ω -invariants	485
6.3	Σ -invariants	486
7	Quasi-projective varieties	487
7.1	Complex algebraic varieties	487
7.2	Cohomology jump loci	488
7.3	Ω -invariants	489
7.4	Σ -invariants	490
7.5	Examples and discussion	491
8	Configuration spaces	492
8.1	Ordered configurations on algebraic varieties	492
8.2	Ordered configurations on the torus	493
9	Hyperplane arrangements	494
9.1	Complement and intersection lattice	494
9.2	Cohomology jump loci	495
9.3	Upper bounds for the Omega- and Sigma-sets	496
9.4	Lower bounds for the Σ -sets	497
9.5	Discussion and examples	498
	References	499

Masahiko Yoshinaga

Minimal stratifications for line arrangements and positive homogeneous presentations for fundamental groups

1	Introduction	503
2	A one-dimensional example	505
3	Basic notation	506
3.1	Setting	506
3.2	Generic flags and numbering of lines	506
3.3	Assumptions on generic flag and numbering	507
3.4	Sails bound to lines	507
3.5	Orientations	508
4	Minimal stratification	510
4.1	Main result	510
5	Dual presentation for the fundamental group	510

5.1	Transversal generators	511
5.2	Chamber relations	511
5.3	Dual presentation	513
5.4	Pivotal argument	513
5.5	Proof of Theorem 5.3	515
5.6	Examples	515
5.7	Twisted minimal chain complex	517
6	Positive homogeneous presentations	518
6.1	Left and right lines	518
6.2	Positive homogeneous relations	519
6.3	Proof of Theorem 6.5	520
7	Proofs of main results	522
7.1	Tangent bundle description	522
7.2	Contractibility of S_i°	524
7.3	Contractibility of U	525
	References	531

Authors' affiliations

A. ADEM – Department of Mathematics, University of British Columbia,
Vancouver BC V6T 1Z2, Canada
adem@math.ubc.ca

M. AMRAM – Shamoon College of Engineering, Bialik/Basel Sts., 84100
Beer-Sheva, Israel
and
Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Is-
rael
meiravt@sce.ac.il, meirav@macs.biu.ac.il

D. ARINKIN – Department of Mathematics, University of North Carolina
at Chapel Hill Chapel Hill, NC 27599-3250, USA
arinkin@email.unc.edu

A. BAHRI – Department of Mathematics, Rider University, Lawrenceville,
NJ 08648, U.S.A.
bahri@rider.edu

E. ARTAL BARTOLO – Departamento de Matemáticas, IUMA, Universi-
dad de Zaragoza, C. Pedro Cerbuna 12, 50009 Zaragoza, Spain
artal@unizar.es

M. BENDERSKY – Department of Mathematics, CUNY, East 695 Park
Avenue New York, NY 10065, U.S.A.
mbenders@xena.hunter.cuny.edu

N. BERLINE – Ecole Polytechnique, Centre de Mathematiques Laurent
Schwartz, 91128 Palaiseau Cedex, France
nicole.berline@math.polytechnique.fr

C.-F. BÖDIGHEIMER – Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
boedigheimer@math.uni-bonn.de

F. CALLEGARO – Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italia
f.callegaro@sns.it

J. I. COGOLLUDO-AGUSTÍN – Departamento de Matemáticas, IUMA, Universidad de Zaragoza, C. Pedro Cerbuna 12, 50009 Zaragoza, Spain
jicogo@unizar.es

D. C. COHEN – Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA
cohen@math.lsu.edu

F. R. COHEN – UR Mathematics, 915 Hylan Building, University of Rochester, RC Box 270138, Rochester, NY 14627, USA
cohf@math.rochester.edu

A. DIMCA – Laboratoire J.A. Dieudonné, UMR du CNRS 6621, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
dimca@unice.fr

M. FALK – Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, Arizona 86011-5717, USA
michael.falk@nau.edu

D. GARBER – Department of Applied Mathematics, Faculty of Sciences, Holon Institute of Technology, 52 Golomb St., PO Box 305, 58102 Holon, Israel
garber@hit.ac.il

S. GITLER – Department of Mathematics, Cinvestav, San Pedro Zacatenco, Mexico, D.F. CP 07360 Apartado Postal 14-740, Mexico
samuel.gitler@gmail.com

C. GIUSTI – Mathematics Department, Willamette University, 219 Ford Hall, 900 State Street Salem, Oregon 97301, USA
cgiusti@willamette.edu

