
The Seventh European Conference on Combinatorics, Graph Theory and Applications

EuroComb 2013

edited by
Jaroslav Nešetřil and Marco Pellegrini



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SERIES

The Seventh European Conference on Combinatorics, Graph Theory and Applications

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Jaroslav Nešetřil and Marco Pellegrini



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Contents

Foreword	xxiii
Invited Speakers	xxv
Erdős problems	1
Roman Glebov, Daniel Král' and Jan Volec	
A problem of Erdős and Sós on 3-graphs	3
1 Introduction	3
2 Sketch of the proof	5
References	7
Zoltán Füredi, Dániel Gerbner and Máté Vizer	
An analogue of the Erdős-Ko-Rado theorem for multisets	9
1 Introduction	9
2 Concluding remarks	11
References	12
Jean-Florent Raymond and Dimitrios M. Thilikos	
Polynomial gap extensions of the Erdős-Pósa theorem	13
1 Introduction	13
2 Excluding packings of planar graphs	15
References	17
Dirk Meierling, Dieter Rautenbach and Thomas Sasse	
The Erdős-Pósa property for long circuits	19
1 Introduction	19
2 Proof of Theorem 1.1	20
References	24
Hypergraphs	25
Dan Hefetz and Peter Keevash	
A hypergraph Turán theorem via lagrangians of intersecting families. Extended abstract	27
1 Introduction	27
2 Lagrangians of intersecting 3-graphs	28

3	An application to a hypergraph Turán problem	29
4	Concluding remarks and open problems	31
	References	32
Andrew Treglown and Yi Zhao		
	Tight minimum degree conditions forcing perfect matchings in uniform hypergraphs	33
1	Introduction	33
2	Overview of the proof of Theorem 1.1	36
	References	37
Daniela Kühn, Deryk Osthus and Timothy Townsend		
	Fractional and integer matchings in uniform hypergraphs	39
	References	43
	Cubic graphs	45
Louis Esperet and Giuseppe Mazzuoccolo		
	On cubic bridgeless graphs whose edge-set cannot be covered by four perfect matchings	47
1	Introduction	47
2	An infinite family of snarks with excessive index 5	49
3	Shortest cycle cover	49
4	Open problems	50
	References	51
Simone Dantas, Felix Joos, Christian Löwenstein, Dieter Rautenbach and Deiwison S. Machado		
	Relating ordinary and total domination in cubic graphs of large girth	53
	References	57
Robert Lukot'ka, Edita Máčajová, Ján Mazák and Martin Škoviera		
	Snarks with large oddness and small number of vertices	59
1	Introduction	59
2	Oddness and resistance ratios	60
3	Lower bounds on oddness ratio	61
4	Constructions	62
	References	63
Robert Lukot'ka and Ján Mazák		
	Non-trivial snarks with given circular chromatic index	65
1	Introduction	65
2	Balanced schemes	67
	References	69

Graphs	71
Henning Bruhn, Pierre Charbit and Jan Arne Telle	
The graph formulation of the union-closed sets conjecture	73
References	77
Henning Bruhn and Oliver Schaudt	
The union-closed sets conjecture almost holds for almost all random bipartite graphs	79
1 Introduction	79
2 Discussion of averaging	80
3 Sketch of the proof of Theorem 2.2	82
References	84
Daniela Kühn, Allan Lo, Deryk Osthus and Katherine Staden	
The robust component structure of dense regular graphs	85
1 Introduction	85
2 Two applications to longest cycles in regular graphs . . .	87
3 Sketch of the proof of Theorems 1.2 and 2.2	89
References	90
Cristina Dalfó and Miquel Àngel Fiol	
The (Δ, D) and (Δ, N) problems in double-step digraphs with unilateral diameter	91
1 Preliminaries	91
2 The unilateral diameter of double-step digraphs with step $a=1$	93
3 The (Δ, D^*) and $(\Delta, N)^*$ problems for double-step digraphs with unilateral diameter	93
References	95
Swee Hong Chan, Henk D. L. Hollmann and Dmitrii V. Pasechnik	
Critical groups of generalized de Bruijn and Kautz graphs and circulant matrices over finite fields	97
1 Introduction	97
2 Preliminaries	98
3 Main results	101
References	103
Noga Alon and Andrey Kupavskii	
Two notions of unit distance graphs	105
1 Introduction	105
2 Results	106
References	109

