Geometry, Structure and Randomness in Combinatorics

edited by
Jiří Matoušek, Jaroslav Nešetřil
and Marco Pellegrini

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Preface

On September 3-7, 2012, as part of the activities of the Mathematics Research Center "Ennio De Giorgi" and on the invitation of its director prof. Mariano Giaquinta, we organized the Workshop "Geometry, Structure and Randomness in Combinatorics" at Scuola Normale Superiore in Pisa. The workshop was organized by Jiří Matousek, Jaroslav Nešetřil (Charles University, Prague) and Marco Pellegrini (CNR, Pisa) and has been supported jointly by SNS and CRM Pisa and DIMATIA centre in Prague.

This workshop intended to reflect some key recent advances in combinatorics, particularly in the area of extremal theory and Ramsey theory. It also aimed to demonstrate the broad spectrum of techniques and its relationship to other fields of mathematics, particularly to geometry, logic and number theory.

Invited speakers included ten of the leading experts. We had the pleasure to invite Prof. Endre Szemerédi, the winner of the Abel Prize in 2012 for his fundamental contributions in the field of discrete mathematics and theoretical computer science. The workshop attracted 48 participants both from Italy and abroad.

The following list is that of the invited lectures at the workshop:

IMRE BARANY, *Tensors, colours, and octahedral*

BÉLA BOLLOBÁS, *Extremal and probabilistic results on bootstrap percolation*

MARIA CHUDNOVSKY, *Extending the Gyarfas-Sumner conjecture*

ZEEV DVIR, *Configurations of points with many collinear triples: going beyond Sylvester-Gallai*

ZOLTAN FUREDI, *Binary codes versus hypergraphs*

JAROSLAV NEŠETŘIL, *A unifying approach to graph limits II*

PATRICE OSSONA DE MENDEZ, *A unifying approach to graph limits I*

ALEX SCOTT, *Discrepancy in graphs, hypergraphs and tournaments*
and (second talk)
Szemerédi regularity lemma for sparse graphs

JOZSEF SOLYMOSI, *Sums vs. products*
and (second talk)
The (7,4)-*conjecture for finite groups*

ENDRE SZEMERÉDI, *On subset sums*

Given the success of both scientific and public workshops, at the end of the event, at the suggestion of Professor Mariano Giaquinta, it has been proposed to organize a volume dedicated to this meeting. This proposal was welcomed by all the speakers. The present volume has been edited for the "CRM Series", with the title "Geometry, Structure and Randomness in Combinatorics" and includes both original scientific articles in extended form or survey articles on results and problems inherent in the themes presented at the workshop. Each article submitted was reviewed.

We thank all the authors for their contribution and again Scuola Normale Superiore and its Centro di Ricerca Matematica Ennio De Giorgi and to DIMATIA Centre of Charles University for their generous support.

Pisa/Prague

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Tensors, colours, octahedra

Imre Bárány

Abstract. Several theorems in combinatorial convexity admit colourful versions. This survey describes old and new applications of two methods that can give such colourful results. One is the octahedral construction, the other is Sarkaria's tensor method.

1 Introduction

Theorems of Carathéodory, Helly, and Tverberg are classical results in combinatorial convexity. They all have coloured versions. Some others involve colours directly. For instance in Kirchberger's theorem [15], the elements of a finite set $X \subset \mathbb{R}^d$ are coloured Red and Blue, and the statement is that the Red and Blue points can be separated by a hyperplane if and only if for every $Y \subset X$ with $|Y| \le d+2$, the Red and Blue points in Y can be separated by a hyperplane.

The aim of this paper is to describe and explain old and new applications of two methods that have turned out to be useful when proving such colourful results. One is the octahedral construction, discovered and first used by László Lovász in 1991, which appeared in [4]. The other is Karinbir Sarkaria's tensor method, originally from [25] and developed further in [5].

In the next section Tverberg's theorem and its colourful version are presented. The octahedral construction is given in Section 3 with applications followed in later sections.

2 Tverberg's theorem and its coloured version

Tverberg's theorem is a gem, one of my favourites. Here is what it says.

Theorem 2.1. *Assume $d \ge 1, r \ge 2$ and $X \subset \mathbb{R}^d$ has $(r-1)(d+1)+1$ elements. Then X has a partition into r parts $X_1, \dots, X_r$ such that* $\bigcap_1^r \operatorname{conv} X_i \neq \emptyset$.

The number $(r-1)(d+1)+1$ is best possible here: for a general position X with one fewer element, the affine hulls of an r-partition do not have a common point (by counting dimensions).

The case $r=2$ is Radon's theorem from 1922 [21] that has a simple proof: Given $x \in \mathbb{R}^d$ we write $(x,1)$ for the $(d+1)$-dimensional vector whose first d components are equal to those of x, and the last one is 1. This time $|X|=d+2$ so the vectors $(x,1) \in \mathbb{R}^{d+1}$ have a nontrivial linear dependence $\sum \alpha(x)(x,1)=(0,0)$. Letting $X_1=\{x \in X : \alpha(x) \ge 0\}$ and $X_2=\{x \in X : \alpha(x)<0\}$ is the partition needed. Indeed, defining $\alpha=\sum_{x\in X_1}\alpha(x)$ and $\alpha^*(x)=\alpha(x)/\alpha$ for $x \in X_1$ and $\alpha^*(x)=-\alpha(x)/\alpha$ for $x \in X_2$, we have convex combinations in

$$z=\sum_{x\in X_1}\alpha^*(x)x=\sum_{x\in X_2}\alpha^*(x)x$$

showing that $z \in \operatorname{conv} X_1 \bigcap \operatorname{conv} X_2$.