E. GODELLE – Université de Caen, Laboratoire LMNO, UMR 6139 du CNRS, Campus II, 14032 Caen cedex, France
eddy.godelle@unicaen.fr

J. M. GÓMEZ – Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
jgomez@math.jhu.edu

A. HENDERSON – School of Mathematics and Statistics, University of Sydney NSW 2006, Australia
anthony.henderson@sydney.edu.au

H. KAMIYA – School of Economics, Nagoya University, Nagoya, 464-8601, Japan
kamiya@soec.nagoya-u.ac.jp

T. Kohno – IPMU, Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
kohno@ms.u-tokyo.ac.jp

G.LEHRER – School of Mathematics and Statistics F07, University of Sydney, NSW 2006, Australia
gustav.lehrer@sydney.edu.au

A. LEVIN – State University - Higher School of Economics, Department of Mathematics, 20 Myasnitskaya Ulitsa, Moscow, 101000, Russia
and
Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova Street, Moscow, 117312, Russia
alevin@hse.ru

A. LIBGOBER – Department of Mathematics, University of Illinois, 851 S. Morgan Str., Chicago, IL 60607
libgober@math.uic.edu

I. MARIN – Université Paris Diderot 175 rue du Chevaleret 75013 Paris
marin@math.jussieu.fr

D. MORONI – Istituto di Scienza e Tecnologia dell'Informazione "Alessandro Faedo" (ISTI), Area della Ricerca CNR di Pisa, Via G. Moruzzi, 1, 56124 Pisa, Italia
davide.moroni@isti.cnr.it

L. PARIS – Université de Bourgogne, Institut de Mathématiques de Bourgogne, UMR 5584 du CNRS, B.P. 47870, 21078 Dijon cedex, France
lparis@u-bourgogne.fr

R. RANDELL – Department of Mathematics, University of Iowa, Iowa City, Iowa 52242, USA
richard-randell@uiowa.edu

M. SALVETTI – Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, 56127 Pisa, Italia
salvetti@dm.unipi.it

J. SHARESHEAN – Department of Mathematics, Washington University, St. Louis, MO 63130, USA
shareshi@math.wustl.edu

D. SINHA – Mathematics Department, University of Oregon, Eugene, OR 97403, USA
dps@math.uoregon.edu

A. I. SUCIU – Department of Mathematics, Northeastern University, Boston, MA 02115, USA
a.suciu@neu.edu

A. TAKEMURA – Department of Mathematical Informatics, University of Tokyo, Bunkyo Tokyo, 113-0033, Japan
takemura@stat.u-tokyo.ac.jp

M. TEICHER – Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
teicher@macs.biu.ac.il

H. TERAOKA – Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
hteraoka00@za3.so-net.ne.jp

U. TILLMANN – Mathematical Institute, Oxford University, 24-29 St Giles, Oxford OX1 3LB, United Kingdom
tillmann@maths.ox.ac.uk

A. VARCHENKO – Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA
anv@email.unc.edu

M. VERGNE – Université Paris 7 Denis Diderot, Institut Mathématique de Jussieu, 175 rue du Chevaleret - 75013 Paris, France
vergne@math.jussieu.fr

A. VILLA – Dipartimento di Matematica “Giuseppe Peano”, Università di Torino, Via Carlo Alberto, 10, 10123 Torino, Italia
andrea.villa@unito.it

M. L. WACHS – Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA
wachs@math.miami.edu

M. YOSHINAGA – Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo, 060-0810, Japan
mhyo@math.kyoto-u.ac.jp

Introduction

The present volume arises from the intensive research period “Configuration spaces: Geometry, Combinatorics and Topology” which took place at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, in May-June 2010.

The program was very intense and included two conferences, nine minicourses and two weekly seminars one of which organized by Filippo Callegaro and run by the younger participants.

The period covered a large number of different topics all centering around the notion of configuration spaces. These included among others:

1. Study of local systems on the complements of a hyperplane arrangements. Characteristic and resonance varieties. Cohomology and monodromy computations. Study of the fundamental group of complements of arrangements. Hodge theoretical aspects.
2. Qualitative and quantitative problems related to the study of partition functions. Relations with the theory of Box Splines and applications to index theory. Enumeration of lattice points in rational polytopes. Relations with toric geometry.
3. Invariants of braids and knots. Topological quantum field theory. Applications to low dimensional topology. Construction of representations of braid groups and related groups.
4. Applications of the theory of hyperplane arrangements and toric arrangements to the study of the combinatorics of matroids and generalizations.
5. Homotopy theory aspects of the study of configuration spaces and moduli spaces. Relations between cohomology of braid and mapping class groups.
6. Combinatorial aspects of the theory of Coxeter and Artin groups. Cohomological computations for both abelian and non abelian local systems.