Aida Abiad, Miquel A. Fiol, Willem H. Haemers and Guillem Perarnau	
An interlacing approach for bounding the sum of Laplacian eigenvalues of graphs	111
1 Introduction	111
2 New results	113
References	116
Yonah Cherniavsky, Avraham Goldstein and Vadim E. Levit	
On the structure of the group of balanced labelings on graphs	117
1 Abstract	117
2 Introduction	118
3 Results	118
4 Conclusions	121
References	121
Rémy Belmonte, Pim van't Hof, Marcin Kamiński and Daniël Paulusma	
The price of connectivity for feedback vertex set	123
1 Introduction	123
2 The proof of Theorem 1.3	126
References	128
Endre Csóka	
A local flow algorithm in bounded degree networks	129
1 Introduction	129
2 Model and results	130
3 Applications on neighborhood distributions	131
References	134
Fabrício Benevides, Victor Campos, Mitre C. Dourado, Rudini M. Sampaio and Ana Silva	
The maximum time of 2-neighbour bootstrap percolation: algorithmic aspects	135
1 Introduction	135
2 Results	137
References	138
Peter Keevash and Richard Mycroft	
A multipartite Hajnal-Szemerédi theorem	141
1 Introduction	141
2 New results	142
3 Rough outline of the proofs	144
4 Future directions	145
References	146

Andrea Jiménez, Mihyun Kang and Martin Loeb	
Directed cycle double covers: hexagon graphs	147
References	151
Peter Whalen and Robin Thomas	
Finding an Odd $K_{3,3}$	153
1 Introduction	153
2 Statement of the theorem	154
3 The proof technique	155
References	156
Jop Briët, Harry Buhrman, Monique Laurent, Teresa Piovesan and Giannicola Scarpa	
Zero-error source-channel coding with entanglement	157
1 Introduction	157
2 Our results	160
References	161
Ramsey Theory	163
Guilherme O. Mota, Gábor N. Sárközy, Mathias Schacht and Anusch Taraz	
Ramsey numbers for bipartite graphs with small bandwidth	165
1 Introduction and Results	165
2 Sketch of the proof of Theorem 1.3	167
References	169
Josef Cibulka, Pu Gao, Marek Krčál, Tomáš Valla and Pavel Valtr	
Polynomial bounds on geometric Ramsey numbers of ladder graphs	171
1 Introduction and basic definitions	171
2 Ladder graphs	172
3 Generalisation to pathwidth-2 outerplanar triangulations	175
References	175
Geometry and Surfaces	177
Ross J. Kang and Tobias Müller	
Arrangements of pseudocircles and circles	179
1 Introduction	179
2 A smallest circleable arrangement	180
3 Circleability and convexibility complexity	181
4 A folklore conjecture	182
References	183

Nick Gravin, Mihail N. Kolountzakis, Sinai Robins and Dmitry Shiryaev	
Extended abstract for structure results for multiple tilings in 3D	185
References	189
Jan Kynčl and Zuzana Safernová	
On the nonexistence of k-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$	191
1 Introduction	191
2 Angles in simplices and Coxeter diagrams	193
3 A simple proof of Theorem 1.1	194
References	195
Imre Bárány and János Pach	
Homogeneous selections from hyperplanes	197
1 The main result	197
2 Proof of Theorem 1.2	199
References	200
Amanda Folsom, Winfried Kohnen and Sinai Robins	
Conic theta functions and their relations to theta functions	203
References	208
Erika M.M. Coelho, Mitre C. Dourado, Dieter Rautenbach and Jayme L. Swarcfiter	
The Carathéodory number of the P_3 convexity of chordal graphs	209
1 Introduction	209
2 Results	210
References	213
Michal Kotrbčík and Martin Škoviera	
Locally-maximal embeddings of graphs in orientable surfaces	215
1 Introduction	215
2 Fundamentals	216
3 Constructions	217
4 Algorithms	218
References	219
Jorge Arocha, Javier Bracho, Natalia García-Colín and Isabel Hubard	
A characterization of triangulations of closed surfaces	221
1 Introduction	221
2 Motivation & contribution	221
3 Preliminaries	223