There are several proofs of Tverberg's theorem, for instance in Tverberg [29,30], Tverberg and Vrećica [31], Roudneff [23], Sarkaria [25], Bárány and Onn [5], Zvagelskii [34], none of them easy. We will give another proof in Section 8 which is from Arocha *et al.* [1].

The coloured version of Tverberg's theorem follows now.

Theorem 2.2. *For every $d \ge 1$ and $r \ge 2$ there is $t=t(r,d)$ with the following property. Given sets $C_1,\dots,C_{d+1} \in \mathbb{R}^d$ (called colours), each of size t, there are r disjoint sets $S_1,\dots,S_r \subset \bigcup_1^{d+1} C_i$ such that $|S_j \cap C_i|=1$ for every i, j and $\bigcap_1^r \operatorname{conv} S_j \neq \emptyset$.*

In other words, given colours $C_1,\dots,C_{d+1} \subset \mathbb{R}^d$ of large enough size, there are r disjoint and colourful sets S_j whose convex hulls have a point in common. Colourful means that S_j is a transversal of the C_i, that is, S_j contains one element from each C_i. The need for this result emerged in connection with the halving plane problem (*c.f.* [3]). It was proved there that $t(3,2)$ is finite. Shortly afterward it was proved by Bárány and Larman [4] that $t(r,2)=r$ for all r, clearly the best possible result. The same paper presents Lovász's proof that $t(2,d)=2$ for all d, the first application of the octahedral method. To simplify notation we write $[k]$ for the set $\{1,2,\dots,k\}$.

3 The octahedral construction

Proof of $t(2,d)=2$. We have $C_i=\{a_i,b_i\} \subset \mathbb{R}^d$, $i \in [d+1]$. Note that we may exchange the names of a_i and b_i later. We want to choose a transversal T from $C_1,\dots,C_{d+1}$ such that the convex hulls of T and of the complementary transversal $\overline{T}$ have a point in common. For this purpose let

$$Q^{d+1} = \text{conv}\{\pm e_1, \ldots, \pm e_{d+1}\}$$

be the standard octahedron in $\mathbb{R}^{d+1}$ (the e_i are the usual basis vectors). We define a map $f : \partial Q^{d+1} \to \mathbb{R}^d$ by setting $f(e_i) = a_i$ and $f(-e_i) = b_i$, and then extend f simplicially to ∂Q^{d+1}, that is, to the facets of Q^{d+1}. Note that f maps the facets of Q^{d+1} to the convex hull of a transversal T exactly, and the opposite facet is mapped to conv $\overline{T}$. So what we need is a pair of opposite facets whose images intersect.

This cries out for the Borsuk-Ulam theorem: ∂Q^{d+1} is homeomorphic to S^d and so f is an $S^d \to \mathbb{R}^d$ map. By a variant of Borsuk-Ulam there are antipodal points $z, -z \in \partial Q^{d+1}$ with $f(z) = f(-z)$. If z lies on a facet F, then $-z$ lies on the opposite facet $\overline{F}$. For simpler writing assume that $F = \text{conv}\{e_1, \ldots, e_{d+1}\}$, then $\overline{F} = \text{conv}\{-e_1, \ldots, -e_{d+1}\}$, and we see that $\text{conv}\{a_1, \ldots, a_{d+1}\}$ and $\text{conv}\{b_1, \ldots, b_{d+1}\}$ have $f(z) = f(-z)$ as a common point.

Actually, more is true: if $z = \sum_1^{d+1} \gamma_i e_i$, then $-z = \sum_1^{d+1} \gamma_i(-e_i)$, and the common point is $\sum_1^{d+1} \gamma_i a_i = \sum_1^{d+1} \gamma_i b_i$. Thus the common point comes with the same coefficients in the convex combinations. □

This is the octahedral method. The basic idea is that facets of the octahedron correspond to transversals of $C_1, \ldots, C_{d+1}$, transversals have the structure of ∂Q^{d+1}, and disjoint transversals come from opposite facets, and the next step is the use of algebraic topology like the Borsuk-Ulam theorem above.

Unfortunately the method does not work for $r \geq 3$. It was conjectured in [4] that $t(r, d) = r$ for all r and d. Finiteness of $t(r, d)$ was proved by Živaljević and Vrećica [33] using equivariant topology. Their result is that $t(r, d) \leq 2r - 1$ if r is a prime (which implies finiteness of $t(r, d)$ for all r). The same was proved by different methods by Björner *et al.* [8] and by Matoušek [17]. More recently Blagojević, Matschke, and Ziegler [9] showed that $t(r, d) = r$ if $r + 1$ is a prime which is again best possible. The strange primality condition in all cases is needed because cyclic groups of prime order behave better in equivariant topology. But the theorem is probably true for every r, the primality condition is required for the method and not for the problem. It is however disappointing (for a convex geometer) that a completely convex (or linear, if you wish) problem does not have a direct convex (or linear) proof, and topology seems a necessity here. Finding a purely geometric proof remains a challenge. The interested reader may wish to read Günter Ziegler's fascinating article [32] about Tverberg's theorem and its colourful version.