7. Toric geometry. Moment angle complexes and applications. Geometry and topology of real and complex toric varieties.

Many of these topics were covered by the minicourses whose list we include here:

Fred Cohen: Moment-angle complexes, their stable structure and cohomology.

Eduard Looijenga: Aspects of the KZ system.

Stefan Papadima, Alex Suciu: Cohomology jumping loci and homological finiteness properties.

Claudio Procesi: Splines and partition functions

Dev Sinha: Hopf rings in topology and algebra.

Alexander Varchenko: The quantum integrable model of an arrangement of hyperplanes.

Michele Vergne: Remarks on Box splines.

Sergey Yuzvinsky: A short introduction to arrangements of hyperplanes

Sergey Yuzvinsky: Resonance varieties for arrangements and their relations to combinatorics and algebraic geometry

We feel that the papers appearing in this volume very well represent the spirit and the topics of the research period.

We very warmly thank both the authors of these papers and the other participants who very much contributed to the success of this activity.

Finally we wish to thank the staff of Centro di Ricerca Ennio De Giorgi and of the Scientific Section of Edizioni della Scuola Normale for a very efficient job both in running our period and in editing the present volume.

Pisa, July 2012

A. Björner, F. Cohen, C. De Concini,
C. Procesi and M. Salvetti

On the structure of spaces of commuting elements in compact Lie groups

Alejandro Adem* and José Manuel Gómez

Abstract. In this note we study topological invariants of the spaces of homomorphisms $\text{Hom}(\pi, G)$, where π is a finitely generated abelian group and G is a compact Lie group arising as an arbitrary finite product of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$.

1 Introduction

Let \mathcal{P} denote the class of compact Lie groups arising as arbitrary finite products of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. In this article we use methods from algebraic topology to study the spaces of homomorphisms $\text{Hom}(\pi, G)$ where π denotes a finitely generated abelian group and $G \in \mathcal{P}$. Our main interest is the computation of invariants associated to these spaces such as their cohomology and stable homotopy type, as well as their equivariant K -theory with respect to the natural conjugation action. The natural quotient space under this action is the space of representations $\text{Rep}(\pi, G)$, which can be identified with the moduli space of isomorphism classes of flat connections on principal G -bundles over M , where M is a compact connected manifold with $\pi_1(M) = \pi$. Thus our results provide insight into these geometric invariants in the important case when $\pi_1(M)$ is a finitely generated abelian group.

Our starting point is the observation (see [3]) that when $G \in \mathcal{P}$ and π is a finitely generated abelian group, the conjugation action of G on the space of homomorphisms $\text{Hom}(\pi, G)$ satisfies the following property: for every element $x \in \text{Hom}(\pi, G)$ the isotropy subgroup G_x is connected and of maximal rank. This property plays a central part in our analysis. Indeed, let $T \subset G$ be a maximal torus; in general if a compact Lie group

*Partially supported by NSERC.

G acts on a compact space X with connected maximal rank isotropy subgroups then there is an associated action of W on the fixed-point set X^T and many properties of the space X are determined by the action of W on X^T (see [3, 8]). For our examples this means that a detailed understanding of the W -action on the subspace $\text{Hom}(\pi, G)^T = \text{Hom}(\pi, T)$ can be used to describe key homotopy-theoretic invariants for the original space of homomorphisms.

This approach can be used for example to obtain an explicit description of the number of path-connected components in $\text{Hom}(\pi, G)$. Indeed we show that if $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group, then the number of path-connected components in $\text{Hom}(\pi, G)$ equals the number of distinct orbits for the action of W on $\text{Hom}(A, T)$.