4	Proof of the theorem	224
	References	226
Massimo Giulietti		
	Quasi-perfect linear codes from singular plane cubics	227
1	Introduction	227
2	Complete caps from bicovering arcs	228
3	Small complete caps from cubic curves	229
	References	231
L. Sunil Chandran, Wilfried Imrich, Rogers Mathew and Deepak Rajendraprasad		
	Boxicity and cubicity of product graphs	233
1	Introduction	233
2	Our Results	235
	References	238
	Planarity	239
Marthe Bonamy		
	Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$-edge-choosable	241
1	Introduction	241
2	Method	242
3	Conclusion	243
	References	244
Martin Derka and Petr Hliněný		
	Planar emulators conjecture is nearly true for cubic graphs	245
1	Introduction	245
2	Cubic planar-emulable graphs	246
3	Conclusions	249
	References	249
Marc Noy and Lander Ramos		
	Random planar graphs with minimum degree two and three	251
1	Main results	251
2	Results for maps	255
	References	258
Robert Lukot'ka, Ján Mazák and Xuding Zhu		
	Degenerated induced subgraphs of planar graphs	259
1	Introduction	259
2	Sketch of the proof	261
	References	264

Mickaël Montassier, Arnaud Pêcher and André Raspaud	
Strong chromatic index of planar graphs with large girth	265
1 Introduction	265
2 On planar graphs with large girth	266
References	269
Reza Naserasr, Edita Rollová and Éric Sopena	
On homomorphisms of planar signed graphs to signed projective cubes	271
1 Preliminaries	271
2 An extension of the Four-Color Theorem	274
References	275
Gábor Korchmáros	
Classification of k-nets embedded in a plane	277
1 Introduction	277
2 k -nets embedded in $PG(2, \mathbb{K})$	277
3 3-nets embedded in $PG(2, \mathbb{K})$	278
References	280
Clemens Huemer and Anna de Mier	
An improved lower bound on the maximum number of non-crossing spanning trees	283
1 Introduction	283
2 The lower bound	284
3 Non-crossing forests of points in convex position	287
4 The upper bound	289
References	289
Cristina G. Fernandes, Tina Janne Schmidt and Anusch Taraz	
On the structure of graphs with large minimum bisection	291
1 Introduction and results	291
2 Trees	293
3 Planar graphs	294
4 Concluding remarks and open questions	295
References	295

Colorings 297

Michał Lasoń, Piotr Micek, Arkadiusz Pawlik and Bartosz Walczak

Coloring intersection graphs of arcwise connected sets in the plane	299
1 Introduction	299
2 Preliminaries	301
3 Proof sketch of Theorem 2.1	302
References	304

Guus Regts

A characterization of edge-reflection positive partition functions of vertex-coloring models	305
1 Introduction	305
2 Partition functions of edge and vertex-coloring models .	306
3 A sketch of the proof of Theorem 2.4	309
References	311

Marthe Bonamy, Nicolas Bousquet and Hervé Hocquard

Adjacent vertex-distinguishing edge coloring of graphs	313
1 Introduction	313
2 Method	315
3 Conclusion and perspectives	317
References	317

Anita Das, P. Suresh and S. V. Subrahmanya

Rainbow path and minimum degree in properly edge colored graphs	319
1 Introduction	319
2 Proof of the main results	321
References	324

Victor Campos, Carlos Lima and Ana Silva

b-coloring graphs with girth at least 8	327
1 Introduction	327
2 Improving tool lemmas	330
3 Comments and open questions	331
References	331

Barbora Candráková and Edita Máčajová

The circular chromatic index of k-regular graphs	333
1 Introduction	333
2 k -Regular graphs with $\chi'_c = k + a/p$	334
References	338