In [1] a stable splitting for the spaces of commuting n -tuples in G , $\text{Hom}(\mathbb{Z}^n, G)$, was derived for any Lie group G that is a closed subgroup of $GL_n(\mathbb{C})$. Here we show that this splitting can be generalized to the spaces of homomorphisms $\text{Hom}(\pi, G)$ when $G \in \mathcal{P}$ and π is any finitely generated abelian group. This is done by constructing a stable splitting on $\text{Hom}(\pi, G)^T = \text{Hom}(\pi, T)$ and proving that this splitting lifts to the space $\text{Hom}(\pi, G)$. Suppose that $\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n)$, where $n \geq 0$ and q_1, \dots, q_n are integers. Here we allow some of the q_i 's to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. Thus $\text{Hom}(\pi, G)$ can be seen as the subspace of G^n consisting of those commuting n -tuples (x_1, \dots, x_n) such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. For $1 \leq r \leq n$ let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \leq m_1 < \cdots < m_r \leq n\}$. Given such a sequence \mathfrak{m} let $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ be a quotient of π . Let $S_1(P_{\mathfrak{m}}(\pi), G)$ be the subspace of $\text{Hom}(P_{\mathfrak{m}}(\pi), G)$ consisting of those r -tuples $(x_{m_1}, \dots, x_{m_r})$ in $\text{Hom}(P_{\mathfrak{m}}(\pi), G)$ for which at least one of the x_{m_i} 's is equal to 1_G .

Theorem 1.1. *Suppose that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Then there is a G -equivariant homotopy equivalence*

$$\Theta : \Sigma \text{Hom}(\pi, G) \rightarrow \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee_{\mathfrak{m} \in J_{n,r}} \text{Hom}(P_{\mathfrak{m}}(\pi), G) / S_1(P_{\mathfrak{m}}(\pi), G) \right).$$

In Section 4 we determine the homotopy type of the stable factors appearing in the previous theorem for certain particular cases. In particular we determine the stable homotopy type of $\text{Hom}(\pi, SU(2))$ for any finitely generated abelian group.

Suppose now that G is any compact Lie group. The fundamental group of the spaces of homomorphisms of the form $\text{Hom}(\mathbb{Z}^n, G)$ was computed

in [7]. Let $\mathbb{1} \in \text{Hom}(\mathbb{Z}^n, G)$ be the trivial representation. If $\mathbb{1}$ is chosen as the base point, then by [7, Theorem 1.1] there is a natural isomorphism $\pi_1(\text{Hom}(\mathbb{Z}^n, G)) \cong (\pi_1(G))^n$. Here we show that the methods applied in [7] can be used to compute $\pi_1(\text{Hom}(\pi, G))$ for any choice of base point if we further require that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Write π in the form $\pi = \mathbb{Z}^n \oplus A$, with A a finite abelian group. Then the space of homomorphisms $\text{Hom}(\pi, G)$ can naturally be identified as a subspace of the product $\text{Hom}(\mathbb{Z}^n, G) \times \text{Hom}(A, G)$. Given $f \in \text{Hom}(A, T)$ let

$$\mathbb{1}_f := \mathbb{1} \times f \in \text{Hom}(\pi, G) \subset \text{Hom}(\mathbb{Z}^n, G) \times \text{Hom}(A, G).$$

Every path-connected component in $\text{Hom}(\pi, G)$ contains some $\mathbb{1}_f$ and thus it suffices to consider the elements of the form $\mathbb{1}_f$ as base points in $\text{Hom}(\pi, G)$. With this in mind we have the following.

Theorem 1.2. *Let $\pi = \mathbb{Z}^n \oplus A$, with A a finite abelian group and let $G \in \mathcal{P}$. Suppose $f \in \text{Hom}(A, T)$ and take $\mathbb{1}_f$ as the base point of $\text{Hom}(\pi, G)$. Then there is a natural isomorphism $\pi_1(\text{Hom}(\pi, G)) \cong (\pi_1(G_f))^n$ where $G_f = Z_G(f)$ is the subgroup of elements in G commuting with $f(x)$ for all $x \in A$.*

In Section 6 we study the equivariant K -theory of the spaces of homomorphisms $\text{Hom}(\pi, G)$ with respect to the conjugation action by G . When π is a finite group, then $\text{Hom}(\pi, G)$ is the disjoint union of homogeneous spaces of the form G/H where H is a maximal rank subgroup. Using this it is easy to see that $K_G^*(\text{Hom}(\pi, G))$ is a free module over the representation ring of rank $|\text{Hom}(\pi, T)|$. This result can be generalized for finitely generated abelian groups of rank 1 in the following way.