Carl Georg Heise, Konstantinos Panagiotou, Oleg Pikhurko and Anusch Taraz	
Coloring d-Embeddable k-Uniform Hypergraphs	339
References	343
Reza Naserasr, Edita Rollová and Éric Sopena	
Homomorphisms of signed bipartite graphs	345
1 Homomorphisms	345
2 Minors	346
3 Hadwiger's conjecture for signed bipartite graphs	347
References	349
Games	351
Tobias Müller and Miloš Stojaković	
A threshold for the Maker-Breaker clique game	353
1 Introduction	353
2 Our results	354
3 Conclusion and open problems	356
References	357
Dennis Clemens and Anita Liebenau	
On the threshold bias in the oriented cycle game	359
1 Introduction and Results	359
2 Outline of the proofs	360
3 Concluding remarks	363
References	363
Dennis Clemens, Asaf Ferber, Roman Glebov, Dan Hefetz and Anita Liebenau	
Building spanning trees quickly in Maker-Breaker games	365
1 Introduction	365
2 Outline of the proofs	367
3 Concluding remarks and open problems	369
References	370
Paul Dorbec, Gabriel Renault, Aaron Siegel and Éric Sopena	
Dicots, and a taxonomic ranking for misère games	371
References	374
Andrzej Grzesik, Mirjana Mikalački, Zoltán Lóránt Nagy, Alon Naor, Balázs Patkós and Fiona Skerman	
Avoider-Enforcer star games	375
1 Introduction	375
2 Results	377
References	379

Algebra and Polynomials **381**

Mirjam Friesen and Dirk Oliver Theis

Fooling-sets and rank in nonzero characteristic **383**

1	Introduction	383
2	Some remarks on the importance of fooling-set matrices	384
3	Fooling-Set Matrices from Linear Recurring Sequences .	386
4	Conclusion	388
	References	389

Tadashi Sakuma and Hidehiro Shinohara

Krasner near-factorizations and 1-overlapped factorizations **391**

1	Introduction	391
2	Preliminaries	393
3	Results	393
	References	395

Vladimir Blinovsky

Correlation inequality for formal series **397**

	References	400
--	----------------------	-----

Witold Kraszkiewicz and Jerzy Weyman

Covariants of spherical Θ -orbits for types E_6, E_7, E_8 **401**

1	Introduction	401
2	Methodology	401
3	Results	404
	References	406

Lorenzo Luperi Baglini

Partition regularity of nonlinear polynomials: a nonstandard approach **407**

	References	412
--	----------------------	-----

Randomness **413**

Florent Foucaud, Guillem Perarnau and Oriol Serra

Random subgraphs make identification affordable **415**

1	Introduction	415
2	Methods and proofs	417
3	Concluding remarks	419
	References	419

Elad Aigner-Horev and Hiệp Hàn

On two-point configurations in subsets of pseudo-random sets	421
1 Introduction	421
2 Sketch of the proof of Theorem 1.1	423
References	424

Michel Bode, Nikolaos Fountoulakis and Tobias Müller

On the giant component of random hyperbolic graphs	425
1 Introduction	425
2 The supercritical regime	427
3 The subcritical regime	428
4 Conclusions - Further directions	429
References	429

Hamed Amini, Nikolaos Fountoulakis and Konstantinos Panagiotou

Discontinuous bootstrap percolation in power-law random graphs	431
1 Introduction	431
2 Models and results	432
References	435

Michael Drmota and Anna Lladó

On a conjecture of Graham and Häggkvist for random trees	437
1 Introduction	437
2 Stable sets of random trees	438
3 The embedding	439
References	441

Katarzyna Rybarczyk

Sharp threshold functions via a coupling method	443
1 Introduction	443
References	447

Élie de Panafieu and Vlady Ravelomanana

Analytic description of the phase transition of inhomogeneous multigraphs	449
1 Model	450
2 Main theorem	451
3 Examples of application of the main theorem	453
References	453

Fixed-point **455**

Mahir Bilen Can, Yonah Cherniavsky and Tim Twelbeck

On the Bruhat-Chevalley order on fixed-point-free involutions **457**

1 Introduction 457

2 Preliminaries 459

3 Results 460

References 461

Elyot Grant and Will Ma

A geometric approach to combinatorial fixed-point theorems: extended abstract **463**

1 Introduction 463

2 Geometric proofs of Sperner-like theorems 466

References 467

Hamiltonicity **469**

Daniela Kühn, John Lapinskas, Deryk Osthus and Viresh Patel

Proof of a conjecture of Thomassen on Hamilton cycles in highly connected tournaments **471**

1 Sketch of the proof of Theorem 2 474

References 475

Béla Csaba, Daniela Kühn, Allan Lo, Deryk Osthus

and Andrew Treglown

Proof of the 1-factorization and Hamilton decomposition conjectures **477**

1 Introduction 477

2 Overview of the proofs of Theorems 1.1 and 1.3 480

References 481

Andrzej Dudek, Alan Frieze, Andrzej Ruciński and Matas Šileikis

Regular hypergraphs: asymptotic counting and loose Hamilton cycles **483**

References 486

Triangles **487**

Tom Bohman and Peter Keevash

Dynamic concentration of the triangle-free process **489**

1 Introduction 489

2 Overview of the proof 491

3 Concluding remarks 493

References 494

Zdeněk Dvořák, Jean-Sébastien Sereni and Jan Volec

Subcubic triangle-free graphs have fractional chromatic number at most $14/5$	497
1 Introduction	497
2 Main result	499
3 Conclusion	500
References	500

Nieke Aerts and Stefan Felsner

Henneberg steps for triangle representations	503
1 Introduction	503
2 Construction steps	505
3 Conclusion and open problems	508
References	508

Cycles and Girth of Graphs **511**

Robert Šámal

Cycle-continuous mappings – order structure	513
1 Introduction	513
2 Properties of cycle-continuous mappings	514
3 The proof	516
4 Concluding remarks	519
References	519

Silvia Messuti and Mathias Schacht

On the structure of graphs with given odd girth and large minimum degree	521
1 Introduction	521
2 Sketch of the proof	523
3 Concluding remarks	525
References	526

Julian Salas and Camino Balbuena

On the order of cages with a given girth pair	527
1 Cages with a given girth pair	527
2 Notation	528
3 Constructions for g odd	529
4 Constructions for g even and h odd	529
References	531

Enumerations, Lattices and Tableaux 533

Axel Bacher

Directed and multi-directed animals on the King's lattice 535

1	Introduction	535
2	Animals on the King's lattice and heaps of segments . .	536
3	Enumeration of directed animals	538
4	Enumeration of multi-directed animals	538
5	Asymptotic results	539
	References	539

Susanna Fishel and Matjaž Konvalinka

Results and conjectures on the number of standard strong marked tableaux 543

1	Introduction	543
2	Preliminaries	544
3	Main results and conjectures	545
4	Strong covers and k -bounded partitions	547
	References	548

Ron Aharoni, Daniel Kotlar and Ran Ziv

On independent transversals in matroidal Latin rectangles 549

	References	550
--	----------------------	-----

Axel Bacher and Gilles Schaeffer

Multivariate Lagrange inversion formula and the cycle lemma 551

1	Introduction	551
2	Generalized cycle lemma	552
3	The bijection	554
	References	555

Combinatorics and algorithms 557

Xavier Goaoc, Jiří Matoušek, Pavel Paták, Zuzana Safernová
and Martin Tancer

Simplifying inclusion - exclusion formulas 559

1	Introduction	559
2	Sketch of a proof of Theorem 1.1	562
	References	564

Dániel Gerbner, Gyula O. H. Katona, Dömötör Pálvölgyi
and Balázs Patkós

Majority and plurality problems 567

1	Introduction	567
---	------------------------	-----

2	Sketches of proofs and additional results	569
	References	571
David Hartman, Jan Hubička and Jaroslav Nešetřil		
	Combinatorial bounds on relational complexity	573
1	Introduction	573
2	Preliminaries	573
3	Relational complexity	574
	References	577
Antoine Deza, Frédéric Meunier and Pauline Sarrazolles		
	A combinatorial approach to colourful simplicial depth	579
1	Colourful simplicial depth	579
2	Combinatorial approach: octahedral systems	580
3	Additional properties of octahedral systems	581
	References	582
Vladimir Shenmaier		
	Complexity and approximation of the smallest k-enclosing ball problem	583
1	Hardness results	584
2	Approximation scheme	586
	References	588
Sourav Chakraborty, Akshay Kamath and Rameshwar Pratap		
	Testing uniformity of stationary distribution	589
1	Introduction	589
2	Preliminaries	590
3	Structure of graphs with uniform stationary distribution	590
4	Application to property testing	592
5	Conclusion	594
	References	594
Lale Özkahya and Brendon Stanton		
	On a covering problem in the hypercube	595
1	Introduction	595
	References	598
Marcin Gašiorek and Daniel Simson		
	A classification of positive posets using isotropy groups of Dynkin diagrams	599
1	Preliminaries and main results	599
2	Outline of the proof via matrix morsifications	601
3	Our calculation technique and concluding remarks	603
	References	604