Theorem 1.3. *Suppose that $G \in \mathcal{P}$ is simply connected and of rank r . Let $\pi = \mathbb{Z} \oplus A$ where A is a finite abelian group. Then $K_G^*(\text{Hom}(\pi, G))$ is a free $R(G)$ -module of rank $2^r \cdot |\text{Hom}(A, T)|$.*

It turns out that $K_G^*(\text{Hom}(\pi, G))$ is not always free as a module over $R(G)$. In fact, as was pointed out in [3], the $R(SU(2))$ -module $K_{SU(2)}^*(\text{Hom}(\mathbb{Z}^2, SU(2)))$ is not free. However, $K_{SU(2)}^*(\text{Hom}(\mathbb{Z}^2, SU(2))) \otimes \mathbb{Q}$ turns out to be free as a module over $R(SU(2)) \otimes \mathbb{Q}$. The next theorem shows that a similar result holds for all the spaces of homomorphisms that we consider here.

Theorem 1.4. *Suppose that $G \in \mathcal{P}$ is of rank r and that π is a finitely generated abelian group written in the form $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group. Then $K_G^*(\text{Hom}(\pi, G)) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank $2^{nr} \cdot |\text{Hom}(A, T)|$.*

The layout of this article is as follows. In Section 2 some general properties of the spaces of homomorphisms $\text{Hom}(\pi, G)$ are determined. In Section 3 we study the cohomology groups with rational coefficients of these spaces. In Section 4 Theorem 1.1 is proved and some explicit examples are computed. In Section 5 the fundamental group of the spaces $\text{Hom}(\pi, G)$ are computed for any choice of base point. Finally, in Section 6 we study the problem of computing $K_G^*(\text{Hom}(\pi, G))$, where G acts by conjugation on $\text{Hom}(\pi, G)$.

Both authors would like to thank the Centro di Ricerca Matematica Ennio De Giorgi at the Scuola Normale Superiore in Pisa for inviting them to participate in the program on Configuration Spaces: Geometry, Combinatorics and Topology during the spring of 2010.

2 Preliminaries on spaces of commuting elements

Let π be a finitely generated discrete group and G a Lie group. Consider the set of homomorphisms from π to G , $\text{Hom}(\pi, G)$. This set can be given a topology as a subspace of a finite product of copies of G in the following way. Fix a set of generators e_1, \dots, e_n of π and let F_n be the free group on n -letters. By mapping the generators of F_n onto the different e_i 's we obtain a surjective homomorphism $F_n \rightarrow \pi$. This surjection induces an inclusion of sets $\text{Hom}(\pi, G) \hookrightarrow \text{Hom}(F_n, G) \cong G^n$. This way $\text{Hom}(\pi, G)$ can be given the subspace topology. It is easy to see that this topology is independent of the generators chosen for π . In case π happens to be abelian, then any map $F_n \rightarrow \pi$ factors through $F_n \rightarrow \mathbb{Z}^n \rightarrow \pi$ yielding an inclusion of spaces $\text{Hom}(\pi, G) \hookrightarrow \text{Hom}(\mathbb{Z}^n, G) \hookrightarrow G^n$. Thus the space of homomorphisms $\text{Hom}(\pi, G)$ can be seen as a subspace of the space of commuting n -tuples in G , $\text{Hom}(\mathbb{Z}^n, G)$.

In this note we collect some facts about these spaces of homomorphisms in the particular case that π is a finitely generated abelian group and G belongs to a suitable family of Lie groups. We are mainly interested in the following family of Lie groups.

Definition 2.1. Let \mathcal{P} denote the collection of all compact Lie groups arising as finite cartesian products of the groups $SU(r)$, $U(q)$ and $Sp(k)$.

Whenever G belongs to the family \mathcal{P} the space of homomorphisms $\text{Hom}(\pi, G)$ satisfies the following crucial condition as we prove below in Proposition 2.3.

Definition 2.2. Let X be a G -space. The action of G on X is said to have connected maximal rank isotropy subgroups if for every $x \in X$, the isotropy group G_x is a connected subgroup of maximal rank; that is, for every $x \in X$ we can find a maximal torus T_x in G such that $T_x \subset G_x$.