Posters	607
Raúl M. Falcón Enumeration and classification of self-orthogonal partial Latin rectangles by using the polynomial method	609
Delia Garijo, Andrew J. Goodall and Jaroslav Nešetřil Polynomial graph invariants from homomorphism numbers	611
Vikram Kamat and Neeldhara Misra An Erdős–Ko–Rado theorem for matchings in the complete graph	613
References	613
Fábio Botler and Yoshiko Wakabayashi A constrained path decomposition of cubic graphs and the path number of cacti	615
Sagnik Sen On push chromatic number of planar graphs and planar p-cliques	617
References	617
Vitor Costa, Simone Dantas, Mitre C. Dourado, Lucia D. Penso and Dieter Rautenbach Firefighting with general weights	619
Eckhard Steffen and Michael Schubert Nowhere-zero flows on signed regular graphs	621
References	622
Bernadette Charron-Bost, Matthias Függer and Thomas Nowak New transience bounds for long walks in weighted digraphs	623
Julien Bensmail Complexity of determining the irregular chromatic index of a graph	625
References	625

Foreword

We are presenting in this volume the collection of extended abstracts that were selected for presentation at the European Conference on Combinatorics, Graph Theory and Applications - Eurocomb 2013, held in Pisa on September 9-13, 2013. Eurocomb conferences are organized biannually, the series was started in Barcelona 2001, and continued with Prague 2003, Berlin 2005, Seville 2007, Bordeaux 2009 and Budapest 2011. Since Prague Eurocomb 2003 the European Prize in Combinatorics is awarded at the meeting. The prize is established to recognize excellent contributions in Combinatorics, Discrete Mathematics and their Applications by young European researchers (eligibility of EU) not older than 35. It is supported by DIMATIA and by private sources.

The role of combinatorics is growing in the contemporary research in mathematics, computer science, engineering, also in physics, life and social sciences. This is reflected in the growing interest in Eurocomb conferences, which became an established forum of the frontier research in these fields. This year a total of 181 contributions were submitted and the Program Committee selected 95 extended abstracts and 9 posters to be presented at the conference. These figures, similar to those of the previous Eurocomb conference, indicate that Eurocomb meeting and its format got firmly established in the spectrum of high level European conferences. We thank all members of the Program Committee for their work. We also thank to all members of the Organizing Committee of Eurocomb 2013 for all their work in preparing the meeting in lovely Pisa.

The conference is organized by the Institute for Informatics and Telematics of the National Research Council of Italy (CNR), with the generous support of the Scuola Normale Superiore - Centro di Ricerca Matematica Ennio De Giorgi, the Dipartimento di Informatica - Università di Pisa, and the Center for Discrete Mathematics, Theoretical Computer Science and Applications (DIMATIA) of Charles University.

Jaroslav Nešetřil and Marco Pellegrini
(PC co-chairs)

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Stefan Felsner
Jacob Fox
Anna Galluccio
Jarek Grytczuk
Ervin Györi
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Invited Speakers

Endre Szemerédi

Packing Trees into a Graph

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Containers for independent sets

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Polynomiality for Packing and Scheduling in the High Multiplicity Setting

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Decomposition Theorems for Graphs Excluding Structure

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Ramsey Theory, Integer Partitions and a New Proof of the Erdős-Szekeres Theorem

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Extremal Combinatorics in Random Structures

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Erdős problems



A problem of Erdős and Sós on 3-graphs

Roman Glebov¹, Daniel Král'² and Jan Volec³

Abstract. We show that for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that every 3-uniform hypergraph on $n \geq n_0$ vertices with the property that every k -vertex subset, where $k \geq \delta n$, induces at least $\left(\frac{1}{4} + \varepsilon\right) \binom{k}{3}$ edges, contains K_4^- as a subgraph, where K_4^- is the 3-uniform hypergraph on 4 vertices with 3 edges. This question was originally raised by Erdős and Sós. The constant $1/4$ is the best possible.