Proposition 2.3. *Suppose that π is a finitely generated abelian group and $G \in \mathcal{P}$. Then the conjugation action of G on $\text{Hom}(\pi, G)$ has connected maximal rank isotropy subgroups.*

Proof. Choose generators e_1, \dots, e_n of π . As pointed out above we can use these generators to obtain an inclusion of G -spaces $\text{Hom}(\pi, G) \hookrightarrow \text{Hom}(\mathbb{Z}^n, G)$. Given this inclusion it suffices to show that the conjugation action of G on $\text{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy groups. In [3, Example 2.4] it was proven that the action of G on $\text{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy subgroups if and only if $\text{Hom}(\mathbb{Z}^{n+1}, G)$ is path-connected. The proposition follows by noting that $\text{Hom}(\mathbb{Z}^k, G)$ is path-connected for all $k \geq 0$ whenever $G \in \mathcal{P}$. \square

Suppose that a compact Lie group G acts on a space X with connected maximal rank isotropy subgroups. Choose a maximal torus T in G and let W be the Weyl group. By passing to the level of T -fixed points, the action of G on X induces an action of the Weyl group W on X^T . Many properties of the action of G on X are determined by the action of W on X^T as explained in [8] and in some situations the former is completely determined by the latter up to isomorphism. For example, we can use this approach to produce G -CW complex structures on the spaces of homomorphisms as is proved next.

Corollary 2.4. *Suppose that π is a finitely generated abelian group and $G \in \mathcal{P}$. Then $\text{Hom}(\pi, G)$ with the conjugation action has the structure of a G -CW complex.*

Proof. Since π is a finitely generated abelian group it can be written in the form $\pi = \mathbb{Z}^n \oplus A$, where A is a finite abelian group. Let $X := \text{Hom}(\pi, G)$ with the conjugation action of G . Note that $X^T = \text{Hom}(\pi, G)^T = T^n \times \text{Hom}(A, T)$. Since $\text{Hom}(A, T)$ is a discrete set, it follows that X^T has the structure of a smooth manifold on which W acts smoothly. In particular, by [9, Theorem 1] it follows that X^T has the structure of a W -CW complex. Since the conjugation action of G on X has connected maximal rank isotropy subgroups then by [3, Theorem 2.2] it follows that this W -CW complex structure on X^T induces a G -CW complex on X . \square

This approach can also be used to determine explicitly the structure of these spaces of homomorphisms whenever π is a finite abelian group.

Proposition 2.5. *Suppose that π is a finite abelian group and $G \in \mathcal{P}$. Then there is a G -equivariant homeomorphism*

$$\Phi : \text{Hom}(\pi, G) \rightarrow \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} G/G_f.$$

Here $[f]$ runs through a system of representatives of the W -orbits in $\text{Hom}(\pi, T)$ and each G_f is a maximal rank subgroup with $W(G_f) = W_f$.

Proof. Consider the G -space $X := \text{Hom}(\pi, G)$. Note that $X^T = \text{Hom}(\pi, T)$ is a discrete set endowed with an action of W . By decomposing X^T into the different W -orbits we obtain a W -equivariant homeomorphism

$$X^T \cong \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} W/W_f.$$

Here $[f]$ runs through a set of representatives for the action of W on $\text{Hom}(\pi, T)$. For each $f \in \text{Hom}(\pi, T)$ let G_f denote the subgroup of elements in G commuting with $f(x)$ for all $x \in \pi$. This group is a maximal rank subgroup in G as $T \subset G_f$. Moreover, by [8, Theorem 1.1] it follows that $W(G_f) = W_f$. Also note that if we let G act on the left on the homogeneous space G/G_f then $(G/G_f)^T = W/W_f$. Let

$$Y = \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} G/G_f.$$

The left action of G on Y has maximal rank isotropy and there is a W -equivariant homeomorphism $\phi : X^T \rightarrow Y^T$. By [8, Theorem 2.1] there is a unique G -equivariant extension $\Phi : X \rightarrow Y$ of ϕ and this map is in fact a homeomorphism. \square

3 Rational cohomology and path-connected components

In this section we explore the set of path connected components and the rational cohomology groups of the spaces of homomorphisms $\text{Hom}(\pi, G)$.

Suppose that G is a compact connected Lie group and let T be a maximal torus in G . Assume that G acts on a space X of the homotopy type of a G -CW complex with maximal rank isotropy subgroups. Consider the continuous map

$$\begin{aligned} \phi : G \times X^T &\rightarrow X \\ (g, x) &\mapsto gx. \end{aligned}$$

Since G acts on X with maximal rank isotropy subgroups for every $x \in X$ we can find a maximal torus T_x in G such that $T_x \subset G_x$. As every pair of maximal tori in G are conjugate it follows that for every $x \in X$ we can find some $g \in G$ such that $gx \in X^T$. This shows that ϕ is a surjective map. The normalizer of T in G , $N_G(T)$ acts on the right on $G \times X^T$ by

$(g, x) \cdot n = (gn, n^{-1}x)$ and the map ϕ is invariant under this action. Thus ϕ descends to a surjective map

$$\begin{aligned} \varphi : G \times_{N_G(T)} X^T &= G/T \times_W X^T \rightarrow X \\ [g, x] &\mapsto gx \end{aligned}$$

The map φ is not injective in general. Indeed, as was proven in [4], given $x \in X$ there is a homeomorphism $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$, where G_x^0 denotes the path-connected component of G_x containing the identity element. Let \mathbb{F} be a field with characteristic relatively prime to $|W|$. Then as observed in [4] the space $G_x^0/N_{G_x^0}(T)$ has \mathbb{F} -acyclic cohomology. The Vietoris-Begle theorem shows that φ induces an isomorphism in cohomology with \mathbb{F} -coefficients. As a consequence we obtain the following proposition (first proved in [4]).

Proposition 3.1. *Suppose that G is a compact connected Lie group acting on a spaces X with maximal rank isotropy subgroups. If \mathbb{F} is a field with characteristic relatively prime to $|W|$ then*

$$H^*(X; \mathbb{F}) \cong H^*(G/T \times_W X^T; \mathbb{F}) \cong H^*(G/T \times X^T; \mathbb{F})^W.$$

Remark 3.2. Suppose that G acts on X with *connected* maximal rank isotropy groups. As pointed out above the map φ is not injective in general since $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$ for $x \in X$. Under the given hypothesis we have $G_x^0 = G_x$. By [8, Theorem 1.1] the assignment $(H) \mapsto (WH)$ defines a one to one correspondence between the set of conjugacy classes of isotropy subgroups of the action of G on X and the set of conjugacy classes of isotropy subgroups of the action of W on X^T . Thus the different isotropy subgroups of the action of W on X^T determine how far the map φ is from being injective. In particular, if W acts freely on X^T then φ is a continuous bijection and thus a homeomorphism if for example X^T is compact.

Suppose now that $G \in \mathcal{P}$ and let π be a finitely generated abelian group. By Proposition 2.3 the conjugation action of G on $\text{Hom}(\pi, G)$ has connected maximal rank isotropy subgroups. In this case $\text{Hom}(\pi, G)^T = \text{Hom}(\pi, T)$. As a consequence of the previous result the following is obtained.

Corollary 3.3. *Suppose that $G \in \mathcal{P}$ and let π be a finitely generated abelian group. Then there is an isomorphism $H^*(\text{Hom}(\pi, G); \mathbb{Q}) \cong H^*(G/T \times \text{Hom}(\pi, T); \mathbb{Q})^W$.*

As an application of Corollary 3.3 the following can be derived.

Corollary 3.4. *Suppose that $G \in \mathcal{P}$ and let π be a finitely generated abelian group written in the form $\pi = \mathbb{Z}^n \oplus A$. Then the number of path-connected components in $\text{Hom}(\pi, G)$ equals the number of different orbits of the action of W on $\text{Hom}(A, T)$*

4 Stable splittings

In this section we show that the fat wedge filtration on a finite product of copies of G induces a natural filtration on the spaces of homomorphisms $\text{Hom}(\pi, G)$. It turns out that this filtration splits stably after one suspension whenever π is a finitely generated abelian group and $G \in \mathcal{P}$.

Suppose that π is a finitely generated abelian group. Using the fundamental theorem of finitely generated abelian groups π can be written in the form

$$\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n),$$

where $n \geq 0$ and q_1, \dots, q_n are integers. Here we allow some of the q_i 's to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. This way we can see $\text{Hom}(\pi, G)$ as the subspace of G^n consisting of those commuting n -tuples (x_1, \dots, x_n) such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. The fat wedge filtration on G^n induces a natural filtration on the space of homomorphisms $\text{Hom}(\pi, G)$. To be more precise, for each $1 \leq j \leq n$ let

$$S_j(\pi, G) = \{(x_1, \dots, x_n) \in \text{Hom}(\pi, G) \subset G^n \mid x_i = 1_G \text{ for at least } j \text{ of the } x_i \text{'s}\}.$$

This way we obtain a filtration of $\text{Hom}(\pi, G)$

$$\{(1_G, \dots, 1_G)\} = S_n(\pi, G) \subset S_{n-1}(\pi, G) \subset \cdots \subset S_0(\pi, G) = \text{Hom}(\pi, G). \quad (4.1)$$