1 Introduction

One of the most influential results in the extremal graph theory is the celebrated theorem of Turán [14] that determines the largest possible number of edges in an n -vertex graph without a complete subgraph of a given size. Erdős, Simonovits, and Stone [5, 7] generalized Turán's theorem by showing that the *extremal number* of an arbitrary graph F , defined as

$\text{ex}(n, F) := \max\{|E(G)| : G \text{ is a graph on } n \text{ vertices with no copy of } F\}$,
is asymptotically determined by its chromatic number. Specifically, for every graph F with at least one edge,

$$\text{ex}(n, F) = \left(\frac{\chi(F) - 2}{\chi(F) - 1} + o(1) \right) \binom{n}{2}.$$

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Note that this problem is dual to determining the minimum number of edges $m(F)$ that guarantees an n -vertex graph G with at least $m(n, F)$ edges to contain a copy of F , since $m(n, F) = ex(n, F) + 1$.

In Ramsey-Turán type problems, which were introduced by Sós in [13], we are again interested in the largest possible number of edges of an n -vertex graph without a copy of F , but under the additional restriction that the graph has to be somewhat “far” from the Turán graph. Typically, such a restriction is expressed by requiring that the graph has only sublinear independence number. For more details on Ramsey-Turán theory, see the survey by Simonovits and Sós [12].

In our problem, we are dealing with an even stronger restriction; we require every linear-size subset of vertices not only to induce at least one edge, but actually to induce a positive proportion of all possible edges. We define the δ -linear density of a graph G to be the smallest density induced on an δ -fraction of vertices, *i.e.*,

$$d(G, \delta) := \min \left\{ \frac{|E(G[A])|}{\binom{|A|}{2}} : A \subseteq V(G), |A| \geq \delta |V(G)| \right\}.$$

Note that for any graph G the function $d(G, \delta)$ is a non-decreasing function of δ taking values in $[0, 1]$.

However, requiring a positive δ -linear density immediately forces large graphs to behave similarly to random graphs in the sense that they contain every given graph as a subgraph, which follows, *e.g.*, from [11, Theorem 1].

Observation 1.1. For every $\varepsilon > 0$ and a fixed graph F , there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that every graph G on at least n_0 vertices with $d(G, \delta) \geq \varepsilon$ contains F as a subgraph.

The notion of the δ -linear density has a natural generalization to a k -uniform hypergraph H , where we define

$$d(H, \delta) := \min \left\{ \frac{|E(H_n[A])|}{\binom{|A|}{k}} : A \subseteq V(H), |A| \geq \delta |V(H)| \right\}.$$

We now focus on 3-uniform hypergraphs. Let K_4 be the complete hypergraph on 4 vertices, and K_4^- the hypergraph on 4 vertices with 3 edges. Erdős and Sós [6, Problem 5] asked whether, analogously to Observation 1.1 for graphs, a sufficiently large hypergraph with positive δ -linear density contains a copy of K_4 , or at least a copy of K_4^- . However, Füredi observed that the following construction of Erdős and Hajnal [4] shows that the situation in 3-uniform hypergraphs is completely different (hence giving a negative answer to the question above).

Example 1.2. Consider a random tournament T_n on n vertices. Let H_n be the 3-uniform hypergraph on the same vertex set consisting of exactly those triples that span an oriented cycle in T_n .

One can check that in every hypergraph obtained in this way, any four vertices span at most two edges, *i.e.*, H_n does not contain K_4^- for every n . On the other hand, for every $\delta > 0$, the δ -linear density of H_n tends to $1/4$ when n goes to infinity. Additional information about this problem and its history can be found in [12, Section 5]. It is also worth noting that in [11], Rödl presented a sequence of K_4 -free (but not K_4^- -free) 3-uniform hypergraphs with δ -linear density tending to $1/2$.

Our main result is the following theorem stating that the bound $1/4$ on δ -linear density for K_4^- -free sequences is in fact the best possible, which answers the explicit question of Erdős [3] positively.

Theorem 1.3. *For every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that every 3-uniform hypergraph H on at least n_0 vertices with $d(H, \delta) \geq 1/4 + \varepsilon$ contains K_4^- as a subgraph.*

2 Sketch of the proof

Suppose that Theorem 1.3 were false, then the following would be true.

There exists $\varepsilon > 0$ and a sequence $(H_n)_{n \in \mathbb{N}}$ of K_4^- -free hypergraphs with order tending to infinity such that $d(H_n, 1/n) \geq \frac{1}{4} + \varepsilon$. (★)

We say that a sequence of 3-uniform hypergraphs $(H_n)_{n \in \mathbb{N}}$ is *convergent* if for every fixed hypergraph F the probabilities $p(F, H_n)$ that random $|F|$ vertices of $V(H_n)$ induce a copy of F converge. We refer to $p(F, H_n)$ as to the *induced density* of F in H_n . By a standard compactness argument, every sequence of hypergraphs have a convergent subsequence. Therefore, we may assume that the sequence given in (★) is convergent.

Our aim is to show that the edge density of the sequence given in (★) must tend to zero, which would contradict having positive δ -linear density.

The main tool used for the proof of Theorem 1.3 is the framework of flag algebras. The framework was introduced by Razborov [10], who was inspired by the theory of dense graph limits from Borgs *et al.* [1, 2] and Lovász and Szegedy [9]. However, unlike just applying the standard flag algebra approach, in order to prove Theorem 1.3 we need to find a way of expressing the δ -linear density condition in this framework. Instead of using the assumption on δ -linear density itself, we establish a certain type of inequalities that are valid for any convergent sequence of K_4^- -free hypergraphs (H_n) such that $d(H_n, f(n)) \geq 1/4$, where $f(n) \rightarrow 0$. Let

us be more specific about that. Fix a K_4^- -free hypergraph F , and consider an arbitrary K_4^- -free superhypergraph F' of F on $|F| + 1$ vertices. For any such a choice of a pair (F, F') , we obtain one inequality as follows. Let S be a fixed copy of F in H_n , and let $U(S)$ be the set of vertices v of $V(H_n) \setminus V(S)$ such that $S \cup \{v\}$ induces F' in H_n . Let T be now three random vertices from H_n not contained in S . The following two outcomes are possible:

- If the size of $U(S)$ is small, *i.e.*, $o(|V(H_n)|)$, then both the probability that T is a subset of $U(S)$ and the probability that T is a subset of $U(S)$ and spans an edge in H_n , are $o(1)$.
- If the size of $U(S)$ is large, *i.e.*, $\Omega(|V(H_n)|)$, then, by the assumption on the δ -linear density, the probability that $T \subseteq U(S)$ and span an edge is at least quarter of the probability that $T \subseteq U(S)$.

Therefore, we get that the following holds for every choice of S

$$4 \cdot P[T \subseteq U(S) \wedge T \in E(H_n)] - P[T \subseteq U(S)] \geq -o(1). \quad (2.1)$$

Note that the little o goes to zero with n tending to infinity. The sought inequality corresponding to the pair (F, F') is then obtained by taking the expectation of (2.1), where the randomness comes from picking a copy S of F in H_n . This expectation can be expressed using the language of flag algebras.

It turned out, that for the proof of Theorem 1.3, it is enough to consider the inequalities for the pairs (F_2, F_3^0) and (F_2, F_3^1) , where F_2 is the only 2-vertex 3-uniform hypergraph, F_3^0 is the empty 3-vertex hypergraph, and F_3^1 is the 3-vertex hypergraph with one edge.

Recall that our aim is to show that edge density of the sequence given in $(\star)$ tends to zero. Unfortunately, we do not know how to apply flag algebras directly to derive this claim. Instead, we use them only to show that induced density of F_5 tends to zero, where F_5 is the 5-vertex hypergraph in Figure 2.1.

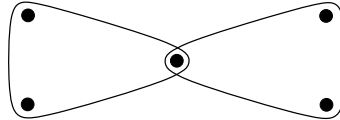


Figure 2.1. The hypergraph F_5 .

To cope with this, we *sparsify* each hypergraph H_n given in $(\star)$, *i.e.*, we remove each edge of H_n independently with a sufficiently small but fixed positive probability, say $\varepsilon/2$. Therefore, we obtain a new sequence of