Note that each $S_j(\pi, G)$ is invariant under the conjugation action of G . In particular each $S_j(\pi, G)$ can be seen as a G -space that has connected maximal rank isotropy subgroups. On the level of the T -fixed points the filtration (4.1) induces a filtration of $\text{Hom}(\pi, G)^T$

$$\{(1_G, \dots, 1_G)\} = S_n(\pi, G)^T \subset S_{n-1}(\pi, G)^T \subset \cdots \subset S_0(\pi, G)^T = \text{Hom}(\pi, G)^T. \quad (4.2)$$

For each $1 \leq i \leq n$ consider $\text{Hom}(\mathbb{Z}/q_i, T) = \{t \in T \mid t^{q_i} = 1\}$. Note that each $\text{Hom}(\mathbb{Z}/q_i, T)$ is a space endowed with the action of W . Whenever $q_i = 0$ we have $\text{Hom}(\mathbb{Z}/q_i, T) = T$ and if $q_i \neq 0$ then $\text{Hom}(\mathbb{Z}/q_i, T)$ is a discrete set. Since T is abelian it follows that

$$\text{Hom}(\pi, G)^T = \text{Hom}(\pi, T) = \text{Hom}(\mathbb{Z}/q_1, T) \times \cdots \times \text{Hom}(\mathbb{Z}/q_n, T).$$

Moreover, the filtration (4.2) is precisely the fat wedge filtration of $\text{Hom}(\pi, G)^T$ where we identify $\text{Hom}(\pi, G)^T$ with the above product. It is well known that the fat wedge filtration on a product of spaces splits stably after one suspension. More precisely, for each $0 \leq j \leq n - 1$ we can find a continuous map

$$r_j : \Sigma S_j(\pi, G)^T \rightarrow \Sigma S_{j+1}(\pi, G)^T$$

in such a way that there is a homotopy h_j between $r_j \circ \Sigma(i_j)$ and $1_{\Sigma(S_{j+1}(\pi, G)^T)}$. Here

$$i_j : S_{j+1}(\pi, G)^T \rightarrow S_j(\pi, G)^T$$

denotes the inclusion map. Moreover, both the map r_j and the homotopy h_j can be arranged in such a way that they are W -equivariant. The W -action that we have in sight is the diagonal action of W on the product $\text{Hom}(\mathbb{Z}/q_1, T) \times \cdots \times \text{Hom}(\mathbb{Z}/q_n, T)$. Consider the action of G on $\Sigma \text{Hom}(\pi, G)$ with G acting trivially on the suspension component. This action has connected maximal rank isotropy subgroups and $(\Sigma \text{Hom}(\pi, G))^T = \Sigma \text{Hom}(\pi, T)$. By [8, Theorem 2.1] we can find a unique G -equivariant extension

$$R_j : \Sigma S_j(\pi, G) \rightarrow \Sigma S_{j+1}(\pi, G)$$

of r_j and a unique G -equivariant homotopy H_j between $R_j \circ \Sigma(I_j)$ and $1_{\Sigma(S_{j+1}(\pi, G))}$ extending h_j . Here $I_j : S_{j+1}(\pi, G) \rightarrow S_j(\pi, G)$ as before denotes the inclusion map.

Let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \leq m_1 < \cdots < m_r \leq n\}$. Note that $J_{n,r}$ contains precisely $\binom{n}{r}$ elements. Given such a sequence \mathfrak{m} , there is an associated abelian group $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ obtained as a quotient of π and also a G -equivariant projection map

$$P_{\mathfrak{m}} : \text{Hom}(\pi, G) \rightarrow \text{Hom}(P_{\mathfrak{m}}(\pi), G)$$

$$(x_1, \dots, x_n) \mapsto (x_{m_1}, \dots, x_{m_r}).$$

The above can be used to prove the following theorem.

Theorem 4.1. *Suppose that $G \in \mathcal{P}$ and that π is a finitely generated abelian group. Then there is a G -equivariant homotopy equivalence*

$$\Theta : \Sigma \text{Hom}(\pi, G) \rightarrow \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee_{\mathfrak{m} \in J_{n,r}} \text{Hom}(P_{\mathfrak{m}}(\pi), G) / S_1(P_{\mathfrak{m}}(\pi), G) \right).$$